SOME GENERALITIES ON G-EQUIVARIANT QUASI-COHERENT $\mathcal{O}_X$ AND $\mathcal{D}_X$-MODULES

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ABSTRACT. We prove some results on G-equivariant $\mathcal{O}_X$- and $\mathcal{D}_X$-modules on not necessarily affine spaces. For example we show that there are enough injectives in the corresponding categories. We also prove the often used result that for a $(G, \mathcal{D}_X)$-module to be G equivariant it is necessary sufficient that the Lie algebra of G acts in the correct way.

1. G-EQUIVARIANT QUASI-COHERENT $\mathcal{O}_X$-MODULES

1.1. **Notations.** In this section we collect some facts concerning G-equivariant quasi-coherent $\mathcal{O}_X$-modules. All of this is well-known, but is seems to be difficult to find a systematic treatment in the literature.

Below k will be a base field. Unadorned tensor and fiber products will be over k. X will be an arbitrary scheme over k and G will be a linear algebraic group over k, acting on X. The category of quasi-coherent $\mathcal{O}_X$-modules will be denoted by $\mathcal{O}_X$-qch.

$\mathcal{O}(G)$ is a Hopf algebra and its comultiplication, counit and antipode will respectively be denoted by Δ, ϵ and S. e will be the unit element of G and the corresponding maximal ideal of $\mathcal{O}(G)$ will be denoted by m_e. We will use the Sweedler convention. I.e. if $h \in \mathcal{O}(G)$ then we write $\Delta h = \sum h_{(1)} \otimes h_{(2)}$.

1.2. **Definitions and some functorial properties.** We start with the following diagram of objects and maps

$$G \times G \times X \;\overset{d_0}{\underset{d_2}{\overset{\xrightarrow{\;d_1\;}}{\longrightarrow}}}\; G \times X \;\overset{d_0}{\underset{d_1}{\overset{\xrightarrow{\;s_0\;}}{\longrightarrow}}}\; X \tag{1}$$

$$\begin{aligned} d_0(g_1, x) &= g_1^{-1}x & \qquad d_0(g_1, g_2, x) &= (g_2, g_1^{-1}x) \\ d_1(g_1, x) &= x & d_1(g_1, g_2, x) &= (g_1 g_2, x) \\ s_0(x) &= (e, x) & d_2(g_1, g_2, x) &= (g_1, x) \end{aligned}$$

Note that one has the following identities :

$$\begin{aligned} d_0^2 &= d_0 d_1 & \qquad d_0 s_0 &= \mathrm{id} \\ d_0 d_2 &= d_1 d_0 & d_1 s_0 &= \mathrm{id} \\ d_1^2 &= d_1 d_2 & & \end{aligned}$$

1991 *Mathematics Subject Classification.* Primary 20G99.
Key words and phrases. G-equivariant $\mathcal{O}_X$-modules, G-equivariant $\mathcal{D}$-modules.
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which express the fact the (1) is part of a simplicial scheme.

Below we will also need the following auxiliary maps

$$\begin{aligned} p &: G\times X \to G\times X : (g_1,x)\mapsto (g_1,g_1x)\\ p_0 &: G\times G\times X \to G\times G\times X : (g_1,g_2,x)\mapsto (g_1,g_2,g_1g_2x)\\ p_1 &: G\times G\times X \to G\times G\times X : (g_1,g_2,x)\mapsto (g_1,g_2,g_1x)\end{aligned} \tag{2}$$

which satisfy the following relations

$$\begin{aligned} d_1p_0 &= pd_1\\ d_2p_1 &= pd_2\end{aligned}$$

Definition 1.2.1. A G-equivariant quasi-coherent $\mathcal{O}_X$-module is a pair $(\mathcal{F},\theta)$ where $\mathcal{F}\in \mathcal{O}_X$-qch and θ is an isomorphism $d_1^*\mathcal{F}\to d_0^*\mathcal{F}$ in $\mathcal{O}_{G\times X}$-qch such that

$$\begin{aligned} d_0^*\theta\circ d_2^*\theta &= d_1^*\theta\\ s_0^*\theta &= \mathrm{id}_{\mathcal{F}}\end{aligned} \tag{3}$$

The corresponding category is denoted by $(G,\mathcal{O}_X)$-qch.

If there is no possibility for confusion we will simply write $\mathcal{F}$ for $(\mathcal{F},\theta)$.

Example 1.2.2. If $\mathcal{F}=\mathcal{O}_X$ then we may take $\theta=\mathrm{id}$. Then (3) is obviously satisfied.

If $\pi: Y\to X$ is a G-equivariant map then there exist functors $L_i\pi^*:(G,\mathcal{O}_X)$-qch $\to (G,\mathcal{O}_Y)$-qch, $R^i\pi_*:(G,\mathcal{O}_Y)$-qch $\to (G,\mathcal{O}_X)$-qch. The reason is that the corresponding functors between $\mathcal{O}_X$-qch and $\mathcal{O}_Y$-qch are compatible with (flat) base-change.

For example $\pi_*(\mathcal{F},\theta)$ is given by $(\pi_*\mathcal{F},\theta')$ where θ' makes the following diagram commutative

$$\begin{array}{ccc} (\mathrm{id}\times\pi)_*d_1^*\mathcal{F} & \xrightarrow{(\mathrm{id}\times\pi)_*(\theta)} & (\mathrm{id}\times\pi)_*d_0^*\mathcal{F}\\ \downarrow & & \downarrow\\ d_1^*\pi_*\mathcal{F} & \xrightarrow{\theta'} & d_0^*\pi_*\mathcal{F}\end{array}$$

Here the vertical maps are the canonical identifications given by base-change.

