IDEAL CLASSES OF THREE DIMENSIONAL SKLYANIN ALGEBRAS

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ABSTRACT. In this paper we classify graded reflexive ideals, up to isomorphism and shift, in certain three dimensional Artin-Schelter regular algebras. This classification is similar to the classification of right ideals in the first Weyl algebra, a problem that was completely settled recently. The situation we consider is substantially more complicated however.

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1. Introduction

This paper is motivated by the recent developments on the classification of right ideals in the first Weyl algebra. We start by recalling the (for now) definite result in this subject, as it was formulated by Berest and Wilson.

Theorem 1.1. [8] *Let $A_1 = \mathbb{C}\langle x, y\rangle/(yx - xy - 1)$ be the first Weyl algebra. Put $G = \operatorname{Aut}(A_1)$ and let $\mathcal{R}$ be the set of isomorphism classes of right A_1-ideals. Then the orbits of the natural G-action on $\mathcal{R}$ are indexed by $\mathbb{N}$, and the orbit corresponding to $n \in \mathbb{N}$ is in natural bijection with the n'th Calogero-Moser space*

$$C_n = \{X, Y \in M_n(\mathbb{C}) \mid \operatorname{rk}(YX - XY - \mathrm{id}) = 1\}/\operatorname{Gl}_n(\mathbb{C}) \tag{1.1}$$

where $\operatorname{Gl}_n(\mathbb{C})$ acts by simultaneous conjugation on (X, Y).

The fact $\mathcal{R}/G \cong \mathbb{N}$ has also been proved by Kouakou in his (unpublished) PhD-thesis [18].

The first proof of Theorem 1.1 used the fact that there is a description of $\mathcal{R}$ in terms of the adelic Grassmanian (due to Cannings and Holland [12]). Using methods from integrable systems, Wilson established a relation between the adelic Grassmanian and the Calogero-Moser spaces [36].

In [7] Berest and Wilson gave a new proof of Theorem 1.1 using noncommutative algebraic geometry [1, 34] (some of the original proofs in [7] were slightly simplified by the second author). See also [6, 16]. That an approach based on noncommutative geometry should be possible was in fact anticipated very early by Le Bruyn who in [20] already came very close to proving Theorem 1.1.

Let us briefly indicate how the methods of noncommutative algebraic geometry may be used to prove Theorem 1.1. We introduce the *homogenized Weyl algebra* $H = k\langle x, y, z\rangle/(zx - xz, zy - yz, yx - xy - z^2)$ and then we consider A_1 as the coordinate ring of an open affine part of a noncommutative space $\mathbb{P}^2_q$, with "homogeneous coordinate ring" H (see below for more precise definitions). The problem of describing $\mathcal{R}$ then becomes equivalent to describing certain objects on the "noncommutative projective plane" $\mathbb{P}^2_q$. Objects on $\mathbb{P}^2_q$ have finite dimensional cohomology groups and these may be used to define moduli spaces, just as in the ordinary commutative case.

The current paper starts from the observation that there are many more noncommutative projective planes than just the one associated to the Weyl algebra (this is in fact a fairly degenerate one) [2, 3, 11]. So below we let $\mathbb{P}^2_q$ be a so-called "elliptic" quantum projective plane. By definition $\mathbb{P}^2_q$ is noncommutative projective scheme which has as homogeneous coordinate ring a graded ring A with generators x, y, z (in degree one) satisfying the relations

$$\begin{cases} ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \\ axy + byx + cz^2 = 0 \end{cases} \tag{1.2}$$

where a, b, c are generic scalars (see below).

Such a $\mathbb{P}^2_q$ is called an elliptic quantum plane because there is an inclusion (in a noncommutative geometry sense) $E \hookrightarrow \mathbb{P}^2_q$ where E is a smooth (commutative) elliptic curve.

We let $\mathcal{R}$ be the set of reflexive graded A-ideals, considered up to isomorphism and shift of grading. We may think of the elements of $\mathcal{R}$ as line bundles on $\mathbb{P}^2_q$. In this paper we prove the following result (see Theorem 5.5.5 below).

Theorem 1.2. *There exist smooth affine varieties D_n of dimension $2n$ such that $\mathcal{R}$ is naturally in bijection with $\coprod_n D_n$.*

We would like to think of the D_n as elliptic Calogero-Moser spaces. We show below that D_0 is a point and D_1 is the complement of E under a natural embedding in $\mathbb{P}^2$.

Remark 1.3. If fact D_n is connected, which we will prove in a subsequent paper [13].

Remark 1.4. A theorem similar to Theorem 1.2 has been announced by [24]. They work in a more general setting where the associated automorphism σ of E may have finite order.

The reader will notice that Theorem 1.2 is weaker than Theorem 1.1 but this is probably unavoidable. Although we have a fairly succinct description of the varieties D_n (see (5.9) below) it is not as explicit as (1.1). And very likely D_n can also not be viewed in a natural way as the orbit of a group.

Our proof of Theorem 1.2 is similar in spirit to the proof of Theorem 1.1. However it is substantially more involved. The reason for this is that the proofs for the Weyl algebra rely heavily on the fact that H contains a central element in degree one (namely z) and the lowest central element in A has degree three.

We also have a result which explicitly describes the elements of $\mathcal{R}$. Recall that a line module over A is a graded A-module of the form A/uA with $u \in A_1 - \{0\}$.

The following theorem can be deduced easily from Theorem 5.6.6 below.

Theorem 1.5. *Let $I \in \mathcal{R}$. Then there exists an $m \in \mathbb{N}$ together with a monomorphism $I(-m) \hookrightarrow A$ such that there exists a filtration of reflexive graded A-ideals $A = M_0 \supset M_1 \supset \cdots \supset M_u = I(-m)$ with the property that the M_i/M_{i+1} are shifted line modules, up to finite length modules.*

It seems plausible that this result may be used to obtain an analogue of the Cannings-Holland classification of ideals in the Weyl algebra (see [12]) but we have not sorted out the details. We hope to come back on this in a subsequent paper.

2. Preliminaries

Throughout we work over an algebraically closed field k of characteristic zero. In this section we recall some basic notions of noncommutative projective geometry. These are collected from [1, 22, 26, 27, 28, 33]. We use the following convention:

Convention 2.1. *If* $\mathrm{XyUvw}(\cdots)$ *denotes an abelian category then* $\mathrm{xyuvw}(\cdots)$ *denotes the full subcategory of* $\mathrm{XyUvw}(\cdots)$ *consisting of noetherian objects.*

To simplify the notations we often use implicitly the following result

Lemma 2.2. *Assume that $\mathcal{C}$ is a locally noetherian category and $\mathcal{C}_f$ is the full subcategory of $\mathcal{C}$ consisting of noetherian objects. Then the natural map $D^b(\mathcal{C}_f) \to D^b_{\mathcal{C}_f}(\mathcal{C})$ is an equivalence of categories.*

Proof. This follows for example from the dual of [17, 1.7.11]. □

2.1. **Graded algebras and modules.** Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded algebra. If $A_i = 0$ for all $i < 0$ we say that A is *positively graded*, and if in addition $A_0 = k$ we say that A is *connected*. Any graded connected Noetherian k-algebra A is *locally finite*, i.e. $\dim_k A_i < \infty$, for all $i \in \mathbb{Z}$.
We write $\operatorname{GrMod}(A)$ for the category of graded right A-modules with morphisms the A-module homomorphisms of degree zero. Let M be a graded right A-module. We use the notation (for all $n \in \mathbb{Z}$) $M_{\geq n} = \oplus_{d\geq n} M_d$ and $M_{\leq n} = \oplus_{d\leq n} M_d$.

We say that M is *left* (resp. *right*) *bounded* if $M_{<n} = 0$ (resp. $M_{>n} = 0$) for some $n \in \mathbb{Z}$. For any integer n, define $M(n)$ as the graded A-module that is equal to M with its original A action, but which is graded by $M(n)_i = M_{n+i}$. We refer to the functor $M \mapsto M(n)$ as the n-th *shift functor*.

Since $\operatorname{GrMod}(A)$ is an abelian category with enough injective objects we may define the functors $\operatorname{Ext}^n_A(M,-)$ on $\operatorname{GrMod}(A)$ as the right derived functors of $\operatorname{Hom}_A(M,-)$. It is convenient to write (for $n \geq 0$)

$$\underline{\operatorname{Ext}}^n_A(M,N) := \bigoplus_{d\in\mathbb{Z}} \operatorname{Ext}^n_A(M,N(d));$$

whence $\underline{\operatorname{Ext}}^n_A(M,-)$ are the right derived functors of $\underline{\operatorname{Ext}}^0_A(M,-) := \underline{\operatorname{Hom}}_A(M,-)$, for $n \geq 1$.

Finally, recall that a module $M \in \operatorname{GrMod}(A)$ is *reflexive* if $M^{**} = M$ where $M^* = \underline{\operatorname{Hom}}_A(M,A)$ is the graded dual of M.

2.2. **Tails.** Let A be a Noetherian connected graded k-algebra. We denote by τ the functor that sends a graded right A-module to the the sum of all its finite dimensional submodules.

Denote by $\operatorname{Tors}(A)$ the full subcategory of $\operatorname{GrMod}(A)$ consisting of all modules such that $\tau M = M$ and write $\operatorname{Tails}(A)$ for the quotient category $\operatorname{GrMod}(A)/\operatorname{Tors}(A)$. We write $\pi : \operatorname{GrMod}(A) \to \operatorname{Tails}(A)$ for the (exact) quotient functor. By localization theory [29] π has a right adjoint which we denote by ω. It is well-known that $\pi \circ \omega = \mathrm{id}$. The object πA in $\operatorname{Tails}(A)$ will be denoted by $\mathcal{O}$ and it is again easy to see that $\omega = \underline{\operatorname{Hom}}_{\operatorname{Tails}}(\mathcal{O},-)$. The objects in $\operatorname{Tails}(A)$ will be denoted by script letters, like $\mathcal{M}$.

The shift functor induces an automorphism $\mathrm{sh} : \mathcal{M} \mapsto \mathcal{M}(1)$ on $\operatorname{Tails}(A)$ which we also call the shift functor (in analogy with algebraic geometry it should perhaps be called the “twist” functor).

When there is no possible confusion we write Hom instead of Hom_A and $\operatorname{Hom}_{\operatorname{Tails}(A)}$. The context will make clear in which category we work.

If $\mathcal{M} \in \operatorname{Tails}(A)$ then $\operatorname{Hom}(\mathcal{M},-)$ is left exact, so we may define its right derived functors $\operatorname{Ext}^n(\mathcal{M},-)$. We also use the notation

$$\underline{\operatorname{Ext}}^n(\mathcal{M},\mathcal{N}) := \bigoplus_{d\in\mathbb{Z}} \operatorname{Ext}^n(\mathcal{M},\mathcal{N}(d))$$

and we set $\underline{\operatorname{Hom}}(\mathcal{M},\mathcal{N}) = \underline{\operatorname{Ext}}^0(\mathcal{M},\mathcal{N})$.

Our Convention 2.1 fixes the meaning of $\operatorname{grmod}(A)$, $\operatorname{tors}(A)$ and $\operatorname{tails}(A)$. It is easy to see that $\operatorname{tors}(A)$ consists of the finite dimensional graded A-modules. Furthermore $\operatorname{tails}(A) = \operatorname{grmod}(A)/\operatorname{tors}(A)$.

If M is finitely generated and N is arbitrary we have

$$\underline{\operatorname{Ext}}^n(\pi M,\pi N) \cong \varinjlim \underline{\operatorname{Ext}}^n_A(M_{\geq n},N). \tag{2.1}$$

If M and N are both finitely generated, then (2.1) implies

$$\pi M \cong \pi N \text{ in } \operatorname{tails}(A) \quad \Leftrightarrow \quad M_{\geq n} \cong N_{\geq n} \text{ in } \operatorname{grmod}(A) \text{ for some } n$$

explaining the word "tails".

For $M \in \operatorname{GrMod}(A)$ there is an exact sequence (see [1], Proposition 7.2)

$$0 \to \tau M \to M \to \omega\pi M \to \varinjlim \underline{\operatorname{Ext}}^1_A(A/A_{\geq n}, M) \to 0. \tag{2.2}$$

An object $\mathcal{M} \in \operatorname{Tails}(A)$ is said to be *reflexive* if $\mathcal{M} = \pi M$ for some reflexive $M \in \operatorname{GrMod}(A)$.

