COHEN-MACAULAYNESS OF MODULES OF INVARIANTS FOR SL_2

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Abstract. Let U, W be finite dimensional representations of $G = SL_2$. We give conditions under which $(U \otimes k[W])^G$ is a Cohen-Macaulay $k[W]^G$-module. In particular we obtain an invariant theoretic proof of the fact that the trace ring of generic 2×2 matrices is Cohen-Macaulay. [11]

1. Introduction.

Let G be a reductive algebraic group over an algebraically closed field of characteristic zero and let W be a finite dimensional representation of G. Then G acts on the polynomial ring $k[W]$ and the Hochster-Roberts theorem tells us that $k[W]^G$ is Cohen-Macaulay [8].

In this paper we study a question that looks very similar. Let U be another finite dimensional representation of G. Then G acts on the free $k[W]$-module $U \otimes k[W]$ and we ask whether $(U \otimes k[W])^G$ is a Cohen-Macaulay module over $k[W]^G$.

Unfortunately the answer to this question is no in general. Stanley gave a complete answer in the case that G is a torus. In this case there are interesting connections with linear diophantine equations [15].

We give a simple example where $(U \otimes k[W])^G$ is not Cohen-Macaulay.

Example 1.1. Let $G = G_m$, $R = k[X, Y, Z]$, $M = k[X, Y, Z]$ and G_m acts on R and M as follows : Let $\alpha \in G_m$, $f \in R$ and $g \in M$. Then $\alpha.f = f(\alpha X, \alpha Y, \alpha^{-1} Z)$ and $\alpha.g = \alpha^{-1} g(\alpha X, \alpha Y, \alpha^{-1} Z)$. Hence $R^G = k[XZ, YZ]$ and $M^G = (XZ, YZ)Z^{-1}$. Clearly M^G is not a Cohen-Macaulay module.

If χ is a generator for $\chi(G_m)$ then this example corresponds to $U = \chi^{-1}$ and $W = \chi \oplus \chi \oplus \chi^{-1}$.

Let us also mention that if $(U \otimes k[W])^G$ is Cohen-Macaulay then the Poincaré series of $(U \otimes k[W])^G$ satisfies a sort of functional equation. In [16, Th. 4.3] Stanley gives a sufficient condition for the existence of such a functional equation.

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Our main motivation for studying $(U \otimes k[W])^G$ lies in trace rings of generic matrices $n \times n$-matrices. (See e.g. [3][11][12][14].)

Fix integers m and n and let $X_k = (x_{ij}^{(k)})_{1\le i,j\le n}$, $1 \le k \le m$ be m $n \times n$-matrices in $M_n(k[x_{ij}^{(k)}])$. Let $\mathbb{G}_{m,n}$ be the k-algebra generated by $X_1, \dots, X_m$. This is called the ring of m generic $n \times n$-matrices.

$\mathbb{G}_{m,n}$ has many fine properties. Among other things, it is an affine prime PI-algebra. It is not Noetherian however. Let $T(\mathbb{G}_{m,n})$ denote the set of all traces of elements in $\mathbb{G}_{m,n}$ (as a subring of $M_n(k[x_{ij}^{(k)}])$). Then $\mathbb{T}_{m,n} = \mathbb{G}_{m,n}T(\mathbb{G}_{m,n})$ is called the trace ring of m generic $n \times n$-matrices. (The notations $\mathbb{T}_{m,n}$, $\mathbb{G}_{m,n}$ are due to L. Lebruyn.)

$\mathbb{T}_{m,n}$ is a affine Noetherian prime PI-algebra, finitely generated over its center. The geometric meaning of $\mathbb{T}_{m,n}$ is that is parametrizes (in a non-commutative way) the irreducible components of the semisimple representations of dimension n of the free algebra $k\langle X_1, \dots, X_m\rangle$ [1][14].

There is a different description of $\mathbb{T}_{m,n}$ that is more suitable for us. Let V be an n-dimensional k-vectorspace and let $W = (V^* \otimes V)^m$, $U = V^* \otimes V$ and $G = SL(V)$. Then $\mathbb{T}_{m,n} = (U \otimes k[W])^G$.

After computations in low dimensions, L. Lebruyn conjectured that $\mathbb{T}_{m,n}$ is always a Cohen-Macaulay module over its center. This was proved by him in the case of 2×2-matrices using the theory of Clifford algebras.

The trace ring of 2×2-matrices is a module of invariants for SL_2. Now the representation theory of SL_2 is almost as simple as the representation theory of a torus, hence it is natural to study the Cohen-Macaulayness of $(U \otimes k[W])^G$ in this case first.