1.3. **Interpretation in terms of R-points.** Let $(\mathcal{F},\theta)\in(G,\mathcal{O}_X)$-qch and let $s:\operatorname{Spec} R\to \operatorname{Spec} k$ be a k-algebra. Then any R/k-point $i_g:\operatorname{Spec} R\to G$ induces an R-automorphism $g: X_R\to X_R$.

Let us denote the map $(i_g,\mathrm{id}): X_R=\operatorname{Spec} R\times X\to G\times X$ also by i_g. Then applying i_g^* to $\theta: d_1^*\mathcal{F}\to d_0^*\mathcal{F}$ yields a map $i_g^*(\theta): s^*\mathcal{F}\to (g^{-1})^*s^*\mathcal{F}$ and the second equation in (3) yields $i_e^*(\theta)=\mathrm{id}_{s^*\mathcal{F}}$.

Let $i_h:\operatorname{Spec} R\to G$ be another R-point of G. Then applying $i_{(g,h)}^*$ to the first equation of (3) yields

$$i_{gh}^*(\theta)=(g^{-1})^*(i_h^*(\theta))i_g^*(\theta)$$

This leads us to the following proposition

Proposition 1.3.1. *The category $(G,\mathcal{O}_X)$-qch is equivalent to the category of quasi-coherent sheaves $\mathcal{F}$ on X equipped with isomorphisms*

$$q_g: s^*\mathcal{F}\to (g^{-1})^*s^*\mathcal{F}$$

for each $s : \operatorname{Spec} R \to \operatorname{Spec} k$ *and for each* R/k*-point* $i_g : \operatorname{Spec} R \to G$ *satisfying*

$$q_e = \mathrm{id} \tag{4}$$
$$q_{gh} = (g^{-1})^*(q_h)q_g$$

in such a way that the (q_g)*'s are compatible with base-change.*

Proof. If $(\mathcal{F}, \theta)$ is in $(G, \mathcal{O}_X)$-qch then we take $q_g = i_g^*(\theta)$.

Conversely, assume that we are given a set of (q_g)'s. We take $R = \mathcal{O}(G)$. Then if "id" denotes the "identity point" $G = \operatorname{Spec} R \to G$ then i_{id} is equal to the map $p : X_R \to X_R$ (see (2)). We put $\theta = q_{\mathrm{id}}$ which goes from d_1^* to $p^{*-1}d_1^*\mathcal{F} = d_0^*\mathcal{F}$.

One easily verifies that this θ has the required properties. □

Remark 1.3.2. This proposition makes it easy to see that canonical objects in $\mathcal{O}_X$-qch such as tangent bundles, sheaves of differential operators etc... are automatically in $(G, \mathcal{O}_X)$-qch.

1.4. **Affine schemes.** If X is affine then the elements of $(G, \mathcal{O}_X)$-qch have a simple interpretation. Recall that rational (or "locally finite") G-actions on a k-vector space V are in one-one correspondence with coactions

$$l : V \to \mathcal{O}(G) \otimes V : v \mapsto \sum v_{(1)} \otimes v_{(2)}$$

via $gv = \sum v_{(1)}(g^{-1})v_{(2)}$.

In particular the action of G on X corresponds to a coaction $l : \mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$ and therefore $\mathcal{O}(X)$ is a rational G-representation.

Let $(\mathcal{F}, \theta) \in (G, \mathcal{O}_X)$-qch. If $f \in \mathcal{F}(X)$ then we put $l(f) = p^*\theta(d_1^*f) \in d_1^*\mathcal{F}(X) = \mathcal{O}(G) \otimes \mathcal{F}(X)$.

By using the method employed in the proof of Theorem 1.5.4 below, or by direct computation, one shows that

$$l : \mathcal{F}(X) \to \mathcal{O}(G) \otimes \mathcal{F}(X)$$

is a coaction of $\mathcal{O}(X)$ on $\mathcal{F}(X)$ and for $a \in \mathcal{O}(X)$, $f \in \mathcal{F}(X)$ one has $l(af) = l(a)l(f)$.

Furthermore, given l, one can reconstruct θ by $\theta(h \otimes f) = hp^{*-1}l(f)$.

Let $(G, \mathcal{O}(X))$-mod be the category of $\mathcal{O}(X)$-modules, equipped with a rational G-action, compatible with the G-action on $\mathcal{O}(X)$. Then we have shown

Proposition 1.4.1. *If* X *is affine then the categories* $(G, \mathcal{O}_X)$-qch *and* $(G, \mathcal{O}(X))$-mod *are equivalent. The equivalence is given by* $(\mathcal{F}, \theta) \mapsto (\mathcal{F}(X), l)$.

1.5. **General schemes.** The previous section gives a satisfying description of $(G, \mathcal{O}_X)$-qch in the case that X is affine. Unfortunately, not every G-scheme may be covered with affine G-schemes. In such a case one may find the objects in $(G, \mathcal{O}_X)$-qch somewhat unpleasant to work with.

Furthermore, even if X is affine, the stalks of $\mathcal{O}_X$ at fixed points of X are usually not rational as G-representations. This shows that $(G, \mathcal{O}_X)$-qch is not closed under some natural operations.

One possible solution is to embed $(G, \mathcal{O}_X)$-qch in the category of $\mathcal{O}_X$-modules equipped with a G-action where G is considered as a discrete group [3]. However this seems to be somewhat inelegant, and in any case it is only justified if G is reduced and has a dense set of k-points.

Below we sketch another approach that works well when G is connected. We embed $(G,\mathcal{O}_X)$-qch in $(\hat{G},\mathcal{O}_X)$-qch where $\hat{G}$ is the *formal* group associated to G.

To be more precise let

$$\mathcal{O}(\hat{G}) = \varprojlim_n \mathcal{O}(G)/m_e^n$$

which is a *topological* Hopf algebra.