We say that A satisfies condition χ if $\dim_k \operatorname{Ext}^j(k, M) < \infty$ for all j and all $M \in \operatorname{grmod}(A)$.

In case A satisfies condition χ then for every $M \in \operatorname{grmod}(A)$ the cokernel of the map $M \to \omega\pi M$ in the exact sequence (2.2) is right bounded. In particular, for $M \in \operatorname{grmod}(A)$ we have $M_{\geq d} \cong (\omega\pi M)_{\geq d}$ for some d.

Every graded quotient of a polynomial ring satisfies condition χ and so do most noncommutative algebras of importance. The condition is essential to get a theory for noncommutative schemes which resembles the commutative theory.

Proposition 2.2.1. [1] *Let A be a right Noetherian connected k-algebra satisfying condition χ. Then $\operatorname{Ext}^j(\mathcal{M}, \mathcal{N})$ is finite dimensional for all j and all $\mathcal{M}, \mathcal{N} \in \operatorname{tails}(A)$.*

2.3. **Serre duality.** It was shown in [38] that under reasonable hypotheses the category $\operatorname{tails}(A)$ satisfies a classical form of Serre duality. However we will need a stronger form of Serre duality introduced by Bondal and Kapranov in [10]. Let $\mathcal{A}$ be a k-linear Ext-finite triangulated category. By this we mean that for all $\mathcal{M}, \mathcal{N} \in \mathcal{A}$ we have $\sum_n \dim_k \operatorname{Hom}(\mathcal{M}, \mathcal{N}[n]) < \infty$. The category $\mathcal{A}$ is said to satisfy Bondal-Kapranov-Serre (BKS) duality if there is an auto-equivalence $F : \mathcal{A} \to \mathcal{A}$ together with for all $A, B \in \mathcal{A}$ natural isomorphisms

$$\operatorname{Hom}(A, B) \to \operatorname{Hom}(B, FA)'$$

(where $(-)'$ denotes the k-dual).

Let $\mathcal{C}$ be an abelian category. We say that $\mathcal{C}$ has *finite global dimension* if there exists an n such that $\operatorname{Ext}^i_{\mathcal{C}}(A, B) = 0$ for all $A, B \in \mathcal{C}$ and for all $i > n$. The minimal such n is called the *global dimension* of $\mathcal{C}$.

In this section we assume that A is a connected graded noetherian ring over a k. By $(-)'$ we denote the functor on graded vector spaces which sends M to $\oplus_n M^*_{-n}$. If we use notations which refer to the left structure of A then we adorn them with a superscript "$\circ$".

We make the following additional assumptions on A

1. A satisfies χ and the functor τ has finite cohomological dimension.
2. A satisfies χ° and the functor τ° has finite cohomological dimension.
3. $\operatorname{tails}(A)$ has finite global dimension.

Note that if A has finite global dimension then so does $\operatorname{tails}(A)$ by (2.1).

Put $R = R\tau(A)'$. According to [32] R is a complex of bimodules with finitely generated cohomology on the left and on the right, which in addition has finite injective dimension, also on the left and on the right. We now have the following result

Theorem 2.3.1. *(Serre duality) The category $D^b(\mathrm{tails}(A))$ satisfies BKS-duality with Serre functor defined by*

$$F(\pi M) = \pi(M \overset{\mathbf{L}}{\otimes} R)[-1]$$

This result is certainly not unexpected but as far as we know a written proof does not exist in the literature. We prove a more general version of Theorem 2.3.1 in Appendix A.

2.4. **Projective schemes.** We use the definition of $\operatorname{Proj} A$ (for a noncommutative algebra A) suggested by Artin and Zhang (see [1]). Let A be a Noetherian graded k-algebra. We define the (polarized) projective scheme $\operatorname{Proj} A$ of A as the triple $(\mathrm{tails}(A), \mathcal{O}, \mathrm{sh})$. In what follows we shall refer to the objects of $\mathrm{tails}(A)$ (resp. $\mathrm{Tails}(A)$) as the *coherent* (resp. *quasicoherent*) sheaves on $X = \operatorname{Proj} A$, even when A is not commutative, and we shall use the notation $\mathrm{coh}(X) := \mathrm{tails}(A)$, $\mathrm{Qcoh}(X) := \mathrm{Tails}(A)$. By analogy we sometimes write $\mathcal{O}_X = \mathcal{O} = \pi A$.

The following definitions agree with the classical ones for projective schemes. If $\mathcal{M}$ is be a quasicoherent sheaf on $X = \operatorname{Proj} A$, we define the *cohomology groups* of $\mathcal{M}$ by

$$H^n(X, \mathcal{M}) := \mathrm{Ext}^n(\mathcal{O}_X, \mathcal{M}).$$

We refer to the graded right A-modules

$$\underline{H}^n(X, \mathcal{M}) := \bigoplus_{d \in \mathbb{Z}} H^n(X, \mathcal{M}(d))$$

as the *full cohomology modules* of $\mathcal{M}$.
Finally, we mention the *cohomological dimension* of X

$$\operatorname{cd} X := \max\{n \in \mathbb{N} \mid H^n(X, -) \neq 0\}.$$

It is easy to prove that

$$\operatorname{cd} X = \max(0, \operatorname{cd} \tau - 1).$$

2.5. **The Grothendieck group, the Euler form and Hilbert series.** In this subsection A will be a Noetherian connected graded k-algebra with finite global (homological) dimension. We recall some basic tools.

2.5.1. *The Euler form.* Let $\mathcal{C}$ be an Ext-finite k-linear abelian category of finite global dimension. We define

$$\chi(A, B) = \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{C}}(A, B)$$

for $A, B \in \mathcal{C}$. It is clear that χ defines a bilinear form $\chi : K_0(\mathcal{C}) \times K_0(\mathcal{C}) \to \mathbb{Z}$ which we call the *Euler form* for $\mathcal{C}$.

2.5.2. *Hilbert series.* The Hilbert series of $M \in \mathrm{grmod}(A)$ is the Laurent power series

$$h_M(t) = \sum_{i=-\infty}^{+\infty} (\dim_k M_i) t^i \in \mathbb{Z}((t)).$$

This definition makes sense since A is right Noetherian.

Let $M \in \mathrm{grmod}(A)$. Given a resolution

$$0 \to P^r \to \ldots \to P^1 \to P^0 \to M \to 0$$

we have

$$h_M(t) = \sum_{i=0}^{r} (-1)^i h_{P^i}(t).$$

Since A is connected, left bounded graded right A-modules are projective if and only if they are free hence isomorphic to a sum of shifts of A. So if we write

$$P^i = \bigoplus_{j=0}^{r_i} A(-l_{ij})$$

we obtain the formula

$$q_M(t) = h_M(t) h_A(t)^{-1} \tag{2.3}$$

where $q_M(t)$ is the *characteristic polynomial of* M, it is defined by

$$q_M(t) = \sum_{i=0}^{r} (-1)^i \sum_{j=0}^{r_i} t^{l_{ij}} \in \mathbb{Z}[t, t^{-1}].$$

2.5.3. *The Grothendieck group* $K_0(X)$ *and the rank function* $K_0(X) \to \mathbb{Z}$. Set $X = \operatorname{Proj} A$. If $\mathcal{M}$ is a coherent sheaf on X, we denote by $[\mathcal{M}]$ its image in $K_0(X)$.
The shift functor on $\operatorname{coh}(X)$ induces an automorphism of $K_0(X)$. Following [22], we view $K_0(X)$ as a $\mathbb{Z}[t, t^{-1}]$-module with t acting as the shift functor $\operatorname{sh}^{-1} : \mathcal{M} \mapsto \mathcal{M}(-1)$. Now $K_0(X)$ may be described in terms of the Hilbert series of A.

Theorem 2.5.1 ([22], Theorem 2.3). *Let A be a Noetherian connected graded k-algebra of finite global dimension. Set $X = \operatorname{Proj} A$ and let $q = q_k(t)$. Then*

$$K_0(X) \cong \mathbb{Z}[t, t^{-1}]/(q)$$

and for each $M \in \operatorname{grmod}(A)$, the isomorphism sends $[\pi M]$ to the characteristic polynomial $q_M(t)$ of M:

$$\theta : [\pi M] \mapsto \overline{q_M(t)}$$

In particular, $[\mathcal{O}(n)]$ is sent to t^{-n}.

Let us assume that A is a domain generated in degree one, so it has a graded division ring of fractions (graded version of Goldie's Theorem; see [23, Ch. C, Cor. I.1.7])

$$\operatorname{Fract}(A) := \{ab^{-1} \mid a, b \in A \text{ homogeneous}, b \neq 0\}.$$

The degree zero component $k(X)$ of $\operatorname{Fract}(A)$ is a division algebra which is called the *function field* of X. $\operatorname{Fract}(A)$ is isomorphic to a skew Laurent extention $k(X)[z, z^{-1}; \sigma]$ where z has degree one (see [23], Chapter A, Corollary I.4.3).
The *rank* of a finitely graded right A-module M is

$$\operatorname{rk} M = \dim_{k(X)} \left(M \otimes_A \operatorname{Fract}(A)\right)_0$$

This also defines an additive rank fuction on $\operatorname{coh}(X)$ and hence a homomorphism $K_0(X) \to \mathbb{Z}$ also denoted by "rk". Obviously $\operatorname{rk} \mathcal{O} = 1$ and $\operatorname{rk} \mathcal{M} = \operatorname{rk} \mathcal{M}(1)$.

2.6. **Artin-Schelter regular algebras.**

Definition 2.6.1. [1] A connected graded k-algebra A is called an *Artin-Schelter regular algebra of dimension* d if it has the following properties:

(i) A has finite global dimension d;
(ii) A has polynomial growth, that is, there exists positive real numbers c, δ such that $\dim_k A_n \le cn^\delta$ for all positive integers n;
(iii) A is Gorenstein, meaning there is an integer l such that
$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_Ak(l) & \text{if } i = d, \\ 0 & \text{otherwise.} \end{cases}$$
where l is called the *Gorenstein parameter* of A.

If A is commutative, then the condition (i) already implies that A is isomorphic to a polynomial ring $k[x_1, \dots x_n]$ with some positive grading. If in this case the grading is standard then $n = l$.
The Gorenstein property determines the full cohomology groups of $\mathcal{O}$.

Theorem 2.6.2. [1] *Let A be a Noetherian Artin-Schelter regular algebra of dimension $d = n + 1$, and let $X = \operatorname{Proj} A$. Then $\operatorname{cd} X = n$, and the full cohomology modules of $\mathcal{O} = \pi A$ are given by*
$$\underline{H}^i(X, \mathcal{O}) \cong \begin{cases} A & \textit{if } i = 0 \\ 0 & \textit{if } i \neq 0, n \\ A'(l) & \textit{if } i = n \end{cases}$$

Let A be an Artin-Schelter regular algebra as in the previous Theorem and put $X = \operatorname{Proj} A$. It is easy to see that A satisfies the hypotheses for Theoreom 2.3.1. In this case the Serre functor has a particularly simple form: indeed in [1] it is shown that $R = (R^{n+1}\tau A)' \cong A[n + 1](-l)$ as left A-modules and in [32] it is proved that $R\tau A \cong R\tau^\circ A$ as complexes of bimodules. Thus we also have that $R = A[n + 1](-l)$ as right A-modules. In other words $R = A_\phi[n + 1](-l)$ where ϕ is some graded automorphism of A. The automorphism $M \mapsto M_\phi$ of $\operatorname{GrMod}(A)$ passes to an automorphism $\operatorname{Tails}(A)$ for which we also use the notation $(-)_\phi$.

We find the the following formula for the Serre functor on $\operatorname{tails}(A)$.
$$F\mathcal{M} = \mathcal{M}_\phi(-l)[n]$$
From this we easily obtain:

Proposition 2.6.3. *One has* $\operatorname{gl}\dim \operatorname{tails}(A) = \operatorname{gl}\dim A - 1$.