This is precisely what we do in this paper. We provide some tools (Th. 3.1, Cor. 5.4 and Lemma 5.6) that make it possible to give a positive answer for certain pairs U, W. In particular we recover the Cohen-Macaulayness of the trace ring of generic 2×2 matrices.

On the other hand we make the assumption that the unstable locus in $\operatorname{proj} k[W]$ is smooth. This puts a severe restriction on the possible W's.

In general however one can always apply Theorem 3.1 to an embedded resolution of the unstable locus. This is the subject of some ongoing research on which I will report in a forthcoming paper.

2. Some preliminaries.

2.1. **Homogeneous bundles.** In this section we describe some of the properties of homogeneous bundles. All these properties are well known and easily proved by faithfully flat descent. I have not been able to locate a convenient reference however.

Here and in the next sections k will be an algebraically closed field of characteristic zero. All schemes will be k-schemes. Fiber products are over $\operatorname{Spec} k$ unless otherwise specified.

If G is an algebraic group and P is an algebraic subgroup of G then the quotient morphism is faithfully flat [5, Expo. VI_{A}, Th. 3.2]. If Y is a scheme with a P action then $G\times^P Y$ is defined informally as $G\times Y/P$ where P acts as $p(g,y)=(gp^{-1},py)$. Formally $G\times^P Y$ is defined by putting appropriate descent data on $G\times Y$.

Projection on the first factor defines a morphism $G\times^P Y\to G/P$ whose fibers are all isomorphic to Y. By construction $G\times_{G/P}(G\times^P Y)\cong G\times Y$.

If S is a scheme and H is a group scheme acting on S then let us denote by Sch_H/S the category of S-schemes with a H-action compatible with the H-action on S.

Then $G\times^P ?$ defines a functor $\mathrm{Sch}_P/\{P\}\to\mathrm{Sch}_G/(G/P)$. To simplify the notation we will often denote this functor by $\tilde{\ }$.

Let $\phi : X'\to G$ be a G-equivariant map. Then $x\to(\phi(x),\phi(x)^{-1}x)$ and $(g,x)\to gx$ define explicit maps between X' and $G\times X'_e$ which are each others inverse.

Similarly if $\phi' : X'\to G$, $\phi'' : X''\to G$ and $f : X'\to X''$ are G-equivariant such that $\phi'' f=\phi'$ then the isomorphisms defined above give rise to a commutative diagram :

(1)
$$\begin{array}{ccc} X' & \xrightarrow{\cong} & G\times X'_e \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1\times f_e} \\ X'' & \xrightarrow[\cong]{} & G\times X''_e \end{array}$$

Hence taking the fiber of e defines an equivalence of categories between Sch_G/G and Sch/e.

Now suppose that we are given $\phi : X\to G/P$, also G-equivariant and assume that the fiber of $\{P\}$ is Y. Then there is a canonical morphism $\pi : G\times^P Y\to X : (g,y)\to gy$ which is an isomorphism on the fibers of $\{P\}$. $G\times_{G/P}\pi$ is a map of G-schemes which is an isomorphism on the fibers of e, so by (1) $G\times_{G/P}\pi$ is an isomorphism, but this means, by faithfully flat descent, that π is also an isomorphism.

Hence $\tilde{\ }$ actually defines an equivalence of categories between $Sch_P/\{P\}$ and $Sch_G/(G/P)$.

Finally assume that we are given a P-equivariant vector bundle $E\to Y$. Then by applying $G\times_{G/P}?$ together with faithfully flat descent one sees that $\tilde{E}\to\tilde{Y}$ is also a vector bundle. Furthermore one verifies that

˜ is compatible with all the usual vector bundle operations f^*, $\otimes$, S^n, Λ^n, exact sequences, etc...

If E is given by its sheaf of sections $\mathcal{E}$ then we will use the notation $\tilde{\mathcal{E}}$ to denote the sheaf of sections of $\tilde{E}$.

What we have shown above implies that a vector bundle on $\tilde{Y}$ is uniquely determined by its fiber over $x = \{P\}$. I.e if $\mathcal{F}$ is a G-equivariant vector bundle on $\tilde{Y}$ then $\widetilde{(\mathcal{F}_x)} \cong \mathcal{F}$. This fact will be used heavily in the sequel.