A coaction of $\mathcal{O}(\hat{G})$ on $\mathcal{F} \in \mathcal{O}_X$-qch is a k-linear sheaf map

$$l : \mathcal{F} \to \mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{F} = \varprojlim \mathcal{O}(G)/m_e^n \otimes \mathcal{F}$$

satisfying the usual associativity conditions (here and below $\hat{\otimes}$ denotes the *completed* tensor product).

Let $(\mathcal{F},\theta)$ be in $(G,\mathcal{O}_X)$. We will show how one may use θ to construct a coaction of $\mathcal{O}(\hat{G})$ on $\mathcal{F}$. Let $U \subset X$ be an affine open and let $f \in \mathcal{F}(U)$. Then $l(f) = p^*(\theta(d_1^* f))$ is a section of $(d_1^*\mathcal{F})(p^{-1}(G\times U)$. Now $p^{-1}(G\times U)$ is a neighborhood of $e \times U$ and hence we may consider $l(f)$ as a section of $s_0^{-1}(d_1^*\mathcal{F})(U)$.

Hence we have defined a map of sheaves on X :

$$l : \mathcal{F} \to s_0^{-1}(d_1^*\mathcal{F})$$

$s_0^{-1}(d_1^*\mathcal{F})$ is embedded in $\mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{F}$ and hence l also defines a map

$$l : \mathcal{F} \to \mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{F}$$

In the proof of Theorem 1.5.4 we will show that this indeed defines a coaction.

We may in particular apply this construction to $(\mathcal{O}_X, \mathrm{id})$ to obtain a “canonical” coaction

$$l : \mathcal{O}_X \to \mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{O}_X$$

and it is almost obvious that for $a \in \mathcal{O}_X(U)$, $f \in \mathcal{F}(U)$ one has $l(af) = l(a)l(f)$. This motivates the following definition

Definition 1.5.1. A quasi-coherent $(\hat{G},\mathcal{O}_X)$-module is a pair $(\mathcal{F}, l)$ where $\mathcal{F}$ is in $\mathcal{O}_X$-qch and

$$l : \mathcal{F} \to \mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{F}$$

is a coaction, compatible with the canonical coaction

$$l : \mathcal{O}_X \to \mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{O}_X$$

I.e. we require for $U \subset X$ open, $a \in \mathcal{O}_X(U)$, $f \in \mathcal{F}(U) : l(af) = l(a)l(f)$.

The category of quasi-coherent $(\hat{G},\mathcal{O}_X)$-modules is denoted by $(\hat{G},\mathcal{O}_X)$-qch.

Hence above we have constructed a functor

$$i : (G,\mathcal{O}_X)\text{-qch} \to (\hat{G},\mathcal{O}_X)\text{-qch}$$

which associates the pair $(\mathcal{F}, l)$ to the pair $(\mathcal{F},\theta)$. We will see in the proof of Theorem 1.5.4 that if G is connected then i is fully faithful (and has other good properties).

The advantage of working with $(\hat{G},\mathcal{O}_X)$-qch rather than with $(G,\mathcal{O}_X)$-qch is that being in $(\hat{G},\mathcal{O}_X)$-qch is a local property. That is if $(\mathcal{F}, l)$ is in $(\hat{G},\mathcal{O}_X)$-qch and $U \subset X$ is open then $(\mathcal{F}|U, l|U)$ is in $(\hat{G},\mathcal{O}_U)$-qch (note that here we are in a slight extension of the present context since $\mathcal{O}(\hat{G})$ coacts on $\mathcal{O}_U$, but this coaction is no longer obtained from an action of G on U).

Furthermore, to do calculations, one can use a variant of the Sweedler notation. That is, for $U \subset X$ affine open, $f \in \mathcal{F}(U)$ we write

$$l(f) = \sum f_{(1)} \hat{\otimes} f_{(2)} \tag{5}$$

where $f_{(1)} \in \mathcal{O}(\hat{G})$, $f_{(2)} \in \mathcal{F}(U)$. The only difference with the ordinary situation is that (5) is now a, usually infinite, convergent sum.

For $h \in \mathcal{O}(\hat{G})$ we also put $\Delta(h) = \sum h_{(1)} \hat{\otimes} h_{(2)}$.

Let us now denote by $\mathcal{O}(\hat{G})^*$ the Hopf algebra

$$\varinjlim_n (\mathcal{O}(G)/m_e^n)^*$$

Note that $\mathcal{O}(\hat{G})^*$ is a real Hopf algebra, not just a topological one. We denote by $\langle -, - \rangle$ the natural pairing between $\mathcal{O}(\hat{G})$ and $\mathcal{O}(\hat{G})^*$.

If $(\mathcal{F}, l)$ is in $(\hat{G}, \mathcal{O}_X)$ then we may construct a left action of $\mathcal{O}(\hat{G})^*$ on $\mathcal{F}$: $l : \mathcal{O}(\hat{G})^* \otimes \mathcal{F} \to \mathcal{F}$ by

$$l(\phi \otimes f) = \sum \langle \phi, S f_{(1)} \rangle f_{(2)} \tag{6}$$

In particular we obtain a canonical left action of $\mathcal{O}(\hat{G})^*$ on $\mathcal{O}_X$ and the actions on $\mathcal{F}$ and $\mathcal{O}_X$ are compatible, in the sense that if $a \in \mathcal{O}_X(U)$, $f \in \mathcal{F}(U)$, $\phi \in \mathcal{O}(\hat{G})^*$ then

$$l(\phi \otimes af) = \sum l(\phi_{(1)} \otimes a) l(\phi_{(2)} \otimes f)$$

Hence if we denote by $(\mathcal{O}(\hat{G})^*, \mathcal{O}_X)$-qch the category of quasi-coherent $\mathcal{O}_X$-modules equipped with a compatible $\mathcal{O}(\hat{G})^*$-action, then (6) defines a functor from $(\hat{G}, \mathcal{O}_X)$-qch to $(\mathcal{O}(\hat{G})^*, \mathcal{O}_X)$-qch and it easy to see that this is an equivalence.