Proof. As above put $\operatorname{gl}\dim A = n + 1$. The inequality $\operatorname{gl}\dim \operatorname{tails}(A) \le n$ follows directly from BKS-duality and the above discussion. The other inequality follows from Theorem 2.6.2. □

2.7. **Dimension and multiplicity.** Let A be a noetherian Artin-Schelter regular algebra. If $0 \neq M \in \operatorname{grmod}(A)$ then the *Gelfand-Kirilov dimension* $\operatorname{GKdim} M$ of M [19] can be computed as the order of the pole of $h_M(t)$ in 1 [3] (in particular it is an integer). If $\operatorname{GKdim} M \le n$ then we define $e_n(M)$ as $\lim_{t\to 1}(1 - t)^n h_M(t)$. Clearly e_n is additive on short exact sequences of objects with $\operatorname{GKdim} \le n$. We have $e_n(M) \ge 0$ and furthermore $e_n(M) = 0$ if and only if $\operatorname{GKdim} M < n$. If $e_n(M) > 0$ then we put $e(M) = e_n(M)$ and we call this the *multiplicity* of M. If $u = \operatorname{GKdim} A$ and if A is a domain generated in degree one then it is easy to see that $\operatorname{rk} M = e_u(M)/e_u(A)$.

If $\mathcal{M} = \pi M \neq 0$ then we put $\dim \mathcal{M} = \operatorname{GKdim} M - 1$ and $e_n(\mathcal{M}) = e_{n+1}(M)$, $e(\mathcal{M}) = e(M)$.

An object in $\operatorname{grmod}(A)$ or $\operatorname{tails}(A)$ is said to be *pure* if it contains no subobjects of strictly smaller dimension. It is *critical* if all non-trivial subobjects have the same multiplicity. It is easy to see that if $M \in \operatorname{grmod}(A)$ is pure or critical then so is πM, and conversely if $\mathcal{M} \in \operatorname{tails}(A)$ is pure or critical then there exists a module $M \in \operatorname{grmod}(A)$ which has the corresponding property such that $\mathcal{M} = \pi M$.

2.8. **Three dimensional Artin-Schelter regular algebras.** There exists a complete classification for Artin-Schelter regular algebras of dimension three [2, 3, 30, 31]:

Theorem 2.8.1. *The Artin-Schelter regular algbras A of dimension three can be classified. They are all Noetherian domains with Hilbert series of a weighted polynomial ring $k[x, y, z]$.*

It is known that three dimensional Artin-Schelter regular algebras have all expected nice homological properties. For example they are both left and right noetherian domains.

In this paper we restrict ourselves to Koszul three dimensional Artin-Schelter regular algebra. These have three generators and three defining relations in degree two. The minimal resolution of k has the form

$$0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$$

hence $q_k(t) = (1-t)^3$ and the Hilbert series of A is the same as that of the commutative polynomial algebra $k[x, y, z]$ with standard grading.

Such algebras are also referred to as *quantum polynomial ring in three variables.* The corresponding $\operatorname{Proj} A$ will be called a *quantum projective plane* and will be denoted by $\mathbb{P}^2_{\mathrm{q}}$.

So let A be a quantum polynomial ring in three variables. A *linear module of dimension d* over A is a cyclic A-module generated in degree zero with Hilbert series $(1-t)^{-d}$. Linear modules of dimension one and two are respectively called point and line modules. The images of these objects in $\operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$ will be called point and line objects respectively. Line and point modules are classified in [2, 3].

Line modules are of the form $A/uA = L$ with $u \in A_1$. Hence line modules correspond naturally to lines in $\mathbb{P}^2$. To classify point modules we write the relations of A as $f_i = \sum_{j=1}^3 m_{ij} x_j$. Set $M = (m_{ij})_{i,j}$. We introduce auxiliary (commuting) variables $x_i^{(p)}$ (for $p \in \mathbb{Z}$) and for a monomial $u = x_{i_0} \cdots x_{i_n}$ we define the *multilinearization* of m as $\widetilde{m}$ as $x_{i_0}^{(0)} \cdots x_{i_n}^{(n)}$. We extend this operation linearly to homogeneous polynomials in the variables $(x_i)_i$.

Let $\Gamma \subset \mathbb{P}^2 \times \mathbb{P}^2$ denote the locus of common zeros of the $\tilde{f}_i$. It turns out that Γ is the graph of an automorphism σ of $E = \operatorname{pr}_1(\Gamma)$, the locus of zeros of the multihomogenized polynomial $\det(\widetilde{M})$. If $\det(\widetilde{M})$ is not identically zero then E is a divisor of degree 3 in $\mathbb{P}^2$. We then say that A is *elliptic*. Otherwise, E is all of $\mathbb{P}^2$ and we call A *linear* in this case.

The connection between E and point modules is as follows: let $P = \sum ke_u$ be a point module where $e_u \in P_u$. Put $e_u x_i = e_{u+1}\lambda_i^{(u)}$ with $\lambda_i^{(u)} \in k$. From the fact that $e_0 f_i = 0$ we deduce that $((\lambda_i^{(0)})_i, (\lambda_i^{(1)})_i) \in \Gamma$ and hence $(\lambda_i^{(0)})_i \in E$. This construction is reversible and defines a bijection between the closed points of E and the point modules over A. If P_q is the pointmodule corresponding to $q \in E$ then we have $P_q(1)_{\geq 0} = P_{\sigma p}$.

Let $j : E \to \mathbb{P}^2$ be the inclusion and put $\mathcal{L} = j^*\mathcal{O}_{\mathbb{P}^2}(1)$. Associated to the geometric data $(E, \sigma, \mathcal{L})$ is a so-called "twisted" homogeneous coordinate ring $B = B(E, \sigma, \mathcal{L})$. This is a special case of a general construction in [1]. See also [4]. Denote the auto-equivalence $\sigma_*(- \otimes_E \mathcal{L})$ by $- \otimes \mathcal{L}_\sigma$. For $\mathcal{M} \in \mathrm{Qcoh}(X)$ put $\Gamma_*(\mathcal{M}) = \oplus_u \Gamma(E, \mathcal{M} \otimes (\mathcal{L}_\sigma)^{\otimes u})$ and $B = \Gamma_*(\mathcal{O}_E)$. It is easy to see that B has a natural ring structure and $\Gamma_*(\mathcal{M})$ is a right B-module. A straightforward verification shows

$$\Gamma_*(\mathcal{O}_q) = P_q.$$

In [3] it is shown that there is a surjective morphism $p : A \to B$ of graded k-algebras. Its kernel is trivial in the linear case and it is generated by a regular normalizing element g of degree three in the elliptic case. All point modules are B-modules. In other words: g annihilates all point modules.

By analogy with the commutative case we may say that $\operatorname{Proj} A$ contains $\operatorname{Proj} B$ as a "closed" subscheme. Though the structure of $\operatorname{Proj} A$ is somewhat obscure, that of $\operatorname{Proj} B$ is well understood.

Indeed it follows from [1, 4] that the functor $\Gamma_* : \mathrm{Qcoh}(E) \to \mathrm{GrMod}(B)$ defines an equivalence $\mathrm{Qcoh}(E) \cong \mathrm{Tails}(B)$. The inverse of this equivalence and its composition with $\pi : \mathrm{GrMod}(B) \to \mathrm{Tails}(B)$ are both denoted by $\widetilde{(-)}$

For further properties of point modules and line modules over three dimensional quantum polynomial algebras we refer to [2, 3].

We will frequently use the following result

Lemma 2.8.2. *Assume that we are in the elliptic case. Let $M \in \mathrm{grmod}(A)$ be such that $M/Mg \in \mathrm{tails}\, A$. Then $\mathrm{GKdim}\, M = 1$. If σ has infinite order then $M \in \mathrm{tails}(A)$.*

Proof. Multiplication induces an isomorphism $M_n \cong M_{n+3}$ for large n. Hence $\mathrm{GKdim}\, M = 1$. Furthermore $(M_g)_0$ is a finite dimensional representation of $(A_g)_0$. It is is shown in [3] that if σ has infinite order then $(A_g)_0$ is a simple ring. In particular it has no finite dimensional representations. Thus $(M_g)_0 = 0$. This implies $M \in \mathrm{tails}(A)$. □

In the sequel it will be useful to cast the relationship between the noncommutative graded ring A and the commutative scheme E into the language of noncommutative algebraic geometry exhibited in [26, 33] although we will use this language only in an intuitive way. Let $X = \operatorname{Proj} A$, $Y = \operatorname{Proj} B$.

We define a map of noncommutative schemes $i : E \to X$ by

$$i^*\pi M = (M \otimes_A B)\tilde{}$$
$$i_*\mathcal{M} = \pi(\Gamma_*(\mathcal{M})_A)$$

We will call $i^*(\pi M)$ the *restriction* of πM to E. i_* is clearly an exact functor. For the left derived functor of i^* we have:

Lemma 2.8.3. *If $M \in D^-(\mathrm{GrMod}(A))$ then $Li^*(\pi M) = (M \overset{\mathbf{L}}{\otimes}_A B)\tilde{}$*

Proof. One shows first that the objects πF where F is a finitely generated graded free A-module are acyclic for i^* in the sense of [14]. Then the lemma follows by replacing M by a resolution of finitely generated free A-modules. □

We easily obtain the following consequence:

Lemma 2.8.4. *Assume that we are in the elliptic case and let $\mathcal{M} \in D^-(\mathrm{Qcoh}(\mathbb{P}^2_q))$. Then there are short exact sequences:*

$$0 \to i^* H^j(\mathcal{M}) \to H^j(Li^*\mathcal{M}) \to L_1 i^* H^{j+1}(\mathcal{M}) \to 0$$

Proof. Take $M \in D^-(\mathrm{GrMod}(A))$ such that $\mathcal{M} = \pi M$. We may assume that M is given by a right bounded complex of graded projective A-modules. The lemma now follows by applying π to the long exact homology sequence associate to the short exact sequence of complexes

$$0 \to Mg \to M \to M/Mg \to 0 \quad \square$$

2.9. **Three dimensional Sklyanin algebras.** Below we are interested in Sklyanin algebras of dimension three which are elliptic Artin-Schelter regular algebras such that the corresponding elliptic curve E is smooth and the automorphism is a translation. More specificly, we are interested in the algebras

$$\mathrm{Skl}_3(a,b,c) = k\{x,y,z\}/(f_1,f_2,f_3)$$

where f_1, f_2, f_3 are the quadratic equations

$$\left\{\begin{array}{l} f_1 = ayz + bzy + cx^2 \\ f_2 = azx + bxz + cy^2 \\ f_3 = axy + byx + cz^2 \end{array}\right. \tag{2.4}$$

and $(a,b,c) \in \mathbb{P}^2 \setminus F$ where

$$F = \{(a,b,c) \in \mathbb{P}^2 \mid abc = 0 \text{ or } a^3 = b^3 = c^3 \text{ or } (3abc)^3 = (a^3+b^3+c^3)^3\}.$$

The algebras $\mathrm{Skl}_3(a,b,c)$ are elliptic quantum polynomial rings. They correspond to Artin-Schelter algebras of dimension three where, in the associated geometric data, E is a smooth elliptic curve and σ is given by translation under the group law. We refer to [2] for the description of E and σ. The regular normalizing element g of degree three turns out to be central in this case.

Put $A = \mathrm{Skl}_3(a,b,c)$. Combining the results in [32] with Theorem 2.3.1 we see that Serre duality for A takes a particularly simple form:

Theorem 2.9.1. *Let $\mathcal{M}, \mathcal{N} \in D^b(\mathrm{tails}(A))$. Then there are natural isomorphisms*

$$\mathrm{Ext}^i(\mathcal{M}, \mathcal{N}) \cong \mathrm{Ext}^{2-i}(\mathcal{N}, \mathcal{M}(-3))^*$$

Corollary 2.9.2. *Let $\mathcal{M} \in D^b(\mathrm{tails}(A))$ and let $\mathcal{P} \in \mathrm{tails}(A)$ be a point object corresponding to $p \in E$. Then*

$$\mathrm{Ext}^i(\mathcal{P}, \mathcal{M}) \cong \mathrm{Ext}^{2-i}(\mathcal{M}, \mathcal{P}')^*$$

where $\mathcal{P}'$ is the point object corresponding to $\sigma^{-3}p$.

3. Cohomology of rank one sheaves on a quantum projective plane

In this section, A will be a quantum polynomial ring in three variables, and $\mathbb{P}^2_q = \mathrm{Proj}\, A$ the associated quantum projective plane. As usual $\mathcal{O} = \pi A$.

We say that a graded right A-module $M \neq 0$ is *torsion* if $\mathrm{rk}\, M = 0$. M is called *torsion-free* if M contains no torsion submodule. This is the same as saying that M is pure three dimensional. We use the same terminology for objects in $\mathrm{coh}(\mathbb{P}^2_q)$.

The graded right ideals of A are, up to isomorphism, precisely the shifts of torsion-free rank one right A-modules.