2.2. **Collapsing of homogeneous bundles.** In the sequel we will encounter the following situation : Y is a closed subvariety of a variety X on which an algebraic group G acts. In general the union of all conjugates of Y is only a constructable set (denoted by GY). We will need a criterion under which GY is nice.

Such a criterion is provided by Kirwan.

If $P \subset G$ are algebraic groups then we will denote by $\mathbf{p} \subset \mathbf{g}$ their respective Lie algebras.

If G-acts on a scheme X then there is an induced action $\zeta \to \zeta_x$ from $\mathbf{g}$ to the tangent space T_xX for each $x \in X$.

Proposition 2.1. *Let Y be a closed subvariety of a variety X on which an algebraic group G acts. Assume that there is a parabolic subgroup of G with the property that for all $y \in Y$*

$$P = \{g \in G \mid gy \in Y\}$$

$$\mathbf{p} = \{\zeta \in \mathbf{g} \mid \zeta_y \in T_yY\}$$

Then the natural map $G \times^P Y \to X : (g, y) \to gy$ is a closed immersion.

Proof. This fact can be distilled from the proof of [10, Th 13.6] □

3. The method.

Let G be a reductive algebraic group over an algebraically closed field k of characteristic zero and let U be an irreducible and W an arbitrary finite dimensional representation of G. (Assuming U irreducible is no restriction since we can always analyze the irreducible components of U separately.)

Define $R = k[W]$, $M = U \otimes k[W]$, $h + 1 = \dim R^G$ and $d + 1 = \dim R = \dim_k W$. Let $X = \operatorname{proj} R$ and let X^u the locus of G-unstable points in X. The defining ideal for X^u is given by the graded ideal $I = rad((R^+)^G R)$ in R. Let $\mathcal{I}$ be the corresponding sheaf of ideals in $\mathcal{O}_X$. Obviously I and $\mathcal{I}$ are G-invariant.

The following criterion for M^G to be Cohen-Macaulay is easily proved :

Theorem 3.1. *If $U \otimes (\Lambda^{d+1}W)^*$ does not occur as a G-representation in $H^j(X^u, (\mathcal{I}^t/\mathcal{I}^{t+1})(l))$ for all t, l and $j = d-h, \ldots, d$ then M^G is Cohen-Macaulay.*

Proof. If $d-h=0$ the there is nothing to prove, so we assume $d-h \geq 1$. A well known criterion for M^G to be a Cohen-Macaulay R^G module is that $H^i_{(R_+)^G}(M^G) = 0$ for $i = 0, \ldots, h$ [15]. Now by a simple generalization of [9, Lemma 4.5] $H^i_{(R^+)^G}(M^G) = (H^i_I(M))^G = H^i_I(U \otimes R)^G = (U \otimes H^i_I(R))^G$ which is non-zero if and only if $U \otimes H^i_I(R)$ does not contain a trivial representation, i.e. if and only if U^* does not occur in $H^i_I(R)$. Now by definition

$$H^i_I(R) = \lim_{\rightarrow} \operatorname{Ext}^i_R(R/I^t, R)$$

Hence any representation that occurs in $H^i_I(R)$ occurs in at least one $\operatorname{Ext}^i_R(I^t/I^{t+1}, R)$.

But by local duality applied to the localization of R at R^+ : $H^{d+1-i}_{R^+}(I^t/I^{t+1}) = \operatorname{Hom}_R(\operatorname{Ext}^i_R(I^t/I^{t+1}, R), J)$ where $J = H^{d+1}_{R^+}(R)$ [7, Thm 6.3].

Let J' be the graded R-module defined by

$$J' = \lim_{\rightarrow} \operatorname{Hom}_k(R/(R^+)^n, k)$$

then one computes (somewhat laboriously) from the definition

$$J = \lim_{\rightarrow} \operatorname{Ext}^{d+1}_R(R/(R^+)^n, R)$$

that $J = (\Lambda^{d+1}W)^* \otimes_k J'$ as G-module.

Hence

$$\begin{aligned} H^{d+1-i}_{R^+}(I^t/I^{t+1}) &= \operatorname{Hom}_R(\operatorname{Ext}^i_R(I^t/I^{t+1}, R), \lim_{\rightarrow} \operatorname{Hom}_k(R/(R^+)^n, k)) \otimes (\Lambda^{d+1}W)^* \\ &= \lim_{\rightarrow} \operatorname{Hom}_k(\operatorname{Ext}^i_R(I^t/I^{t+1}, R) \otimes R/(R^+)^n, k) \otimes (\Lambda^{d+1}W)^* \end{aligned}$$

So U^* will not occur in $\operatorname{Ext}^i_R(I^t/I^{t+1}, R)$ if and only if $(\Lambda^{d+1}W)^* \otimes U$ does not occur in $H^{d+1-i}_{R^+}(I^t/I^{t+1})$.