Let $\mathfrak{g} = (m_e/m_e^2) \subset \mathcal{O}(\hat{G})^*$. $\mathfrak{g}$ consists of primitive elements and hence it is a Lie algebra. If we denote by $(\mathfrak{g}, \mathcal{O}_X)$-qch the category of quasi-coherent $\mathcal{O}_X$-modules, equipped with a compatible $\mathfrak{g}$-action then restriction defines a functor $(\mathcal{O}(\hat{G})^*, \mathcal{O}_X)$-qch $\to (\mathfrak{g}, \mathcal{O}_X)$-qch. If $\operatorname{char} k = 0$ then $\mathcal{O}(\hat{G})^* = U(\mathfrak{g})$ [4] and hence we obtain an equivalence.

Before we summarize our constructions in Theorem 1.5.4 below. We introduce the following notion which will considerably shorten the statements of results further on

Definition 1.5.2. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is a right closed embedding if F is exact and has a right adjoint G such that for the induced natural transformations $\mathrm{id} \to GF$, $FG \to \mathrm{id}$ one has that the first one is an isomorphism and the second one is a monomorphism.

Lemma 1.5.3. *Assume that $F : \mathcal{A} \to \mathcal{B}$ is a right closed embedding with right adjoint G. Then*

1. *F is fully faithful;*
2. *The essential image of F is closed under subquotients and direct limits.*
3. *If $\mathcal{B}$ has enough injectives then so does $\mathcal{A}$.*

If a right closed embedding $F : \mathcal{A} \to \mathcal{B}$ exists then informally we say that $\mathcal{A}$ is a right closed subcategory of $\mathcal{B}$. Note that in the terminology of [2] we would say that $\mathcal{A}$ is a closed subcategory of $\mathcal{B}$.

Theorem 1.5.4. *There are functors*

$$(G,\mathcal{O}_X)\text{-qch} \xrightarrow{i} (\hat{G},\mathcal{O}_X)\text{-qch} \to (\mathcal{O}(\hat{G})^*,\mathcal{O}_X)\text{-qch} \to (\mathfrak{g},\mathcal{O}_X)\text{-qch}$$

The first one is a right closed embedding if G is connected; the second one is an equivalence and the third one is an equivalence if $\operatorname{char} k = 0$.

Remark 1.5.5. One may introduce the sheaf of rings $\mathcal{O}_X\#\mathcal{O}(\hat{G})^*$. As a sheaf of vectorspaces this is just $\mathcal{O}_X\otimes\mathcal{O}(\hat{G})^*$ and the multiplication is given as follows : let $a,b\in\mathcal{O}_X(U)$, $\phi,\psi\in\mathcal{O}(\hat{G})^*$. Then

$$(a\#\phi)(b\#\psi)=\sum a\phi_{(1)}(b)\#\phi_{(2)}(b)\psi$$

where we have written $\phi_{(1)}(b)$ for $l(\phi_{(1)}\otimes b)$. It is easy to see that one has

$$(\mathcal{O}(\hat{G})^*,\mathcal{O}_X)\text{-qch}\cong\mathcal{O}_X\#\mathcal{O}(\hat{G})^*\text{-qch}$$

which realizes $(G,\mathcal{O}_X)$-qch as a right closed subcategory of the category of quasi-coherent modules over a sheaf of rings (if G is connected).

Proof of Theorem 1.5.4. The only thing we still have to do is to prove that i has the desired properties.

Let $(\mathcal{F},\theta)$ be in $(G,\mathcal{O}_X)$-qch and let $l:\mathcal{F}\to\mathcal{O}(\hat{G})\mathbin{\hat{\otimes}}\mathcal{F}$ be constructed as above. We first have to show that l defines a coaction. Let $f\in\mathcal{F}(U)$. Applying $d_0^*\theta\circ d_2^*\theta=d_1^*\theta$ to $d_1^{*2}f$ yields

$$(d_0^*\theta)(d_2^*\theta)(d_1^{*2}f)=(d_1^*\theta)(d_1^{*2}f)$$

Using the fact that $d_1^2=d_1d_2$, this yields

$$(d_0^*\theta)(d_2^*(\theta(d_1^*f))=d_1^*(\theta(d_1^*f))$$

Applying p_0^* and using $d_1p_0=pd_1$ yields

$$p_0^*(d_0^*\theta)(d_2^*(\theta(d_1^*f)))=d_1^*(l(f)) \tag{7}$$

We may rewrite the left hand side of (7) as follows :

$$\begin{aligned}\text{LHS}(7)&=(p_0^*\circ d_0^*\theta\circ d_2^*\circ p^{*-1})(p^*(\theta(d_1^*f))\\&=(p_0^*\circ d_0^*\theta\circ p_1^{*-1}\circ d_2^*)(l(f))\\&=(p_0^*\circ p_1^{*-1}\circ p_1^*\circ d_0^*\theta\circ p_1^{*-1}\circ d_2^*)(l(f))\\&=((p_1^{-1}p_0)^*\circ(d_0p_1)^*\theta\circ d_2^*)(l(f))\\&=(\text{pr}_{23}^*(p)^*\circ\text{pr}_{23}^*(\theta)\circ\text{pr}_{13}^*)(l(f))\\&=(1\otimes l)(l(f))\end{aligned} \tag{8}$$

So finally we find

$$d_1^*(l(f))=(1\otimes l)(l(f))$$

Completing this relation in (e,e,X) yields that l is indeed coassociative. The other relation we need is $(\epsilon\mathbin{\hat{\otimes}}1)l(f)=1\otimes f$, but this is easy.