A torsion-free rank one graded A-module I gives rise to a torsion-free coherent sheaf $\mathcal{I} = \pi I$ on $\mathbb{P}^2_{\mathrm{q}}$ of rank one. Conversely, every torsion-free $\mathcal{I} \in \mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ determines a torsion-free rank one graded A-module $\omega\mathcal{I}$.

Any shift l of a torsion-free rank one graded A-module I gives rise to a torsion-free rank one coherent sheaf $\mathcal{I}(l) = \pi I(l)$ on $\mathbb{P}^2_{\mathrm{q}}$. Our first aim is to normalize this shift.

We will use the following natural basis for $K_0(\mathbb{P}^2_q)$.

Proposition 3.1. *Let P be a point module and S a line module over A. Denote the corresponding objects in $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$ by $\mathcal{P}$ and $\mathcal{S}$.*
Then $\{[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]\}$ is a $\mathbb{Z}$-module basis of $K_0(\mathbb{P}^2_q)$, which does not depend on the particular choice of S and P, and the action of the shift functor on that basis is

$$\begin{aligned} [\mathcal{O}(1)] &= [\mathcal{O}] + [\mathcal{S}] + [\mathcal{P}] \\ [\mathcal{S}(1)] &= [\mathcal{S}] + [\mathcal{P}] \\ [\mathcal{P}(1)] &= [\mathcal{P}] \end{aligned} \tag{3.1}$$

Proof. It follows from Theorem 2.5.1 that the class in $K_0(\mathbb{P}^2_{\mathrm{q}})$ of an object πM depends only on the Hilbert series of M. Thus $[\mathcal{S}]$ and $[\mathcal{P}]$ are indeed independent of the particular choice of S and P.

Using a computation with Hilbert series we see that the images of $[\mathcal{O}], [\mathcal{S}]$ and $[\mathcal{P}]$ under the isomorphism θ of Theorem 2.5.1

$$\theta : K_0(\mathbb{P}^2_{\mathrm{q}}) \to \mathbb{Z}[t, t^{-1}]/(1-t)^3$$

are respectively $\overline{1}$, $\overline{1-t}$, $\overline{(1-t)^2}$. Furthermore the shift functor corresponds to multiplication by t^{-1}. This easily yields what we want. □

From now on, we fix such a $\mathbb{Z}$-module basis $\{[\mathcal{O}], [\mathcal{S}], [\mathcal{P}]\}$ of $K_0(\mathbb{P}^2_q)$. For any coherent sheaf $\mathcal{J}$ on $\mathbb{P}^2_{\mathrm{q}}$ we may write

$$[\mathcal{J}] = r[\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{P}]$$

where r is the rank of $\mathcal{J}$.
It follows from (3.1) that we have

$$[\mathcal{J}(l)] = r[\mathcal{O}] + (a + lr)[\mathcal{S}] + (\frac{1}{2}l(l+1)r + la + b)[\mathcal{P}] \tag{3.2}$$

for all integers l.

Proposition 3.2. 1. *Let $\mathcal{I}$ be a coherent sheaf on $\mathbb{P}^2_{\mathrm{q}}$ of rank one, and write $[\mathcal{I}] = [\mathcal{O}] + a[\mathcal{S}] + b[\mathcal{P}]$. Then there is an unique shift c (namely $-a$) and an integer n such that*

$$[\mathcal{I}(c)] = [\mathcal{O}] - n[\mathcal{P}].$$

Moreover, $n = \frac{1}{2}a(a+1) - b$.

2. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^2_{\mathrm{q}}$ of rank zero, and write $[\mathcal{F}] = u[\mathcal{S}] + v[\mathcal{P}]$. Then $u = e_1(\mathcal{F})$. If $u = 0$, then $v = e_0(\mathcal{F})$.*

Proof. For the first part, use (3.2). The uniqueness is easy to see.
For the second statement, take the image of $[\mathcal{F}] = u[\mathcal{S}] + v[\mathcal{P}]$ under the isomorphism θ of Theorem 2.5.1. Take F such that $\mathcal{F} = \pi F$. We obtain

$$q_F(t) = uq_L(t) + vq_P(t) + f(t)q_k(t)$$

for a suitable $f(t) \in \mathbb{Z}[t, t^{-1}]$. Multiplying both sides with $h_A(t) = q_k(t)^{-1}$ yields (see (2.3))

$$h_F(t) = uh_L(t) + vh_P(t) + f(t)$$

We find $e_2(F) = \lim_{t\to 1}(1-t)^2 h_F(t) = u$ and if $u = 0$ then $e_1(F) = \lim_{t\to 1}(1-t)h_F(t) = v$. □

We call the integer n appearing in Proposition 3.2 the "invariant" of $\mathcal{I}$ (or of the corresponding torsion-free rank one graded A-module I such that $\mathcal{I} = \pi I$). Note that two torsion-free rank one graded A-modules I, J have the same invariant if and only if $\dim_k I_i = \dim_k J(d)_i$ for $i \gg 0$ and for a fixed integer d.

We will call a torsion-free rank one coherent sheaf $\mathcal{I}$ on $\mathbb{P}^2_{\mathrm{q}}$ *normalized* if $[\mathcal{I}] = [\mathcal{O}] - n[\mathcal{P}]$ for an integer n. We will prove later that this n is actually positive.

We will call a torsion-free reflexive rank one sheaf on $\mathbb{P}^2_{\mathrm{q}}$ a *line bundle*. Our aim is to classify line bundles on $\mathbb{P}^2_{\mathrm{q}}$ up to shift. By the above discussion this is equivalent to classifying normalized line bundles up to isomorphism.

It is also easy to see that through the functors π and ω classifying line bundles up to shift is equivalent to classifying reflexive torsion-free rank one graded A-modules, also up to shift.

We recall two elementary lemmas.

Lemma 3.3. *Let $\mathcal{M}, \mathcal{N}$ be torsion-free coherent sheaves on $\mathbb{P}^2_{\mathrm{q}}$ of rank one. Then every nonzero morphism in* $\operatorname{Hom}(\mathcal{M}, \mathcal{N})$ *is injective.*

Proof. $\mathcal{M}$ and $\mathcal{N}$ are critical of the same dimension. It is well-known that this implies that any map between them must be injective [3]. □

Lemma 3.4. *Let $\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$. Then $\mathcal{M}$ is reflexive if and only if $\mathcal{M}$ is torsion-free and* $\operatorname{Ext}^1(\mathcal{N}, \mathcal{M}) = 0$ *for all* $\mathcal{N} \in \operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$ *of dimension zero.*

Proof. Assume that $\mathcal{M}$ is a reflexive coherent sheaf on $\mathbb{P}^2_{\mathrm{q}}$. By (2.1) we need to prove the corresponding statement for $\operatorname{grmod}(A)$. Thus assume that M is a reflexive A-module and $\operatorname{GKdim} N \leq 1$. Assume that there is a non-split exact sequence

$$0 \to M \to M' \to N \to 0 \tag{3.3}$$

By [3, Theorem 4.1] one has $\underline{\operatorname{Ext}}^1(N, A) = 0$. Hence we obtain $M'^* = M^*$ and thus $M = M^{**} = M'^{**}$. Thus the composition of $M \to M' \to M'^{**}$ is an isomorphism, implying that the first map splits. This contradicts the non-triviality of the extention (3.3).

For the other implication, let $\mathcal{M} \in \operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$ be torsion-free and $\operatorname{Ext}^1(\mathcal{N}, \mathcal{M}) = 0$ for all $\mathcal{N} \in \operatorname{coh}(\mathbb{P}^2_{\mathrm{q}})$ of dimension zero. $M = \omega\mathcal{M}$ is pure and $\operatorname{GKdim} M = 3$ since $\mathcal{M}$ is pure two dimensional. By [3, Corollary 4.2] there is a canonical map $\mu : M \to M^{**}$ and $\ker \mu$ is the maximal submodule of M which has $\operatorname{GKdim} < 3$. Hence μ is injective, and we have an exact sequence

$$0 \to M \to M^{**} \to \operatorname{coker} \mu \to 0 \tag{3.4}$$

where $\operatorname{GKdim}(\operatorname{coker} \mu) \leq 1$. Applying π on (3.4) yields

$$0 \to \mathcal{M} \to \pi M^{**} \to \mathcal{N} \to 0 \tag{3.5}$$

where $\mathcal{N} = \pi \operatorname{coker} \mu$. Now $\mathcal{N}$ must be zero, otherwise $\dim \mathcal{N} = 0$ and since $\operatorname{Ext}^1(\mathcal{N}, \mathcal{M}) = 0$ the sequence (3.5) would split, which is impossible because M^{**} is pure three dimensional. Hence $\mathcal{M} = \pi M = \pi M^{**}$ and thus $\mathcal{M}$ is reflexive. □

Now we can partially compute the cohomology of line bundles on $\mathbb{P}^2_q$. This computation is similar to the one for the homogenized Weyl algebra in [20]. However the computations for the homogenized Weyl algebra rely on the existense of a central element in degree one. So they do not apply in a straightforward way to the case we consider.

Theorem 3.5. *Let $\mathcal{M}$ be a rank one torsion-free coherent sheaf on $\mathbb{P}^2_q$ where $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$. Assume that $\mathcal{M} \not\cong \mathcal{O}$. Then*

1. $H^0(\mathbb{P}^2_q, \mathcal{M}(l)) = 0$ *for* $l \leq 0$,
   $H^2(\mathbb{P}^2_q, \mathcal{M}(l)) = 0$ *for* $l \geq -2$;
2. $\chi(\mathcal{O}, \mathcal{M}(l)) = \frac{1}{2}(l+1)(l+2) - n$ *for all* $l \in \mathbb{Z}$;
3. $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}) = n-1$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) = n$
   $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) = n$
4. $H^j(\mathbb{P}^2_q, \mathcal{M}) = 0$ *for* $j \geq 3$.

As a consequence, n is positive and nonzero.

If $\mathcal{M}$ is a line bundle then we have in addition: $H^2(\mathbb{P}^2_q, \mathcal{M}(l)) = 0$ for $l = -3$ and $\dim_k H^1(\mathbb{P}^2_q, \mathcal{M}(-3)) = n-1$.

Proof. That $H^j(\mathbb{P}^2_q, \mathcal{M}) = 0$ for $j \geq 3$ is part of Theorem 2.6.2.

To prove the rest of the current theorem we first let $l \leq 0$. Suppose f is a nonzero morphism in $\operatorname{Hom}(\mathcal{O}, \mathcal{M}(l))$. By lemma 3.3 f is injective and from the exact sequence

$$0 \to \mathcal{O} \to \mathcal{M}(l) \to \operatorname{coker} f \to 0 \tag{3.6}$$

we get $[\operatorname{coker} f] = l[\mathcal{S}] + (l(l+1)/2 - n)[\mathcal{P}]$. Using Proposition 3.2 gives $l \geq 0$, thus $l = 0$ and $[\operatorname{coker} f] = -n[\mathcal{P}]$. Hence by the discussion in §2.7 together with Proposition 3.2 we obtain $\dim \operatorname{coker} f = 0$. By lemma 3.4 $\operatorname{Ext}^1(\operatorname{coker} f, \mathcal{O}) = 0$. This means that the exact sequence (3.6) splits hence $\mathcal{M}$ is not torsion-free. A contradiction. We conclude that $\operatorname{Hom}(\mathcal{O}, \mathcal{M}(l)) = 0$ for $l \leq 0$.
Second, let $l \geq -2$. Serre duality (Theorem 2.9.1) yields

$$\operatorname{Ext}^2(\mathcal{O}, \mathcal{M}(l))^* \cong \operatorname{Hom}(\mathcal{M}(l+3), \mathcal{O}).$$

If g is a nonzero morphism in $\operatorname{Hom}(\mathcal{M}(l+3), \mathcal{O})$ then g is injective, and from the exact sequence

$$0 \to \mathcal{M}(l+3) \to \mathcal{O} \to \operatorname{coker} g \to 0 \tag{3.7}$$

we get $[\operatorname{coker} g] = u[\mathcal{S}] + v[\mathcal{P}]$ where $u = -(l+3)$ and $v = n - (l+3)(l+4)/2$. By Proposition 3.2 $u \geq 0$ but $l \geq -2$ implies $u < 0$. This yields a contradiction.

Assume now $l \geq -3$ and $\mathcal{M}$ reflexive. By the same reasoning as above we obtain $l = -3$ and thus the dimension of $\omega \operatorname{coker} g$ is zero. By lemma 3.4 it follows that (3.7) splits. But this contradicts the fact that $\mathcal{O}$ is torsion-free.