But by [4, Ch III] $H^{d+1-i}_{R^+}(I^t/I^{t+1})$ is a quotient of the graded R-module $\oplus_{l \in \mathbb{Z}} H^{d-i}(X, \mathcal{I}^t/\mathcal{I}^{t+1}(l))$ (if $d-i \geq 1$ this is even an isomorphism).

It suffices now to note that $H^{d-i}(X, \mathcal{I}^t/\mathcal{I}^{t+1}(l)) = H^{d-i}(X^u, \mathcal{I}^t/\mathcal{I}^{t+1}(l))$ to complete the proof of 3.1. □

4. The description of the unstable locus.

Our aim is now to apply Theorem 3.1. For this we have to understand the unstable locus in X. This is accomplished by the Hilbert-Mumford criterion which we will briefly recall in this section.

We keep the notations of the previous sections. In addition we define $X^* = \operatorname{Spec} k[W]$ and $X^{*u} = \operatorname{Spec} k[W]/I$. The k-points of X^* are in one-one correspondence with the elements of the vector space W^*.

If $\lambda : k^* \to G$ is a one-parameter subgroup then we can choose a basis in W^* such that the action of λ is diagonal. Hence λ is given by $z \to \operatorname{diag}(z^{r_1}, \dots, z^{r_{d+1}})$. If $x = (x_1, \dots, x_{d+1}) \in W^*$ then one defines $m(x;\lambda) = \min\{r_j \mid x_j \neq 0\}$. In [13, Th. 2.1] Mumford proves $x \in X^{*u} \iff m(x;\lambda) > 0$ for some λ. This is the so-called Hilbert-Mumford criterion.

By elementary theory of algebraic groups it follows that any one-parameter subgroup of G can be factored through a maximal torus. Since all maximal tori are conjugate we can write any one-paramater subgroup of G as $g^{-1}\lambda g$ where λ is a one-parameter subgroup of some fixed maximal torus T.

$X^{*u}_\lambda = \{x \in X^* \mid m(x;\lambda) > 0\}$ is clearly a linear subspace of X^*. Since $m(x; g^{-1}\lambda g) = m(gx;\lambda)$ we see that $X^{*u}_{g^{-1}\lambda g} = gX^{*u}_\lambda$. Hence $X^{*u} = \bigcup_\lambda GX^{*u}_\lambda$ where λ runs over the one-parameter subgroups of T. Projectivizing one obtains a similar statement $X^u = \bigcup_\lambda GX^u_\lambda$.

In the sequel we will restrict ourselves to $G = SL(V)$ where V is a two dimensional k-vector space. The representation theory of $SL(V)$ is particulary simple. All irreducible representations of $SL(V)$ are of the form S^kV, $k \geq 0$.

Lemma 4.1. *Let $G = SL(V)$. Then $X^{*u} = GX^{*u}_\lambda$ and $X^u = GX^u_\lambda$ where λ is given by $z \to diag(z, z^{-1})$.*

Proof. A general λ is of the form $z \to diag(z^a, z^{-a})$ but it is immediately verified that there are only two different X^{*u}_λ's, one corresponding to $a > 0$ and one corresponding to $a < 0$. They are transformed into each other by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ □

Lemma 4.2. *Let $G = SL(V)$ and assume that W contain only direct summands (as G-representation) of the form V or S^2V. Then there is a Borel subgroup P of G acting on X^u_λ such that the natural map $G \times^P X^u_\lambda \to GX^u_\lambda = X^u$ is an isomorphism of varieties.*

Proof. If $G = SL(V)$ one verifies immediately (using the Hilbert-Mumford criterion) that the stabilizer of X^u_λ is a Borel subgroup conjugate to $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$ Furthermore the hypothesis for Proposition 2.1 are easily checked in the case that $W = V$ of $W = S^2V$. But then they are also true for direct sums of representations of this form. □

5. The computation of $\mathcal{I}^t/\mathcal{I}^{t+1}$.

In this section we keep the notation of the previous sections. G will now be $SL(V)$ where V is a two-dimensional vector space over k. W is a finite dimensional G-representation containing only direct summands of the form V or $S^2(V)$. In section 4 we have seen that $X^u = GX^u_\lambda \cong G \times^P X^u_\lambda$ where X^u_λ is a linear subspace of X and P is a Borel subgroup of G.