Now we show that the coaction of $\mathcal{O}(\hat{G})$ on $\mathcal{F}$ is compatible with the coaction of $\mathcal{O}(\hat{G})$ on $\mathcal{O}_X$ obtained from $(\mathcal{O}_X,\text{id})\in(G,\mathcal{O}_X)$-qch.

Let $a \in \mathcal{O}_X(U)$, $f \in \mathcal{F}(U)$. Then

$$\begin{aligned} l(af) &= p^*(\theta(d_1^*(af)) \\ &= p^*(\theta(d_1^*a\, d_1^*f)) \\ &= p^*(d_1^*a\, \theta(d_1^*f)) \\ &= p^*(d_1^*a)(p^*(\theta(d_1^*f)) \\ &= l(a)l(f) \end{aligned}$$

Now we construct a right adjoint $j : (\hat{G}, \mathcal{O}_X)$-qch $\to (G, \mathcal{O}_X)$-qch to i. Let $(\mathcal{F}, l)$ be in $(\hat{G}, \mathcal{O}_X)$-qch. First we extend l to a $\mathcal{O}(\hat{G})$-linear map $\hat{\theta} : \mathcal{O}(\hat{G}) \hat{\otimes} \mathcal{F} \to \mathcal{O}(\hat{G}) \hat{\otimes} \mathcal{F}$.

Note that for $U \subset X$ one has embeddings of $d_1(\mathcal{F})(d_1^{-1}(U))$ and $d_0(\mathcal{F})(d_1^{-1}(U))$ inside $\mathcal{O}(\hat{G}) \hat{\otimes} \mathcal{F}(U)$ (this uses the fact that G is connected). Let $\mathcal{G} \subset \mathcal{F}$ be a subsheaf in $\mathcal{O}_X$-qch. By running the computation (8) in reverse one sees that if for all $U \subset X$ affine one has $\hat{\theta}(d_1^*\mathcal{G}(d_1^{-1}(U))) \subset d_0^*\mathcal{G}(d_1^{-1}(U))$ then $\hat{\theta}$ restricts to a map $\theta : d_1^*\mathcal{G} \to d_0^*\mathcal{G}$ and the corresonding pair $(\mathcal{G}, \theta)$ is an object of $(G, \mathcal{O}_X)$-qch. We now let $j(\mathcal{G}, l)$ be the pair $(\mathcal{G}, \theta)$, where $\mathcal{G} \subset \mathcal{F}$ is maximal with the property that θ exists. It is easy to show that j has the required properties. □

Corollary 1.5.6. *Assume that X is quasi-compact and quasi-separated. Then $(G, \mathcal{O}_X)$-qch has enough injectives.*

Proof. The usual restriction-corestriction argument reduces us to the case that G is connected. Then, by Theorem 1.5.4 and remark 1.5.5, $(G, \mathcal{O}_X)$-qch is a right closed subcategory of $\mathcal{O}_X \# \mathcal{O}(\hat{G})^*$-qch and it is standard that the category of quasi-coherent modules over a quasi-coherent sheaf of rings over a quasi-compact quasi-separated scheme has enough injectives (see [1, Prop. VI.2.1] for the case of $\mathcal{D}$-modules). We then apply lemma 1.5.3. □

Remark 1.1. An interesting question is when $(G, \mathcal{O}_X)$-qch is closed under extensions in $(\mathcal{O}(\hat{G})^*, \mathcal{O}_X)$-qch. Comparison with the affine case suggest that this should be true if $\operatorname{char} k = 0$ and G is semisimple. However I have no proof of this.

2. G-EQUIVARIANT $\mathcal{D}_X$-MODULES

In this section we treat G-equivariant quasi-coherent $\mathcal{D}_X$-modules. Our main aim is to prove Proposition 2.6 below, which occurs frequently in the literature, but as far as I know, each time without proof.

Below k will be an algebraically closed field of characterstic zero, X will be a smooth k-scheme and G will be a linear algebraic group over k, which acts on X. By $\mathcal{D}_X$ we denote the sheaf of differential operators on X and by $\mathcal{D}(X)$ we denote its global sections.

A G-equivariant quasi-coherent $\mathcal{D}$-module is a pair $(\mathcal{F}, \theta)$ where $\mathcal{F}$ is in $\mathcal{D}_X$-qch and $\theta : d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ is in $\mathcal{D}_{G\times X}$-qch satisfying $d_1^*\theta = d_0^*\theta \circ d_2^*\theta$. Note that this makes sense since both $d_1^*\mathcal{F}$ and d_0^* lie in $\mathcal{D}_{G\times X}$-qch [1, VI.§4].

The category G-equivariant quasi-coherent $\mathcal{D}_X$-modules is denoted by $(G, \mathcal{D}_X)$-qch.

If in the above pair $(\mathcal{F}, \theta)$, θ only lies in $\mathcal{O}_G \boxtimes \mathcal{D}_X$-qch then one says that $(\mathcal{F}, \theta)$ is a *weakly* G-equivariant quasi-coherent $\mathcal{D}_X$-module. The corresponding category is denoted by $(G, \mathcal{D}_X)$-wqch.