For the second part we use Theorem 2.6.2 to obtain

$$\chi(\mathcal{O}, \mathcal{O}(l)) = \frac{1}{2}(l+1)(l+2) \text{ for all } l \in \mathbb{Z}$$

and from Proposition 3.1 we deduce

$$\begin{aligned} [\mathcal{S}] &= -[\mathcal{O}(2)] + 3[\mathcal{O}(1)] - 2[\mathcal{O}] \\ [\mathcal{P}] &= [\mathcal{O}(2)] - 2[\mathcal{O}(1)] + [\mathcal{O}] \end{aligned}$$

Combining these results yields $\chi(\mathcal{O},\mathcal{S})=1$ and $\chi(\mathcal{O},\mathcal{P})=1$. Now we use (3.2) to obtain

$$\begin{aligned}\chi(\mathcal{O},\mathcal{M}(l)) &= \chi(\mathcal{O},\mathcal{O})+l\chi(\mathcal{O},\mathcal{S})+\left(\frac{1}{2}l(l+1)-n\right)\chi(\mathcal{O},\mathcal{P})\\ &= \frac{1}{2}(l+1)(l+2)-n\end{aligned}$$

Finally, we combine the first two results of the theorem. If $-2\le l\le 0$ (or $-3\le l\le 0$ if $\mathcal{M}$ is reflexive) the first statement gives

$$\begin{aligned}\chi(\mathcal{O},\mathcal{M}(l)) &= \dim_k H^0(\mathbb{P}^2_{\mathrm{q}},\mathcal{M}(l))-\dim_k H^1(\mathbb{P}^2_{\mathrm{q}},\mathcal{M}(l))+\dim_k H^2(\mathbb{P}^2_{\mathrm{q}},\mathcal{M}(l))\\ &= -\dim_k H^1(\mathbb{P}^2_{\mathrm{q}},\mathcal{M}(l))\end{aligned}$$

and comparing with the expression $\chi(\mathcal{O},\mathcal{M}(l))=\frac{1}{2}(l+1)(l+2)-n$ completes the proof. □

Using Theorem 3.5 the torsion-free rank one graded A-modules having invariant zero are easy to determine.

Corollary 3.6. *Let $\mathcal{I}$ be a torsion-free coherent sheaf of rank one on $\mathbb{P}^2_{\mathrm{q}}$ with invariant n. Then*

$$n=0\Leftrightarrow \mathcal{I}\cong\mathcal{O}(d)$$

for some integer d.

Proof. If $\mathcal{I}\cong\mathcal{O}(d)$ then clearly $n=0$. Assume conversely $n=0$. We may assume that $\mathcal{I}$ is normalized. If $\mathcal{I}\not\cong\mathcal{O}$ then by Theorem 3.5 $n>0$. Since $n=0$ we obtain $\mathcal{I}\cong\mathcal{O}$ by contraposition. □

4. Restriction of coherent sheaves

In this section, A will be a Sklyanin algebra $\mathrm{Skl}_3(a,b,c)$ as defined in §2.9. We recycle the notations of sections §2.6-§2.9. In particular the symbols $\mathcal{O},E,\sigma,\mathcal{L},B,i$ have their usual meaning.

Note that E is a smooth elliptic curve. We fix a grouplaw on E. Then σ is a translation by some element $\xi\in E$.

The dimension of objects in $\mathrm{grmod}(B)$ or $\mathrm{tails}(B)$ will be computed in $\mathrm{grmod}(A)$ or $\mathrm{tails}(A)$. The dimension of objects in $\mathrm{coh}(E)$ is the dimension of their support.

There is a group homomorphism

$$K_0(\mathbb{P}^2_{\mathrm{q}})\to K_0(E):[\mathcal{M}]\mapsto[i^*\mathcal{M}]-[L_1 i^*\mathcal{M}]$$

which as usual is also denoted by i^*.

Lemma 4.1. *We have*

$$\begin{aligned} i^*[\mathcal{O}] &= [\mathcal{O}_E] & &\\ i^*[\mathcal{S}] &= [\mathcal{O}_u]+[\mathcal{O}_v]+[\mathcal{O}_w] & &u,v,w \textit{ arbitrary but colinear}\\ i^*[\mathcal{P}] &= [\mathcal{O}_p]-[\mathcal{O}_{p^{\sigma^{-3}}}] & &p \textit{ arbitrary}\end{aligned}$$

Proof. This follows easily from lemma 2.8.3 □

According to [15, Ex II. 6.11] we have $K_0(E) \cong \mathbb{Z} \oplus \mathrm{Pic}(E)$. The projection $K_0(E) \to \mathbb{Z}$ is given by the rank and the projection $K_0(E) \to \mathrm{Pic}(E)$ is given the first Chern class. If $\mathcal{E}$ is a vector bundle on E then $c_1(\mathcal{E}) = \wedge^{\mathrm{rk}\,\mathcal{E}} \mathcal{E}$. We also have for $q \in E$: $c_1(\mathcal{O}_q) = \mathcal{O}_E(q)$.

There is a homomorphism $\deg : \mathrm{Pic}(E) \to \mathbb{Z}$ which assigns to a line bundle its degree. For simplicity we will denote the composition $\deg \circ c_1$ also by deg. If $\mathcal{U}$ is a line bundle then $\deg[\mathcal{U}] = \deg \mathcal{U}$. If $F \in \mathrm{coh}(E)$ has finite length then $\deg[F] = \operatorname{length} F$ [15, Ex. 6.12]. From lemma 4.1 we deduce that if $[\mathcal{M}] = a[\mathcal{O}] + b[\mathcal{S}] + c[\mathcal{P}]$ then

$$\operatorname{rk} i^*[\mathcal{M}] = a = \operatorname{rk} \mathcal{M} \tag{4.1}$$

$$\deg i^*[\mathcal{M}] = 3b \tag{4.2}$$

Lemma 4.2. 1. *If $M \in \mathrm{grmod}(B)$ is pure two dimensional then $\tilde{M} \in \mathrm{coh}(E)$ is pure one dimensional.*
2. *If $\mathcal{N} \in \mathrm{coh}(E)$ is pure one dimensional then $\Gamma_*(\mathcal{N})$ is pure two dimensional.*

Proof. The indecomposable objects in $\mathrm{coh}(E)$ are vector bundles and finite length objects. Using Riemann-Roch it is easy to see that if $0 \neq \mathcal{U} \in \mathrm{coh}(E)$ then $\operatorname{GKdim} \Gamma_*(\mathcal{U}) = \dim \mathcal{U} + 1$. From this we deduce that if $V \in \mathrm{grmod}(B)$ is not in $\mathrm{tors}(B)$ then $\operatorname{GKdim} V = \dim \tilde{V} + 1$. The lemma now easily follows. ☐

We deduce

Proposition 4.3. 1. *If $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q)$ is reflexive then $i^*\mathcal{M}$ is a vector bundle on E and $L_j i^*\mathcal{M} = 0$ for $j > 0$.*
2. *If $\mathcal{M}$ is a line bundle then so is $i^*\mathcal{M}$.*
3. *If $\mathcal{M}$ is a line bundle then $\mathcal{M}$ is normalized if and only if $\deg i^*\mathcal{M} = 0$.*
4. *If $\mathcal{M}$ is a normalized line bundle with invariant n then*

$$c_1(i^*\mathcal{M}) = \mathcal{O}((o) - (3n\xi))$$

where "o" is the origin for the group law.

Proof. 1. We have $\mathcal{M} = \pi M$ where M is reflexive. In particular M is torsion-free. By lemma 2.8.3 it follows that $L_j i^*\mathcal{M} = 0$ for $j > 0$ and $i^*\mathcal{M} = (M/Mg)\tilde{}$.

If M/Mg contains a nonzero submodule N/Mg of GK-dimension ≤ 1 then N represents an element of $\mathrm{Ext}^1(N/Mg, Mg)$ which must be zero by lemma 3.4 (or rather its proof). Thus $N/Mg \subset N \subset M$. This is impossible since M is torsion-free.

Hence M/Mg is pure of GK-dimension 2. By the previous lemma it follows that $(M/Mg)\tilde{}$ is a vector bundle.
2. This follows from (4.1).
3. This follows from (4.2).
4. We have $[\mathcal{M}] = [\mathcal{O}] - n[\mathcal{P}]$. By lemma 4.1 we obtain $[i^*\mathcal{M}] = [\mathcal{O}_E] - n[\mathcal{O}_p] + n[\mathcal{O}_{p^{\sigma^{-3}}}]$. Hence $c_1(i^*\mathcal{M}) = \mathcal{O}(n(p^{\sigma^{-3}}) - n(p))$. Now $n(p^{\sigma^{-3}}) - n(p)$ and $(o) - (3n\xi)$ are both divisors of degree zero which have the same sum for the group law. Hence they are linearly equivalent by [15, IV Thm 4.13B]. This finishes the proof. ☐

Now we prove a converse of Proposition 4.3.

Proposition 4.4. *Assume that σ has infinite order and that $\mathcal{M} \in D^b(\mathrm{coh}(\mathbb{P}^2_q))$ is such that $Li^*\mathcal{M}$ is a vector bundle on E. Then $\mathcal{M}$ is a reflexive object in* $\mathrm{coh}(\mathbb{P}^2_q)$.

Proof. It follows from lemma 2.8.4 that $i^*H^j(\mathcal{M}) = 0$ for $j \neq 0$. Then it follows from lemma 2.8.2 that $\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q)$ and $L_1 i^*\mathcal{M} = 0$, using lemma 2.8.4 again.

Pick an object M in $\mathrm{grmod}(A)$ such that $\pi M = \mathcal{M}$. We may assume that M contains no subobject in $\mathrm{tors}(A)$. By lemma 2.8.3 we have $L_1 i^*\mathcal{M} = \ker(M(-3) \xrightarrow{\times g} M)\tilde{}$. Thus $\ker(M(-3) \xrightarrow{\times g} M) \in \mathrm{tors}(A)$. Since M contains no subobject in $\mathrm{tors}(A)$ it follows that M is g-torsion free. Furthermore by lemma 4.2 $\Gamma_*(i^*\mathcal{M}) = \Gamma_*((M/Mg)\tilde{})$ is pure two dimensional. If T is the maximal submodule of M/Mg which is in $\mathrm{tails}(A)$ then since $(M/Mg)/T \subset \Gamma_*((M/Mg)\tilde{})$ we obtain that $(M/Mg)/T$ is pure two dimensional.

We now claim that M is pure three dimensional. Let N be the maximal submodule of M of dimension ≤ 2. Then $C = M/N$ is pure three dimensional and in particular g-torsion-free. Hence we have a short exact sequence

$$0 \to N/Ng \to M/Mg \to C/Cg \to 0$$

By the purity of $(M/Mg)/T$ it follows that $N/Ng \subset T$ and hence $N/Ng \in \mathrm{tails}(A)$. It follows from lemma 2.8.2 than $N \in \mathrm{tails}(A)$ and hence $N = 0$. This shows that M is pure.

Put $Q = M^{**}/M$. Thus we obtain an exact sequence

$$0 \to \mathrm{Tor}_1^A(Q, B)\tilde{} \to (M \otimes_A B)\tilde{} \to (M^{**} \otimes_A B)\tilde{} \to (Q \otimes_A B)\tilde{} \to 0$$

By [3] we have $\mathrm{GKdim}\, Q \leq 1$. Thus we have $\mathrm{GKdim}\, \mathrm{Tor}_1^A(Q, B) \leq \mathrm{GKdim}\, Q \leq 1$. So by the proof of lemma 4.2, $\dim \mathrm{Tor}_1^A(Q, B)\tilde{} \leq 0$. Since $(M \otimes_A B)\tilde{}$ is a vector bundle by hypotheses it contains no finite dimensional subobjects and we obtain $\mathrm{Tor}_1^A(Q, B)\tilde{} = 0$. Thus $\mathrm{Tor}_1^A(Q, B) \in \mathrm{tails}(A)$. Thus, in high degree, multiplication by g is an isomorphism on Q. But then by lemma 2.8.2 $Q \in \mathrm{tails}(A)$. Hence $\mathcal{M} = \pi M = \pi M^{**}$ and thus $\mathcal{M}$ is reflexive. □

5. Elliptic quantumspaces

5.1. **Generalities.** Let A be a Sklyanin algebra $\mathrm{Skl}_3(a, b, c)$. We use again our standard notations as in the previous section.