To soften the notation we will put $Y = X^u_\lambda$, $S = G/P$. There is a natural map $X^u \cong G \times^P X^u_\lambda \to G/P = S$ which will be denoted by ϕ. Y will be identified with the fiber of some closed point $x \in G/P$. X^u is a projective bundle on S and hence it will be of the form $\mathbb{P}_S(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on S.

Let $\mathcal{I}'$ be the ideal sheaf of Y in X and let $I' \subset R$ be the corresponding graded ideal. Then I' is generated by some linear subspace $W' \subset W$. Put $W'' = W/W'$.

Finally let $\mathcal{O}_X(1)$ be the line bundle associated to a hyperplane in X. This bundle restricts to a line bundle on X^u which as usual is denoted by $\mathcal{O}_{X^u}(1)$. In this case this leads to an annoying notation conflict. Since $X^u = \mathbb{P}_S(\mathcal{E})$ there is a twisting sheaf on X^u which is classically denoted by $\mathcal{O}_{X^u}(1)$ too [6, pp 160]. To avoid confusion let us momentarily denote this twisting sheaf by $\mathcal{O}_{X^u}(1)'$. It is immediately verified that $\mathcal{O}_{X^u}(1)$ and $\mathcal{O}_{X^u}(1)'$ agree on the fibers of ϕ. Hence $\mathcal{O}_{X^u}(1)' = \mathcal{O}_{X^u}(1) \otimes \phi^*\mathcal{L}$ [6, Ex II.5.9] for some line bundle $\mathcal{L}$ on S. By changing $\mathcal{E}$ into $\mathcal{E} \otimes \mathcal{L}$ we can then assume that $\mathcal{O}_{X^u}(1) = \mathcal{O}_{X^u}(1)'$. This is the assumption that will be made in the sequel.

Lemma 5.1. *With assumptions as above* $\mathcal{E} = \tilde{W}''$

Proof. As usual $\mathcal{E} = \phi^*\mathcal{O}_{X^u}(1)$. In this case however we can take the fiber for $x \in S$ [4, par. 7]. Hence $\mathcal{E}_x = \phi^*\mathcal{O}_Y(1)$ and since $Y = \operatorname{proj} k[W'']$ one sees that $\mathcal{E}_x = W''$. Hence $\mathcal{E} = \tilde{W}''$ □

Before we continue we state a standard lemma.

Lemma 5.2. *Let $U \subset V \subset W$ be schemes such that U is a local complete intersection in V and V is a local complete intersection in W. Assume that the ideal sheaves defining U in V, V in W and U in W are respectively $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$. Then there is an exact sequence of vector bundles on U :*

$$0 \to \mathcal{J}/\mathcal{J}^2 \otimes \mathcal{O}_U \to \mathcal{K}/\mathcal{K}^2 \to \mathcal{I}/\mathcal{I}^2 \to 0 \tag{2}$$

where the maps are defined in the obvious way.

Proof. It suffices to prove this in the case that $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$, $W = \operatorname{Spec} C$ where C is local.

There are surjective maps $C \xrightarrow{\pi} B \xrightarrow{\pi'} A$ associated to the inclusions $U \subset V \subset W$. Let $J = \ker \pi$, $I = \ker \pi'$, $K = \ker(\pi'\pi)$. Clearly $K = \pi^{-1} I$.

Associated to (2) there is a complex of A modules

$$0 \to J/J^2 \otimes_C A \xrightarrow{i} K/K^2 \to I/I^2 \to\to 0 \tag{3}$$

Now $K/K^2 \cong K \otimes_C C/K \cong K \otimes_C A$, $J/J^2 \otimes_C A \cong J \otimes_C B \otimes_C A \cong J \otimes_C A$, $I/I^2 \cong I \otimes_C B/I \cong I \otimes_C A$. Then one verifies that (3) is obtained by tensoring the exact sequence

$$0 \to J \hookrightarrow K \xrightarrow{\pi} I \to 0$$

with A. Hence (3) will always be right exact. To show that i is injective we compute $J/J^2 \otimes_C A \cong J/J^2 \otimes_C C/K = J/(JK + J^2) = J/JK$ since $J \subset K$.