Proposition 2.1. *The inclusion functor* $(G, \mathcal{D}_X)$-qch $\to$ $(G, \mathcal{D}_X)$-wqch *is right closed.*

Proof. Clear. □

We observe that if $\pi : Y \to X$ is a G-equivariant map of smooth k-schemes and $(\mathcal{M}, \theta) \in (G, \mathcal{D}_X)$-(w)qch, $(\mathcal{N}, \theta) \in (G, \mathcal{D}_Y)$-(w)qch then for all i, $H^i\pi^!\mathcal{M}$, $H^i\pi^+\mathcal{M} \in \mathcal{D}_X$-qch, $H^i\pi_!\mathcal{M}$, $H^i\pi_+\mathcal{M} \in \mathcal{D}_Y$-qch carry natural G-structures and hence they define objects in $(G, \mathcal{D}_X)$-(w)qch and $(G, \mathcal{D}_Y)$-(w)qch. This is because these functors commute with (smooth) base change (see §1 for the corresponding statement about $(G, \mathcal{O}_X)$-qch and $(G, \mathcal{O}_Y)$-qch).

There is the following obvious analogue to Proposition 1.3.1

Proposition 2.2. *The category* $(G, \mathcal{D}_X)$-wqch *is equivalent with the category of quasi-coherent* $\mathcal{D}_X$*-modules* $\mathcal{F}$ *on* X *equipped with isomorphisms in* $\mathcal{D}_{X_R/R}$-qch

$$q_g : s^*\mathcal{F} \to (g^{-1})^* s^*\mathcal{F}$$

for each $s : \operatorname{Spec} R \to \operatorname{Spec} k$ *and for each* R/k*-point* $i_g : \operatorname{Spec} R \to G$ *satisfying*

$$\begin{aligned} q_e &= \mathrm{id} \\ q_{gh} &= (g^{-1})^*(q_h)q_g \end{aligned} \tag{9}$$

in such a way that the (q_g)*'s are compatible with base-change.*

Our next aim is now to embed $(G, \mathcal{D}_X)$-wqch in a corresponding local category $(\mathfrak{g}, \mathcal{D}_X)$-qch.

We start by observing that $\mathcal{D}_X$ itself lies $(G, \mathcal{D}_X)$-wqch (but not in $(G, \mathcal{D}_X)$-qch!). To see this we have to define

$$q_g : \mathcal{D}_{X_R/R} \to (g^{-1})^*\mathcal{D}_{X_R/R}$$

satisfying (9). It will be more convenient however to define $r_g = g^* \circ q_g$. Note that by definition, for every open $U \subset X_R$, r_g should be a map from $\mathcal{D}_{X_R/R}(U)$ to $\mathcal{D}_{X_R/R}(g^{-1}U)$. Condition (9) translates into $r_{gh} = r_h r_g$ and $r_e = \mathrm{id}$.

For $D \in \mathcal{D}_{X_R/R}(U)$ we define $r_g(D) = g^*D$, where by definition for every $f \in \mathcal{O}_X(g^{-1}U)$ one has $(g^*D) * f = D * (f \circ g^{-1}) \circ g$ (note the use of "$*$" for the action of a differential operator). It is clear that r_g has the required properties.

Thus $\mathcal{D}_X$ defines a corresponding object $(\mathcal{D}_X, \theta)$ in $(G, \mathcal{D}_X)$-wqch. As before let $\mathrm{id} : G \to G$ be the identity point. The automorphism of $X_{\mathcal{O}(G)}$ corresponding to id is p. According to the proof of Proposition 1.3.1, $\theta = q_{\mathrm{id}}$. Hence by the above definitions

$$(p^* \circ \theta)(D) * f = D * (f \circ p^{-1}) \circ p$$

for $D \in \mathcal{D}_{G\times X/G}(U)$, $f \in \mathcal{O}_{G\times X}(U)$ with $U \subset G \times X$ open. We conclude

$$l(D) * f = (1 \otimes D) * (f \circ p^{-1}) \circ p$$

Now let $f \in \mathcal{O}_X(U)$, $D \in \mathcal{D}_X(U)$. One computes

$$\begin{aligned} l(D) * (1 \mathbin{\hat{\otimes}} f) &= [(1 \otimes D) * ((1 \mathbin{\hat{\otimes}} f) \circ p^{-1})] \circ p \\ &= (1 \otimes D) * (\textstyle\sum Sf_{(1)} \mathbin{\hat{\otimes}} f_{(2)}) \circ p \\ &= (\textstyle\sum Sf_{(1)} \mathbin{\hat{\otimes}} D * f_{(2)}) \circ p \\ &= \textstyle\sum Sf_{(1)}(D * f_{(2)})_{(1)} \mathbin{\hat{\otimes}} (D * f_{(2)})_{(2)} \end{aligned}$$

Hence we obtain

$$\sum D_{(1)} \hat{\otimes} D_{(2)} * f = \sum Sf_{(1)}(D * f_{(2)})_{(1)} \hat{\otimes} (D * f_{(2)})_{(2)} \tag{10}$$

Recall that by Theorem 1.5.4 there is a natural action $\mathfrak{g}$ on $\mathcal{O}_X$. This action is by derivations and hence one obtains a natural map $\mathfrak{g} \to \mathcal{D}(X)$. If $v \in \mathfrak{g}$ then we denote the corresponding differential operator by D_v. I.e. for $f \in \mathcal{O}_X(U)$ one has $vf = D_v * f$.