We set $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and $D = \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{E}) = \bigoplus_{i,j=0}^2 \mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{O}(i), \mathcal{O}(j))$ the algebra of endomorphisms of $\mathcal{E}$. We consider the left exact functor $\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, -)$ which takes coherent sheaves on $\mathbb{P}^2_q$ to right D-modules.

$\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, -)$ extends to a functor $\mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, -)$ on bounded derived categories

$$\mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, -) : \mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2_q)) \to \mathrm{D}^b(\mathrm{mod}(D)). \tag{5.1}$$

This is done as follows: $\mathrm{Qcoh}(\mathbb{P}^2_q)$ has enough injectives and this yields a functor $\mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, -) : \mathrm{D}^b_{\mathrm{coh}(\mathbb{P}^2_q)}(\mathrm{Qcoh}(\mathbb{P}^2_q)) \to D^b_{\mathrm{mod}(D)}(\mathrm{Mod}(D))$. Now $\mathrm{coh}(\mathbb{P}^2_q)$ and $\mathrm{mod}(D)$ are noetherian abelian categories and this yields equivalences $\mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2_q)) \cong \mathrm{D}^b_{\mathrm{coh}(\mathbb{P}^2_q)}(\mathrm{Qcoh}(\mathbb{P}^2_q))$ and $\mathrm{D}^b(\mathrm{mod}(D)) \cong D^b_{\mathrm{mod}(D)}(\mathrm{Mod}(D))$ (lemma 2.2). The functor (5.1) is obtained by composing with these equivalences.

In a similar way as in [9, Theorem 6.2] one shows that $\mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_q}(\mathcal{E}, -)$ is an equivalence of derived categories. The inverse functor is given by $- \overset{\mathbf{L}}{\otimes}_D \mathcal{E}$. For a non-negative integer i the equivalence restricts to an equivalence between $\mathcal{X}_i$ and $\mathcal{Y}_i$ where $\mathcal{X}_i \subset \mathrm{coh}(\mathbb{P}^2_q)$ is the full subcategory with objects

$$\mathcal{X}_i = \{\mathcal{M} \in \mathrm{coh}(\mathbb{P}^2_q) \mid \mathrm{Ext}^j_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M}) = 0 \text{ for } j \neq i\}$$

and $\mathcal{Y}_i \subset \mathrm{mod}(D)$ the full subcategory with objects

$$\mathcal{Y}_i = \{M \in \mathrm{mod}(D) \mid \mathrm{Tor}^D_j(M, \mathcal{E}) = 0 \text{ for } j \neq i\}.$$

The inverse equivalences between these categories are given by $\mathrm{Ext}^i_{\mathbb{P}^2_q}(\mathcal{E}, -)$ and $\mathrm{Tor}^D_i(-, \mathcal{E})$.

Let (Δ, R) be the quiver

$$-2 \quad \begin{matrix} \xrightarrow{X_{-2}} \\ \xrightarrow{Y_{-2}} \\ \xrightarrow{Z_{-2}} \end{matrix} \quad -1 \quad \begin{matrix} \xrightarrow{X_{-1}} \\ \xrightarrow{Y_{-1}} \\ \xrightarrow{Z_{-1}} \end{matrix} \quad 0 \tag{5.2}$$

with relations

$$\begin{cases} aY_{-2}Z_{-1} + bZ_{-2}Y_{-1} + cX_{-2}X_{-1} = 0 \\ aZ_{-2}X_{-1} + bX_{-2}Z_{-1} + cY_{-2}Y_{-1} = 0 \\ aX_{-2}Y_{-1} + bY_{-2}X_{-1} + cZ_{-2}Z_{-1} = 0 \end{cases} \tag{5.3}$$

We write $\mathrm{Mod}(\Delta)$ for the category of representations of the quiver Δ (representations are always assumed to satisfy the relations (5.3)). If $i = -2, -1, 0$ then we denote by P_i, S_i respectively the projective representation and the simple representation corresponding to i.

It is easy to see that $D \cong k\Delta/(R)$. Since the category $\mathrm{Mod}(\Delta)$ of representations of Δ is equivalent to the category of right $k\Delta/(R)$-modules we deduce $\mathrm{Mod}(\Delta) \cong \mathrm{Mod}(D)$.

Let $\mathcal{M} \in \mathcal{X}_1$ and $M = \mathrm{Ext}^1_{\mathbb{P}^2_q}(\mathcal{E}, \mathcal{M})$. By functoriality, multiplication by $x, y, z \in A$ induces linear maps $M(\lambda_{-1}) : \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{M}) \to \mathrm{Ext}^1(\mathcal{O}, \mathcal{M})$ and $M(\lambda_{-2}) : \mathrm{Ext}^1(\mathcal{O}(2), \mathcal{M}) \to \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{M})$ $(\lambda = X, Y, Z)$. Hence M is determined by the following representation of Δ

$$H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) \quad \begin{matrix} \xrightarrow{M(X_{-2})} \\ \xrightarrow{M(Y_{-2})} \\ \xrightarrow{M(Z_{-2})} \end{matrix} \quad H^1(\mathbb{P}^2_q, \mathcal{M}(-1)) \quad \begin{matrix} \xrightarrow{M(X_{-1})} \\ \xrightarrow{M(Y_{-1})} \\ \xrightarrow{M(Z_{-1})} \end{matrix} \quad H^1(\mathbb{P}^2_q, \mathcal{M})$$

For further use we note that the Euler form $\chi(S_i, S_j)$ is given by the following matrix

$$\begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.4}$$

where i refers to the columns and j refers to the rows.

Let Δ^0 be the full subquiver of Δ consisting of the vertices $-2, -1$ and let $\mathrm{Res} : \mathrm{Mod}(\Delta) \to \mathrm{Mod}(\Delta^0)$ be the obvious restriction functor. Res has a left adjoint which we denote by Ind. If e is the sum of the vertices of Δ^0 then $\mathrm{Ind} = - \otimes_{k\Delta^0} ek\Delta$. Note that $\mathrm{Res} \circ \mathrm{Ind} = \mathrm{id}$.

The following was already observed by Le Bruyn in the case of the homogenized Weyl algebra.

Lemma 5.1.1. *If $\mathcal{M} \neq \mathcal{O}$ is a normalized line bundle on $\mathbb{P}^2_q$ and $M = \operatorname{Ext}^1(\mathcal{E}, \mathcal{M})$ then $M = \operatorname{Ind}\operatorname{Res} M$.*

Proof. This follows from an argument by Baer [5, Corollary 7.2]. For the convenience of the reader we repeat this argument.

We say that two objects A, B in an abelian category are orthogonal ($A \perp B$) if $\operatorname{Hom}(A, B) = \operatorname{Ext}^1(A, B) = 0$.

We have $\mathbf{R}\operatorname{Hom}(\mathcal{E}, \mathcal{M}) = M[-1]$, $\mathbf{R}\operatorname{Hom}(\mathcal{E}, \mathcal{O}) = S_0$. Thus $\operatorname{Ext}^i(\mathcal{M}, \mathcal{O}) = \operatorname{Ext}^i(M[-1], S_0) = \operatorname{Ext}^{i+1}(M, S_0)$. In particular $\operatorname{Hom}(M, S_0) = 0$ and $\operatorname{Ext}^1(M, S_0) = \operatorname{Hom}(\mathcal{M}, \mathcal{O}) = H^2(\mathbb{P}^2_q, \mathcal{M}(-3))^* = 0$ where we have used Serre duality and Theorem 3.5. We conclude by lemma 5.1.2 below. □

Lemma 5.1.2. *Let $M \in \operatorname{mod}\Delta$. Then $M = \operatorname{Ind}\operatorname{Res} M$ if and only if $M \perp S_0$.*

Proof. First assume $M = \operatorname{Ind}\operatorname{Res} M$. Put $M^0 = \operatorname{Res} M$ and take a projective resolution

$$0 \to F_1^0 \to F_0^0 \to M^0 \to 0$$

Applying Ind we get a projective resolution of M of the form

$$0 \to S_0^a \to F_1 \to F_0 \to M \to 0$$

for some $a \in \mathbb{N}$ where $F_i = \operatorname{Ind} F_i^0$. The fact that $\operatorname{Hom}(F_1, S_0) = \operatorname{Hom}(F_0, S_0) = 0$ (by adjointness) implies $\operatorname{Hom}(M, S_0) = 0$ and $\operatorname{Ext}^1(M, S_0) = 0$.

To prove the converse let $N = \operatorname{Ind}\operatorname{Res} M$. By adjointness we have a map $p : N \to M$ whose kernel K and cokernel C are direct sums of S_0. We have $\operatorname{Hom}(M, S_0) = 0$ and hence $\operatorname{Hom}(C, S_0) = 0$. Thus $C = 0$ and p is surjective.

Applying $\operatorname{Hom}(-, S_0)$ to the short exact sequence

$$0 \to K \to N \to M \to 0$$

and using $\operatorname{Hom}(N, S_0) = 0$ (by adjointness) yields $\operatorname{Hom}(K, S_0) = 0$ and hence $K = 0$. Thus p is an isomorphism and we are done. □

Lemma 5.1.3. *Let $p = (\alpha, \beta, \gamma) \in E$ and put $(\alpha_i, \beta_i, \gamma_i) = p^{\sigma^i}$. p corresponds to a point module P of A. Put $\mathcal{P} = \pi P$ and $\bar{P} = \omega \mathcal{P}$.*

1. *$H^i(\mathbb{P}^2_q, \mathcal{P}(n)) = 0$ for all n and $i > 0$. In particular $\mathcal{P} \in \mathcal{X}_0$.*
2. *$\dim \bar{P}_n = 1$ for all n and $\bar{P}_{\geq n}$ is a shifted point module for all n. In particular $\bar{P}_{\geq 0} = P$.*
3. *$H^0(\mathbb{P}^2_q, \mathcal{P}(n)) = \bar{P}_n$.*
4. *The representation of Δ corresponding to $\mathcal{P}$ is*

$$k \quad \begin{matrix} \xrightarrow{\alpha_{-2}} \\ \xrightarrow{\beta_{-2}} \\ \xrightarrow{\gamma_{-2}} \end{matrix} \quad k \quad \begin{matrix} \xrightarrow{\alpha_{-1}} \\ \xrightarrow{\beta_{-1}} \\ \xrightarrow{\gamma_{-1}} \end{matrix} \quad k$$

5. *Denote the representation in the previous diagram also by p. We have $p = \operatorname{Ind}\operatorname{Res} p$.*

Proof. 1. Since the $\mathcal{P}(n)$ are all obtained from point modules, it suffices to treat the case $n = 0$. We use lemma 2.8.3 and the discussion before that. We have $\mathcal{P} = i_* \mathcal{O}_p$ and hence $\operatorname{Ext}^j(\mathcal{O}, \mathcal{P}) = \operatorname{Ext}^j_E(Li^* \mathcal{O}, \mathcal{O}_p) = \operatorname{Ext}^j_E(\mathcal{O}_E, \mathcal{O}_p) = 0$ for $j > 0$.

2. This is easy to check.

3. Use $\omega = \underline{\mathrm{Hom}}_{\mathrm{Tails}}(\mathcal{O}, -)$.
4. This follows from the previous step.
5. According to lemma 5.1.2 we need $\mathrm{Ext}^i(p, S_0) = 0$ for $i \leq 1$. This follows from the fact that we have $\mathrm{Ext}^i(p, S_0) = \mathrm{Ext}^i(\mathcal{P}, \mathcal{O}) = 0$ for $i \leq 1$ by lemma 3.4. □

To simplify the discussion below we define $\mathcal{R}_n$ (for $n \geq 1$) as the category in which the objects are the normalized line bundles on $\mathbb{P}^2_{\mathrm{q}}$ with invariant n and the morphisms are the isomorphisms in $\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}})$. Thus $\mathcal{R}_n$ is a groupoid. Note that we do not know yet if $\mathcal{R}_n \neq \emptyset$. This question will be addressed below.

5.2. **$\mathcal{R}_n$ is non-empty.** To simplify things we assume that σ has infinite order. The following result is necessary for the dimension computations in §5.5.

Lemma 5.2.1. *The set $\mathcal{R}_n$ is not empty.*

Proof. Let $\mathcal{S}$ be a line object on $\mathbb{P}^2_{\mathrm{q}}$. Writing $\mathcal{S}$ as the cokernel of a map $\mathcal{O}(-1) \to \mathcal{O}$ we find by Theorem 2.6.2 that if $n \geq -1$ then $H^0(\mathbb{P}^2_{\mathrm{q}}, \mathcal{S}(n))$ has dimension $n+1$.