Hence i will be injective iff $J \cap K^2 = JK$. It is easily verified that is true using the fact that J and K are generated by regular sequences. □

Proposition 5.3. *With notations as above there is an exact sequence*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \phi^* \tilde{W}' \otimes \mathcal{O}_{X_u}(-1) \to \phi^* \Omega_{S/k} \to 0 \tag{4}$$

Proof. Since Y, X^u and X are smooth, Y and X^u are local complete intersections. Hence we can use Lemma 5.2 to describe $\mathcal{I}/\mathcal{I}^2$. Let m_x be the maximal ideal in $\mathcal{O}_S$ defining x. Then Y in X^u is defined by the ideal $\phi^*(m_x)$. Hence (2) translated to the present situation reads as

$$0 \to (\mathcal{I}/\mathcal{I}^2)_x \to \mathcal{I}'/\mathcal{I}'^2 \to \phi^*(m_x)/\phi^*(m_x^2) \to 0 \tag{5}$$

Then one makes the following observations

- Since I' is generated by a linear subspace W' of W one computes that $\mathcal{I}'/\mathcal{I}'^2 = \phi^*(W') \otimes \mathcal{O}_Y(-1) = \phi^*(W') \otimes \mathcal{O}_{X^u}(-1)_x$
- Since ϕ is flat $\phi^*(m_x)/\phi^*(m_x)^2 \cong \phi^*(m_x/m_x^2)$ but $m_x/m_x^2 \cong (\Omega_{S/k})_x$ [6, II.8.7]. Hence $\phi^*(m_x)/\phi^*(m_x^2) \cong \phi^*(\Omega_{S/k})_x$.

Applying $\tilde{}$ to (5) yields (4) □

Corollary 5.4. *For $t \geq 0$ and l arbitrary, there are exact sequences*

(6)

$$0 \to (\mathcal{I}^t/\mathcal{I}^{t+1})(l) \to \phi^*(S^t \tilde{W}')(l-t) \to \phi^*(S^{t-1}\tilde{W}' \otimes \Omega_{S/k})(l-t+1) \to 0$$

(In the case t=0 we follow the convention that $S^{-1}(?) = 0$).

Proof. Since X^u is smooth, X^u is a local complete intersection in X. Hence $\mathcal{I}^t/\mathcal{I}^{t+1} \cong S^t(\mathcal{I}/\mathcal{I}^2)$.

The case $t = 0$ is a tautology. If $t > 0$ then (6) is obtained by taking symmetric powers of (4) (using the fact that $\Omega_{S/k}$ is a line bundle since $S \cong \mathbb{P}^1$). □

The sequences (6) will be used to compute the cohomology of $\mathcal{I}^t/\mathcal{I}^{t+1}(l)$. To do this we need another standard lemma.

Lemma 5.5. *Let S be a scheme and let $X = \mathbb{P}_S(\mathcal{E})$ where $\mathcal{E}$ is some vector bundle of rank r on S. Let ϕ denote the structure map $X \to S$ and let $\mathcal{F}$ be some other vector bundle on S. Then*

$$H^i(X, (\phi^*\mathcal{F})(l)) =$$
$$\begin{array}{ll} H^i(S, \mathcal{F} \otimes_{\mathcal{O}_S} S^l\mathcal{E}) & \text{if } l \geq 0 \\ 0 & \text{if } -r < l < 0 \\ H^{i-r+1}(S, \mathcal{F} \otimes_{\mathcal{O}_S} (\Lambda^r\mathcal{E})^* \otimes_{\mathcal{O}_S} (S^{-l-r}\mathcal{E})^*) & \text{if } l \leq -r \end{array}$$

Proof. This follows from the Leray spectral sequence for ϕ and the fact that

$$\begin{array}{rcll} R^j\phi_*(\phi^*\mathcal{F}(l)) & = & \mathcal{F} \otimes_{\mathcal{O}_S} S^l\mathcal{E} & \text{if } j = 0 \\ & = & 0 & \text{if } j \neq 0, r-1 \\ & = & \mathcal{F} \otimes_{\mathcal{O}_S} (\Lambda^r\mathcal{E})^* \otimes_{\mathcal{O}_S} (S^{-l-r}\mathcal{E})^* & \text{if } j = r-1 \end{array}$$

[6, Ex III.8.3,8.4]. Here as usual a negative symmetric power is to be interpreted as 0. □

From this Lemma we deduce that the cohomology of the last two terms in (6) only lives in degrees 0, 1, $r-1$, r where r is the rank of W''.

If we assume that G acts generically free on X then $d - h = 3$ and hence the cohomology in degrees 0, 1 has no influence on the cohomology of $\mathcal{I}^t/\mathcal{I}^{t+1}(l)$ in degrees $d - h$ and higher.

So by Theorem 3.1 we only have to look in degrees $r-1$, r.