To understand the coaction l better we look at the corresponding left action of $\mathfrak{g}$ on $\mathcal{D}_X$:

$$l : \mathfrak{g} \otimes \mathcal{D}_X \to \mathcal{D}_X : v \otimes D \mapsto \sum \langle v, SD_{(1)} \rangle D_{(2)} = -\sum \langle v, D_{(1)} \rangle D_{(2)}$$

Using (10) we find

$$\begin{aligned} l(v \otimes D) * f &= -\sum \langle v, Sf_{(1)}(D * f_{(2)})_{(1)} \rangle (D * f_{(2)})_{(2)} \\ &= \sum -\langle v, Sf_{(1)} \rangle \epsilon((D * f_{(2)})_{(1)})(D * f_{(2)})_{(2)} - \epsilon(Sf_{(1)}) \langle v, (D * f_{(2)})_{(1)} \rangle (D * f_{(2)})_{(2)} \\ &= \sum -\langle v, Sf_{(1)} \rangle (D * f_{(2)}) + \epsilon(f_{(1)}) D_v * (D * f_{(2)}) \\ &= \sum -D * (\langle v, Sf_{(1)} \rangle f_{(2)}) + D_v * (D * \epsilon(f_{(1)}) f_{(2)}) \\ &= -D * (D_v * f) + D_v * (D * f) \\ &= [D_v, D] * f \end{aligned}$$

So we find

$$l(v \otimes D) = [D_v, D]$$

Now we define some categories

Definition 2.3. (1) A quasi-coherent $(\hat{G}, \mathcal{D}_X)$-module is a pair $(\mathcal{F}, l)$ where $\mathcal{F} \in \mathcal{D}_X$-qch and

$$l : \mathcal{F} \to \mathcal{O}(\hat{G}) \hat{\otimes} \mathcal{F}$$

is a coaction compatible with the canonical coaction

$$l : \mathcal{D}_X \to \mathcal{O}(\hat{G}) \hat{\otimes} \mathcal{D}_X$$

in the sense that if $D \in \mathcal{D}_X(U)$, $f \in \mathcal{F}(U)$ then $l(D * f) = l(D) * l(f)$.

The category of quasi-coherent $(\hat{G}, \mathcal{D}_X)$-modules is denoted by $(\hat{G}, \mathcal{D}_X)$-wqch.

(2) A quasi-coherent $(\mathfrak{g}, \mathcal{D}_X)$-module is a pair $(\mathcal{F}, l)$ where $\mathcal{F} \in \mathcal{D}_X$-qch and

$$l : \mathfrak{g} \otimes \mathcal{F} \to \mathcal{F} : v \otimes f \mapsto vf$$

is a left action compatible with the canonical left action

$$l : \mathfrak{g} \otimes \mathcal{D}_X \to \mathcal{D}_X : v \otimes D \mapsto [D_v, D]$$

in the sense that if $v \in \mathfrak{g}$, $D \in \mathcal{D}_X(U)$, $f \in \mathcal{F}(U)$ then

$$v(D * f) - D * (vf) - [D_v, D] * f = 0$$

The category of quasi-coherent $(\mathfrak{g}, \mathcal{D}_X)$-modules is denoted by $(\mathfrak{g}, \mathcal{D}_X)$-qch.

Then we have the following result

Theorem 2.4. *There are functors*

$$(G, \mathcal{D}_X)\text{-wqch} \xrightarrow{i} (\hat{G}, \mathcal{D}_X)\text{-wqch} \to (\mathfrak{g}, \mathcal{D})\text{-qch}$$

The first one is a right closed embedding if G is connected, and the second one is an equivalence.

Proof. As in Theorem 1.5.4 □

Remark 2.5. As before one has

$$(\mathfrak{g}, \mathcal{D}_X)\text{-qch} \cong \mathcal{D}_X \# U(\mathfrak{g})\text{-qch}$$

This shows for example that $(G, \mathcal{D}_X)$-wqch has enough injectives (as in corollary 1.5.6).

Now we concentrate on $(G, \mathcal{D}_X)$-qch. We prove the following result.

Proposition 2.6. *Assume that G is connected and $(\mathcal{F}, \theta) \in (G, \mathcal{D}_X)$-wqch. Then $(\mathcal{F}, \theta) \in (G, \mathcal{D}_X)$-qch if and only if for all $v \in \mathfrak{g}$ the action of v on $\mathcal{F}$ coincides with the action of D_v on $\mathcal{F}$. I.e for all open affine $U \subset X$ and for all $f \in \mathcal{F}(U)$ one has $D_v * f = vf$.*

Proof. We have to express the fact that $\theta : d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ is $\mathcal{D}_G \boxtimes \mathcal{D}_X$-linear, given that it is $\mathcal{O}_G \boxtimes \mathcal{D}_X$-linear.

We recall that $\mathcal{D}(G) = \mathcal{O}(G)[\mathfrak{g}]$ where we assume that $\mathfrak{g}$ acts by left invariant derivations on $\mathcal{O}(G)$. That is, if $h \in \mathcal{O}(G)$, $v \in \mathfrak{g}$ then

$$v * h = \sum h_{(1)} \langle v, h_{(2)} \rangle$$

Let $v \in \mathfrak{g}$. It is sufficient to express the condition that $\theta : d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ is compatible with the action of all such v.

Let $U \subset X$ be affine open, $f \in \mathcal{F}(U)$, $h \in \mathcal{O}(G)$. We have to study the condition

$$p^*((v \otimes 1) * \theta(h \otimes f)) = p^*(\theta((v \otimes 1) * (h \otimes f))) \tag{11}$$

Obviously

$$\text{RHS}(11) = \sum (v * h) f_{(1)} \hat{\otimes} f_{(2)}$$

so we concentrate on the left hand side of (11).

We have

$$p^*((v \otimes 1) * \theta(h \otimes f)) = p^*(v \otimes 1) * p^*(\theta(h \otimes f)) \tag{12}$$

(see lemma 2.7 for a precise statement of the principle we use here).