By [3] there exist at most three line objects $\mathcal{S}'$ such that $\mathcal{S}'(-1)$ is a subobject of $\mathcal{S}$ and furthermore these three line objects contain in turn any other object contained in $\mathcal{S}$.

Hence if $n \geq 0$ then we may pick an epimorphism $f : \mathcal{O} \to \mathcal{S}(n)$ (a generic f will do). Put $\mathcal{I} = (\ker f)(1)$. Using Proposition 3.1 we find $[\mathcal{I}(-1)] = [\mathcal{O}] - ([\mathcal{S}] + n[\mathcal{P}])$ and hence $[\mathcal{I}] = [\mathcal{O}] - n[\mathcal{P}]$. It is easy to see that $\mathcal{I}$ is reflexive. Thus $\mathcal{I} \in \mathcal{R}_n$. □

Below we will show that $\mathcal{R}_n$ is parametrized by an algebraic variety of dimension $2n$. The amount of freedom in the construction exhibited in the proof of lemma 5.2.1 is less than or equal to $2(\text{choice of } \mathcal{S}) + n(\text{choice of } f)$ parameters, hence for $n > 2$ this construction can not possibly yield all elements of $\mathcal{R}_n$. In §5.6 we will exhibit a related construction which works for all n.

5.3. **First description of $\mathcal{R}_n$.** Let $\mathcal{C}_n$ be the image of $\mathcal{R}_n$ under the equivalence $\mathcal{X}_1 \cong \mathcal{Y}_1$.

Theorem 5.3.1. *Let $n \geq 1$. There is an equivalence of categories*

$$\mathcal{R}_n \underset{\mathrm{Tor}_1^D(-,\mathcal{E})}{\overset{\mathrm{Ext}^1_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}_n \tag{5.5}$$

where

$$\mathcal{C}_n = \{M \in \mathrm{mod}(\Delta) \mid \underline{\dim} M = (n, n, n-1) \textit{ and } \\ \mathrm{Hom}_\Delta(M, p) = 0, \mathrm{Hom}_\Delta(p, M) = 0 \textit{ for all } p \in E\}.$$

Proof. First, let $\mathcal{M}$ be an object of $\mathcal{R}_n$. By Proposition 4.3 we have $i^*\mathcal{M} \cong \mathcal{N}$ for a line bundle $\mathcal{N}$ of degree zero on E. Hence (for all $p \in E$) we have $\mathbf{R}\mathrm{Hom}_E(Li^*\mathcal{M}, \mathcal{O}_p) = k$. Since

$$\mathbf{R}\mathrm{Hom}_E(Li^*\mathcal{M}, \mathcal{O}_p) \cong \mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{M}, i_*\mathcal{O}_p)$$

we obtain $\mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{M}, \mathcal{P}) = k$ where $\mathcal{P} = i_*\mathcal{O}_p$ is the corresponding point object on $\mathbb{P}^2_{\mathrm{q}}$. Writing $M = \mathrm{Ext}^1_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{E}, \mathcal{M})$ this means that $\mathbf{R}\mathrm{Hom}_D(M[-1], p) = k$, proving that $\mathrm{Hom}_D(M, p) = 0$ and $\mathrm{Ext}^2_D(M, p) = 0$. By BKS-duality (Theorem 2.9.1) we obtain $\mathrm{Hom}_D(p', M) = 0$ for some other point p' determined by p (and determining p). Hence $M \in \mathcal{C}_n$.

Conversely, let M be an object of $\mathcal{C}_n$. Thus (using Serre duality on $\mathbb{P}^2_{\mathrm{q}}$ again) $\mathrm{Hom}_D(M, p) = \mathrm{Ext}^2_D(M, p) = 0$ for all $p \in E$. Now $\mathrm{gl}\dim D = 2$ so we may compute $\dim \mathrm{Ext}^1_D(M, p)$ using the Euler form (5.4) on $\mathrm{mod}(D)$. We obtain $\mathrm{Ext}^1_D(M, p) = k$. In other words $\mathbf{R}\mathrm{Hom}_D(M[-1], p) = k$.

Put $\mathcal{M} = M[-1] \overset{\mathbf{L}}{\otimes}_D \mathcal{E}$. By the category equivalence between $D^b(\mathrm{coh}(\mathbb{P}^2_{\mathrm{q}}))$ and $D^b(\mathrm{mod}(D))$ we obtain $\mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_{\mathrm{q}}}(\mathcal{M}, \mathcal{P}) = k$, giving (by adjointness) $\mathbf{R}\mathrm{Hom}_E(Li^*\mathcal{M}, \mathcal{O}_p) = k$. Since E is a smooth elliptic curve it is easy to see that this implies that $Li^*\mathcal{M}$ is a line bundle. Hence by Propositions 4.3 and 4.4 the same is true for $\mathcal{M}$. □

5.4. **Application.** Using the material in the previous sections it is now easy to parametrize the line bundles on $\mathbb{P}^2_{\mathrm{q}}$ with invariant one.

Theorem 5.4.1. *The representations in $\mathcal{C}_1$ are the representations*

$$k \;\overset{\xrightarrow{\alpha}}{\underset{\xrightarrow{\gamma}}{\xrightarrow{\beta}}}\; k \;\overset{\xrightarrow{0}}{\underset{\xrightarrow{0}}{\xrightarrow{0}}}\; 0 \tag{5.6}$$

for some $(\alpha, \beta, \gamma) \in \mathbb{P}^2 - E$

Proof. First let $F \in \mathcal{C}_1$. F is given by a representation as in (5.6). Then the condition $\mathrm{Hom}_\Delta(p, F) = 0$ for $p \in E$ implies $(\alpha, \beta, \gamma) \notin E$.

Conversely let F be as in (5.6) with $(\alpha, \beta, \gamma) \notin E$. Then we immediately have $\mathrm{Hom}_\Delta(p, F) = \mathrm{Hom}_\Delta(F, p) = 0$ for $p \in E$. □

5.5. **Second description of $\mathcal{R}_n$.** Although the category $\mathcal{C}_n$ has a fairly elementary description, it is not so easy to handle. In particular the analogy with the Weyl algebra case is not obvious. We will now give another description of $\mathcal{R}$ which is more similar to the one used for the Weyl algebra. In particular it will follow that the isomorphism classes of objects in $\mathcal{R}_n$ are parametrized by smooth affine varieties of dimension $2n$.

In general, let Q be a quiver without oriented cycles and write Q_0, Q_1 for respectively the set of vertices and edges of Q. Let "$\cdot$" be the standard scalar product on $\mathbb{Z}^{Q_0}$: $(\alpha_v)_v \cdot (\beta_v)_v = \sum_v \alpha_v \beta_v$.

Let $\theta \in \mathbb{Z}^{Q_0}$. A representation F of Q is called *θ-semistable* (resp. *stable*) if $\theta \cdot \underline{\dim} F = 0$ and $\theta \cdot \underline{\dim} N \geq 0$ (resp. > 0) for every non-trivial subrepresentation N of F.

We have $K_0(\mathrm{mod}\, Q) = \mathbb{Z}^{Q_0}$, canonically. It is a fundamental fact [25] that F is semistable for some θ if and only there exists $G \in \mathrm{mod}(Q)$ such that $F \perp G$. The relation between θ and $\underline{\dim} G$ is such that the forms $- \cdot \theta$ and $\chi(-, \underline{\dim} G)$ are proportional.

Fix a dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ and let $\mathrm{Rep}(Q, \alpha)$ be the corresponding representation space, i.e. $\mathrm{Rep}(Q, \alpha) = \prod_{i \in Q_1} M_{\alpha_{h(i)} \times \alpha_{t(i)}}(k)$ where the maps $h, t : Q_1 \to Q_0$ associate to an arrow its begin and end vertex. The isomorphism class of representations of dimension vector α are in one-one correspondence with the orbits of the group $\mathrm{Gl}(\alpha) = \prod_{v \in Q_0} \mathrm{Gl}_{\alpha_v}(k)$ acting on $\mathrm{Rep}(Q, \alpha)$ by conjugation.

Associated to $G \in \mathrm{mod}(Q)$ there is a semi-invariant function ϕ_G on $\mathrm{Rep}(Q, \alpha)$ such that the set

$$^{\perp}G = \{F \in \mathrm{Rep}(Q, \alpha) \mid F \perp G\} \tag{5.7}$$

coincides with $\{\phi_G \neq 0\}$. In particular (5.7) is affine.

Lemma 5.5.1. *There exists $V \in \mathrm{mod}(\Delta^0)$ with $\underline{\dim} V = (6, 3)$ such that*

1. *for all $M \in \mathcal{C}_n$ we have $M^0 \perp V$ where $M^0 = \mathrm{Res}\, M$ and*
2. *if $p \in E$ then $\mathbf{R}\mathrm{Hom}_{\Delta^0}(\mathrm{Res}\, p, V) \neq 0$.*

Proof. 1. Pick a degree zero line bundle $\mathcal{U}$ on E which is not of the form $\mathcal{O}((o) - (3n\xi))$ for $n \in \mathbb{N}$ (where o, ξ are as in Proposition 4.3).

Let $\mathcal{M} \in \mathcal{R}_n$. Then we have by adjointness $\mathbf{R}\mathrm{Hom}(\mathcal{M}, i_*\mathcal{U}) = \mathbf{R}\mathrm{Hom}_E(Li^*\mathcal{M}, \mathcal{U})$. By Proposition 4.3 we have $Li^*\mathcal{M} = \mathcal{O}((o) - (3n\xi))$. We conclude by Serre duality for E that $\mathbf{R}\mathrm{Hom}(\mathcal{M}, i_*\mathcal{U}) = 0$. Now put $M = \mathrm{Ext}^1(\mathcal{E}, \mathcal{M})$ and $U' = \mathbf{R}\mathrm{Hom}(\mathcal{E}, i_*\mathcal{U})$. We obtain $\mathbf{R}\mathrm{Hom}_D(M[-1], U') = 0$.

What is U'? By adjointness we have $\mathbf{R}\mathrm{Hom}(\mathcal{E}, i_*\mathcal{U}) = \mathbf{R}\mathrm{Hom}_E(Li^*\mathcal{E}, \mathcal{U})$. An easy verification shows that $Li^*\mathcal{E} = \sigma_*^2(\mathcal{L}) \otimes \sigma_*(\mathcal{L}) \oplus \sigma_*\mathcal{L} \oplus \mathcal{O}_E$. Thus by Riemann-Roch and Serre duality $U' = U[-1]$ where $\underline{\dim} U = (6, 3, 0)$. Put $V = \mathrm{Res}\, U$. Thus $\underline{\dim} V = (6, 3)$.

Replacing M with a projective resolution it is easy to see that $\mathbf{R}\mathrm{Hom}_\Delta(M, U) = \mathbf{R}\mathrm{Hom}_{\Delta^0}(M^0, V)$. It follows that $\mathrm{Hom}_{\Delta^0}(M^0, V) = 0$ and $\mathrm{Ext}^1_{\Delta^0}(M^0, V) = 0$.

2. Put $Q = \mathrm{Res}\, p$ for $p \in E$. Then $\mathbf{R}\mathrm{Hom}_{\Delta^0}(Q, V) = \mathbf{R}\mathrm{Hom}_\Delta(p, U) = \mathbf{R}\mathrm{Hom}_\Delta(p[-1], U') = \mathbf{R}\mathrm{Hom}_{\mathbb{P}^2_q}(i_*\mathcal{O}_p[-1], i_*\mathcal{U}) = \mathbf{R}\mathrm{Hom}_E(Li^*i_*\mathcal{O}_p[-1], \mathcal{U})$. Now $Li^*i_*\mathcal{O}_p[-1]$ is a nonzero complex whose homology has finite length. It is easy to deduce from this $\mathbf{R}\mathrm{Hom}_E(Li^*i_*\mathcal{O}_p[-1], \mathcal{U}) \neq 0$. Hence we are done □

We obtain the following consequence.