To simplify the notation we define

$$\begin{array}{rcl} \mathcal{A}_{l,t} & = & \phi^*(S^t\tilde{W}')(l-t) \\ \mathcal{B}_{l,t} & = & \phi^*(S^{t-1}\tilde{W}' \otimes \Omega_{S/k})(l-t+1) \end{array}$$

Lemma 5.6. *Let $i = 0$, 1.*
If $t - l - r \geq 0$ then

$$H^{r-1+i}(X^u, \mathcal{A}_{l,t}) = H^i(S, S^t\tilde{W}' \otimes (\Lambda^r\tilde{W}'')^* \otimes (S^{t-l-r}\tilde{W}'')^*)$$

If $t - l - r - 1 \geq 0$ then

$$H^{r-1+i}(X^u, \mathcal{B}_{l,t}) = H^i(S, S^{t-1}\tilde{W}' \otimes (\Lambda^r\tilde{W}'')^* \otimes (S^{t-1-l-r}\tilde{W}'')^* \otimes \Omega_{S/k})$$

$H^{r-1+i}(X^u, \mathcal{A}_{l,t})$ and $H^{r-1+i}(X^u, \mathcal{B}_{l,t})$ are zero in the other cases.

Proof. Apply Lemma 5.5 □

It remains to compute the cohomology of $\tilde{U}$ where U is some P representation. In the case that U irreducible this is accomplished by Bott's theorems [2]. In the case that U is not irreducible we can construct a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$ such that U_{i+1}/U_i is one-dimensional.

Then there is a similar filtration $0 = \tilde{U}_0 \subset \cdots \subset \tilde{U}_n = \tilde{U}$ such that $\tilde{U}_{i+1}/\tilde{U}_i = \widetilde{U_{i+1}/U_i}$ is a line bundle and hence the cohomology of $\tilde{U}$ must be contained in the cohomology of $\oplus\widetilde{U_{i+1}/U_i}$.

Now let $P = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \subset G$ and let T be the maximal torus $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ in G. Then the character group of T is generated by the character $\chi : \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \to z$.

We will identify χ with a one-dimensional representation of P. Then one easily verifies that the one-dimensional subquotients of W'' are of the form χ, χ^2 and the one-dimensional subquotients of W' are of the form χ^{-2}, χ^{-1}, 1.

Also using the fact that $(\Omega_{S/k}) = m_x/m_x^2$ [6, II.8.7] we verify that $(\Omega_{S/k})_x = \tilde{\chi}^{-2}$. Hence the direct summands as G-module of the cohomology of $\mathcal{A}_{l,t}$ and $\mathcal{B}_{l,t}$ are among the direct summands of the cohomology of certain tensor powers of $\tilde{\chi}$. Furthermore since χ is dominant and a generator of $\chi(T)$ we obtain as a trivial application of Bott's theory [2] that $\tilde{\chi} = \mathcal{O}_S(1)$ with some suitable G-action and $H^0(S, \tilde{\chi}^n) = S^n V$.

We will now use this method in the case that $W = (S^2V)^m$. This leads to the main application of this paper. It is clear however that similar computations can be made in more general cases.

Theorem 5.7. *Let $W = (S^2V)^m$ where $m \geq 2$. Then $(S^jV \otimes k[W])^G$ is a Cohen-Macaulay $k[W]^G$-module if $0 \leq j \leq 2*m-3$ or if j is odd.*

Proof. First note that $r = m$ in this case.

We list the one-dimensional subquotients of the vector bundles occurring in Lemma 5.6

$S^t\tilde{W}'$	$\tilde{\chi}^{-2t}, \tilde{\chi}^{-2t+2}, \ldots, \tilde{\chi}^{-2}, 1$
$\Lambda^r\tilde{W}''$	$\tilde{\chi}^{2r}$
$S^{t-l-r}\tilde{W}''$	$\tilde{\chi}^{2(t-l-r)}$
$S^{t-1}\tilde{W}'$	$\tilde{\chi}^{-2t+2}, \tilde{\chi}^{-2t+4}, \ldots, \tilde{\chi}^{-2}, 1$
$S^{t-l-r-1}\tilde{W}''$	$\tilde{\chi}^{2(t-l-r-1)}$
$S^t\tilde{W}' \otimes (\Lambda^r\tilde{W}'')^* \otimes (S^{t-l-r}\tilde{W}'')^*$	$\tilde{\chi}^{2l-4t}, \tilde{\chi}^{2l-4t+2}, \ldots, \tilde{\chi}^{2l-2t}$
$S^{t-1}\tilde{W}' \otimes (\Lambda^r\tilde{W}'')^* \otimes (S^{t-l-r-1}\tilde{W}'')^* \otimes \Omega_{S/k}$	$\tilde{\chi}^{2l-4t+2}, \tilde{\chi}^{2l-4t+4}, \ldots, \tilde{\chi}^{2l-2t}$