Recall that $p^*(v \otimes 1)$ is an element of $\mathcal{D}(G) \otimes \mathcal{D}(X)$. To know precisely which element we choose $r \otimes s \in \mathcal{O}(G) \otimes \mathcal{O}_X(U)$, and we compute $p^*(v \otimes 1) * (r \otimes s)$ on the intersection $d_1^{-1}U \cap d_0^{-1}U$.

$$\begin{aligned}
p^*(v \otimes 1) * (r \otimes s) &= (v \otimes 1) * ((r \otimes s) \circ p^{-1}) \circ p \qquad \text{(see (18))} && (13)\\
&= \sum (v * (rSs_{(1)}) \hat{\otimes} s_{(2)}) \circ p \\
&= \sum v * (rSs_{(1)}) s_{(2)} \hat{\otimes} s_{(3)} \\
&= \sum (v * r) Ss_{(1)} s_{(2)} \hat{\otimes} s_{(3)} + r(v * Ss_{(1)}) s_{(2)} \hat{\otimes} s_{(3)} && (14)
\end{aligned}$$

The first term of (14) is equal to $(v * r) \otimes s$, so we concentrate on the second term, and more in particular on the subexpression $\sum (v * Ss_{(1)})s_{(2)}$. We find

$$\begin{aligned}\sum (v * Ss_{(1)})s_{(2)} &= \sum v * ((Ss_{(1)})s_{(2)}) - \sum Ss_{(1)}(v * s_{(2)}) \\ &= -\sum Ss_{(1)}s_{(2)}\langle v, s_{(3)}\rangle \\ &= -\sum \epsilon(s_{(1)})\langle v, s_{(2)}\rangle \\ &= -\sum \langle v, \epsilon(s_{(1)})s_{(2)}\rangle \\ &= -\langle v, s_{(1)}\rangle\end{aligned}$$

Hence we find that

$$\begin{aligned}(14) &= (v * r) \otimes s - \sum r\langle v, s_{(1)}\rangle \hat{\otimes} s_{(2)} \\ &= (v * r) \otimes s - \sum r \hat{\otimes} \langle v, s_{(1)}\rangle s_{(2)} \\ &= (v * r) \otimes s + r \otimes D_v * s\end{aligned}$$

Finally we find that

$$p^*(v \otimes 1) = v \otimes 1 + 1 \otimes D_v$$

(It is possible to give easier proofs of this by looking at tangent vectors.)

Now we use (12). Since $p^*(\theta(h \otimes f)) = hl(f)$ is a section of $d_1^*\mathcal{F} = \mathcal{O}_G \boxtimes \mathcal{F}$, we can write down how $p^*(v \otimes 1)$ acts on $p^*(\theta(h \otimes f))$. We find

$$\begin{aligned}\text{LHS}(11) &= (v \otimes 1 + 1 \otimes D_v) * (\sum hf_{(1)} \hat{\otimes} f_{(2)}) \\ &= \sum v * (hf_{(2)}) \hat{\otimes} f_{(2)} + hf_{(1)} \hat{\otimes} D_v * f_{(2)}\end{aligned}$$

So we find finally that (11) is equivalent to

$$\sum (v * h)f_{(1)} \hat{\otimes} f_{(2)} = \sum v * (hf_{(1)}) \hat{\otimes} f_{(2)} + hf_{(1)} \hat{\otimes} D_v * f_{(2)}$$

which simplifies to

$$\sum h(v * f_{(1)}) \hat{\otimes} f_{(2)} = -\sum hf_{(1)} \hat{\otimes} D_v * f_{(2)} \tag{15}$$

Hence (11) is equivalent to having (15) for all h. However having (15) for all h is clearly equivalent to having it for $h = 1$. Thus it is necessary and sufficient to have :

$$\sum v * f_{(1)} \hat{\otimes} f_{(2)} = -\sum f_{(1)} \hat{\otimes} D_v * f_{(2)} \tag{16}$$

v act by left invariant derivations on $\mathcal{O}(G)$ and hence we have

$$\begin{aligned}\text{LHS}(16) &= \sum (v * f_{(1)}) \hat{\otimes} f_{(2)} \\ &= \sum f_{(1)}\langle v, f_{(2)}\rangle \hat{\otimes} f_{(3)} \\ &= \sum f_{(1)} \hat{\otimes} \langle v, f_{(2)}\rangle f_{(3)} \\ &= -\sum f_{(1)} \hat{\otimes} vf_{(2)}\end{aligned}$$

So finally we obtain that (11) is equivalent to having

$$\sum f_{(1)} \hat{\otimes} vf_{(2)} = \sum f_{(1)} \hat{\otimes} D_v * f_{(2)} \tag{17}$$

This is certainly true if v acts in the same way as D_v, and conversely by applying $\epsilon \otimes 1$ to (17) we find $vf = D_v * f$. □

We have used the following lemma.

Lemma 2.7. *Assume that we have a commutative diagram of smooth k-schemes.*

$$\begin{array}{ccc} Y & \xrightarrow{p} & Z \\ {\scriptstyle d}\downarrow & & {\scriptstyle e}\downarrow \\ X & = & X \end{array}$$

where p is an isomorphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{D}_X$-module. Then according to [1, VI.§4], *$\mathcal{D}_Y$ acts on $d^*\mathcal{F}$ and $\mathcal{D}_Z$ acts on $e^*\mathcal{F}$. Let $U \subset Z$ be open and let $D \in \mathcal{D}_Z(U)$. Define p^*D by $(p^*D)(h) = D * (h \circ p) \circ p^{-1}$. Then for $f \in (e^*\mathcal{F})(U)$ we have the following identity in $(d^*\mathcal{F})(p^{-1}U)$*

$$p^*(D * f) = p^*D * p^*f \tag{18}$$

Proof. This is an exercise on the use of the chain rule which is left to the reader. □

Corollary 2.8. *Assume G connected. Then the forgetful functor $(G, \mathcal{D}_X)$-qch $\rightarrow$ $\mathcal{D}_X$-qch is a right closed embedding.*

Proof. This is proved in a similar way as Theorem 1.5.4, using Proposition 2.6. □

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