Hence if $t-l-r \geq 0$, $t \geq 0$, $i = 0, 1$ then the indecomposable summands of $H^{r-1+i}(X^u, \mathcal{A}_{l,t})$ are among

$$H^i(S, \mathcal{O}_S(2l-4t)), \ldots, H^i(S, \mathcal{O}_S(2l-2t)) \tag{7}$$

Similarly if $t-1-l-r \geq 0$, $t \geq 1$, $i = 0, 1$ then the indecomposable summands of $H^{r-1+i}(X^u, \mathcal{B}_{l,t})$ are among

$$H^i(S, \mathcal{O}_S(2l-4t+2)), \ldots, H^i(S, \mathcal{O}_S(2l-2t)) \tag{8}$$

In the other cases $H^{r-1+i}(X^u, \mathcal{A}_{l,t}) = 0$, $H^{r-1+i}(X^u, \mathcal{B}_{l,t}) = 0$.

If $t-l-r \geq 0$ then we can write $l = t-r-h$, $h \geq 0$. Plugging this in (7) we see that $H^{r-1}(X^u, \mathcal{A}_{l,t}) = 0$.

On the other hand, by Serre duality and (7), the indecomposable direct summands of $H^r(X^u, \mathcal{A}_{l,t})$ are among

$$\begin{array}{rclcl} H^0(S, \mathcal{O}_S(4t-2l-2)) & = & S^{4t-2l-2}V & = & S^{2t+2r+2h-2}V \\ \vdots & & \vdots & & \vdots \\ H^0(S, \mathcal{O}_S(2t-2l-2)) & = & S^{2t-2l-2}V & = & S^{2r+2h-2}V \end{array}$$

If $t-l-1-r \geq 0$ then again $l = t-1-r-h$ where $h \geq 0$. In the same way as above we see that $H^{r-1}(X^u, \mathcal{B}_{l,t}) = 0$.

The indecomposable direct summands of $H^r(X^u, \mathcal{B}_{l,t})$ are among

$$\begin{array}{rclcl} H^i(S, \mathcal{O}_S(4t-2l-4)) & = & S^{4t-2l-4}V & = & S^{2t+2r+2h-2}V \\ \vdots & & \vdots & & \vdots \\ H^i(S, \mathcal{O}_S(2t-2l-2)) & = & S^{2t-2l-2}V & = & S^{2r+2h}V \end{array}$$

Hence it is clear that S^jV, for $j = 0, \ldots, 2r-3$ or j odd, does not occur among the direct summands of $H^{r-1+i}(X^u, \mathcal{A}_{l,t})$, $H^{r-1+i}(X^u, \mathcal{B}_{l,t})$ where $i = 0, 1$. Hence these representations will also not occur in the cohomology of $\mathcal{I}^t/\mathcal{I}^{t+1}(l)$ by (6). It then follows from Theorem 3.1 that $(S^jV \otimes k[W])^G$ is a Cohen-Macaulay $k[W]^G$-module. □

Corollary 5.8. $\mathbb{T}_{m,2}$ *is Cohen-Macaulay*

Proof. It is well known that $\mathbb{T}_{2,2}$ is Cohen-Macaulay [3][12]. (This follows also from (6) if one notices that in this case only the cohomology of $\mathcal{B}_{l,t}$ is important by Theorem 3.1.) Hence we may assume that $m \geq 3$.

We have already mentioned that $\mathbb{T}_{m,2} = (U \otimes k[W])^G$ where $U = V^* \otimes V$, $W = (V^* \otimes V)^m$. But $V^* \otimes V = k \oplus S^2V$ where k is the trivial G-module. Then it is easy to see that $\mathbb{T}_{m,2}$ is a polynomial ring over $\mathbb{T}^0_{m,2} = (U \otimes k[(S^2V)^m])^G = k[(S^2V)^m]^G \oplus (S^2V \otimes k[(S^2V)^m])^G$.

Hence it suffices to prove our claim for $(U \otimes k[W])^G$ where $W = (S^2V)^m$ and $U = k$, S^2V. But in these cases Theorem 5.7 applies. □

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