Lemma 5.5.2. *If $M \in \mathcal{C}_n$ and $M^0 = \mathrm{Res}\, M$ then M^0 is θ-semistable for $\theta = (-1, 1)$.*

Proof. This is a straightforward verification. □

Lemma 5.5.3. *Assume that σ has infinite order. Let N be a representation of Δ^0 of dimension vector (n, n), $n \geq 1$. If $\mathrm{Hom}_{\Delta^0}(N, \mathrm{Res}\, p) = \mathrm{Hom}_{\Delta^0}(\mathrm{Res}\, p, N) = 0$ for all $p \in E$ then $\dim(\mathrm{Ind}\, N)_0 \leq n - 1$.*

Proof. Assume the lemma is false. Thus $\dim(\mathrm{Ind}\, N)_0 \geq n$. Then we may construct a surjective map $\mathrm{Ind}\, N \to W$ where $\underline{\dim} W = (n, n, n)$. We will consider $W \overset{\mathbf{L}}{\otimes}_D \mathcal{E}$ and $Li^*(W \overset{\mathbf{L}}{\otimes}_D \mathcal{E})$. Note that since E is smooth, $Li^*(W \overset{\mathbf{L}}{\otimes}_D \mathcal{E})$ is the sum of its homology.

We have for $p \in E$:

$$\begin{aligned} \mathrm{Ext}^j_E(Li^*(W \overset{\mathbf{L}}{\otimes}_D \mathcal{E}), \mathcal{O}_p) &= \mathrm{Ext}^j_{\mathbb{P}^2_q}(W \overset{\mathbf{L}}{\otimes}_D \mathcal{E}, i_*\mathcal{O}_p) \\ &= \mathrm{Ext}^j_\Delta(W, p) \end{aligned} \tag{5.8}$$

Now a simple computation shows that $\chi(W, p) = 0$. Furthermore we have $\mathrm{Hom}_\Delta(W, p) \subset \mathrm{Hom}_\Delta(\mathrm{Ind}\, N, p) = \mathrm{Hom}_{\Delta^0}(N, \mathrm{Res}\, p) = 0$. Finally by Serre duality on $\mathbb{P}^2_q$ (see Theorem 2.9.1) we have $\mathrm{Ext}^2_\Delta(W, p) = \mathrm{Hom}_\Delta(p', W)^* = 0$. We conclude that also $\mathrm{Ext}^1_\Delta(W,p)=0$. It follows from (5.8) that $Li^*(W \overset{\mathbf{L}}{\otimes}_D \mathcal{E})=0$. Hence by lemmas 2.8.4 and 2.8.2 we deduce $W \overset{\mathbf{L}}{\otimes}_D \mathcal{E}=0$ and hence $W=0$ which is a contradiction. □

We can now prove our main result.

Theorem 5.5.4. *Assume that σ has infinite order. Let $V \in \mathrm{mod}(\Delta^0)$ be as in lemma 5.5.1.*

1. *The functors* Res *and* Ind *define inverse equivalences between $\mathcal{C}_n$ and the following category*
$$\mathcal{D}_n=\{F\in \mathrm{mod}(\Delta^0)\mid \underline{\dim} F=(n,n), F\perp V, \dim(\mathrm{Ind}\,F)_0\ge n-1\}$$
2. *The representations in $\mathcal{D}_n$ are θ-stable for $\theta=(-1,1)$.*

Proof. Below we use often implicitly the already proved equivalence $\mathcal{C}_n\cong\mathcal{R}_n$ (Theorem 5.3.1).

Step 1. $\mathrm{Res}(\mathcal{C}_n)\subset\mathcal{D}_n$. This follows from lemmas 5.1.1 and 5.5.1.

Step 2. Let $F\in\mathcal{D}_n$. If $F'\subset F$ is such that $\underline{\dim}F'=(m,m)$ then $\mathrm{Hom}_{\Delta^0}(F',V)=\mathrm{Ext}^1_{\Delta^0}(F',V)=\mathrm{Hom}_{\Delta^0}(F/F',V)=\mathrm{Ext}^1_{\Delta^0}(F/F',V)=0$. This follows easily by using the Euler form.

Step 3. Let $F\in\mathcal{D}_n$ and $p\in E$. Then $\mathrm{Hom}_{\Delta^0}(F,\mathrm{Res}\,p)=\mathrm{Hom}_{\Delta^0}(\mathrm{Res}\,p,F)=0$. Both claims are similar so we only consider the first one. We have $F\perp V$ hence F is θ-semistable and $\mathrm{Res}\,p$ is obviously stable. So if $\mathrm{Hom}_{\Delta^0}(F,\mathrm{Res}\,p)\neq 0$ then there is an epimorphism $F\to\mathrm{Res}\,p$. By Step 2 we obtain $\mathbf{R}\mathrm{Hom}_{\Delta^0}(\mathrm{Res}\,p,V)=0$. But this contradicts the choice of V, finishing the argument.

Step 4. $\mathrm{Ind}(\mathcal{D}_n)\subset\mathcal{C}_n$. Let $F\in\mathcal{D}_n$. By Step 3 and lemma 5.5.3 we obtain $\dim(\mathrm{Ind}\,F)_0=n-1$.
It remains to show that for $p\in E$ we have $\mathrm{Hom}_\Delta(\mathrm{Ind}\,F,p)=\mathrm{Hom}_\Delta(p,\mathrm{Ind}\,F)=0$. By lemma 5.1.3 we have $p=\mathrm{Ind}\,\mathrm{Res}\,p$. Thus $\mathrm{Hom}_\Delta(\mathrm{Ind}\,F,p)=\mathrm{Hom}_{\Delta^0}(F,\mathrm{Res}\,p)=0$ and similarly $\mathrm{Hom}_\Delta(p,\mathrm{Ind}\,F)=\mathrm{Hom}_{\Delta^0}(\mathrm{Res}\,p,\mathrm{Res}\,\mathrm{Ind}\,F)=\mathrm{Hom}_{\Delta^0}(\mathrm{Res}\,p,F)=0$ where we have used Step 3 again.

Step 5. Ind and Res are inverses to each other. To prove this we only need to show $\mathrm{Ind}\circ\mathrm{Res}(F)=F$ for $F\in\mathcal{C}_n$. This follows from lemma 5.1.1.

Step 6. Let $F\in\mathcal{D}_n$. Then F is θ-stable. Put a filtration $0=F_0\subsetneq F_1\subsetneq\cdots\subsetneq F_{m-1}\subsetneq F_m=F$ on F such that F_i/F_{i-1} is θ-stable. With the same proof as Step 3 it follows that $\mathrm{Hom}_{\Delta^0}(F_i/F_{i-1},\mathrm{Res}\,p)=\mathrm{Hom}_{\Delta^0}(\mathrm{Res}\,p,F_i/F_{i-1})=0$ for $p\in E$. Assume $\underline{\dim}F_i/F_{i-1}=(d_i,d_i)$. Then by lemma 5.5.3 we have $\dim(\mathrm{Ind}(F_i/F_{i-1}))_0\le d_i-1$. From the right exactness of Ind we deduce $\dim(\mathrm{Ind}\,F)_0\le n-m$. Hence $m=1$ and thus F is stable. □

Below we will define some varieties. We take the classical viewpoint. So they are always reduced.

Let V as in lemma 5.5.1. Let $\alpha=(n,n)$ and put

$$\begin{aligned}\tilde{D}_n&=\{F\in\mathrm{Rep}(\Delta^0,\alpha)\mid F\in\mathcal{D}_n\}\\&=\{F\in\mathrm{Rep}(\Delta^0,\alpha)\mid \phi_V(F)\neq 0,\dim(\mathrm{Ind}\,F)_0\ge n-1\}.\end{aligned}\tag{5.9}$$

It is clear that $\tilde{D}_n$ is a closed subset of $\{\phi_V\neq 0\}$ so in particular $\tilde{D}_n$ is affine. Put $D_n=\tilde{D}_n/\!\!/\,\mathrm{Gl}(\alpha)$.

Theorem 5.5.5. *The affine variety D_n is smooth of dimension $2n$. The isomorphism classes in $\mathcal{D}_n$ (and hence in $\mathcal{C}_n$ and $\mathcal{R}_n$) are in natural bijection with the points in D_n.*

Proof. Since all represention in $\tilde{D}_n$ are stable by Theorem 5.5.4, all $\mathrm{Gl}(\alpha)$-orbits on $\tilde{D}_n$ are closed and so D_n is really the orbit space for the $\mathrm{Gl}(\alpha)$ action on $\tilde{D}_n$. This proves that the isomorphism classes in $\mathcal{D}_n$ are in natural bijection with the points in D_n.

To prove that D_n is smooth it suffices to prove that $\tilde{D}_n$ is smooth (this follows for example using the Luna slice theorem [21]).

We first estimate the dimension of $\tilde{D}_n$. We write the equations of A in the usual form $M(xyz)^t$. Given $n \times n$-matrices X, Y, Z let $M(X,Y,Z)$ be obtained from M by replacing (x,y,z) by X,Y,Z (thus $M(X,Y,Z)$ is a $3n \times 3n$-matrix). Then $\tilde{D}_n$ has the following alternative description:

$$\tilde{D}_n = \{(X,Y,Z) \in M_n(k)^3 \mid \phi_V(X,Y,Z) \neq 0 \text{ and } \operatorname{rk} M(X,Y,Z) \leq 3n-(n-1)\}.$$

By §5.2 $\tilde{D}_n$ is non-empty. The triples satifying $\phi_V(X,Y,Z) \neq 0$ are a dense open subset of $M_n(k)^3$ and hence they represent a variety of dimension $3n^2$. Imposing that $M(X,Y,Z)$ should have corank $\geq n-1$ represents $(n-1)^2$ independent conditions. So the irreducible components of $\tilde{D}_n$ have dimension $\geq 3n^2-(n-1)^2$.

Define $\tilde{C}_n$ by

$$\{G \in \operatorname{Rep}(\Delta,\tilde{\alpha}) \mid G \cong \operatorname{Ind}\operatorname{Res} G, \operatorname{Res} G \in \tilde{D}_n\}$$

where $\tilde{\alpha} = (n,n,n-1)$ (as usual we assume the points of $\operatorname{Rep}(\Delta,\tilde{\alpha})$ to satisfy the relation imposed on Δ).

To extend $F \in \tilde{D}_n$ to a point in $\tilde{C}_n$ we need to choose a basis in $(\operatorname{Ind} F)_0$. Thus $\tilde{C}_n$ is a principal $\mathrm{Gl}_{n-1}(k)$ fiber bundle over $\tilde{D}_n$. In particular $\tilde{C}_n$ is smooth if and only $\tilde{D}_n$ is smooth and the irreducible components of $\tilde{C}_n$ have dimension $\geq 3n^2-(n-1)^2+(n-1)^2 = 3n^2$. Note that by the description of $\mathcal{C}_n$ in Theorem 5.3.1 it follows that $\tilde{C}_n$ is an open subset of $\operatorname{Rep}(\Delta,\tilde{\alpha})$.

Let $x \in \tilde{C}_n$. The stabilizer of x consists of scalars thus if we put $G = \mathrm{Gl}(\tilde{\alpha})/k^*$ then we have inclusions $\operatorname{Lie}(G) \subset T_x(\tilde{C}_n) = T_x(\operatorname{Rep}(\Delta,\tilde{\alpha}))$. Voigt in [35, Ch. 2, §3.4] has shown that there is a natural inclusion $T_x(\operatorname{Rep}(\Delta,\tilde{\alpha}))/\operatorname{Lie}(G) \hookrightarrow \operatorname{Ext}^1_\Delta(x,x)$ (Voigt actually obtains an isomorphism since he is not assuming his representation spaces to be reduced). Now x corresponds to some line bundle $\mathcal{H}$ on $\mathbb{P}^2_q$ and we have $\operatorname{Ext}^1_\Delta(x,x) = \operatorname{Ext}^1(\mathcal{H},\mathcal{H})$. An easy computation shows $\chi(\mathcal{H},\mathcal{H}) = \chi(x,x) = 1-2n$. We have $\operatorname{Hom}(\mathcal{H},\mathcal{H}) = k$ and by Serre duality $\operatorname{Ext}^2(\mathcal{H},\mathcal{H}) = \operatorname{Hom}(\mathcal{H},\mathcal{H}(-3)) = 0$. Thus $\dim \operatorname{Ext}^1(\mathcal{H},\mathcal{H}) = 2n$.

Hence we obtain $3n^2 \leq \dim T_x(\tilde{C}_n) \leq 2n+\dim G = 2n+2n^2+(n-1)^2-1 = 3n^2$. Thus $\dim T_x(\tilde{C}_n) = 3n^2$ is constant and hence $\tilde{C}_n$ is smooth. We also obtain $\dim \tilde{D}_n = 3n^2-(n-1)^2$.

The dimension of D_n is equal to $\dim \tilde{D}_n - \dim \mathrm{Gl}(\alpha)+1 = 3n^2-(n-1)^2-2n^2+1 = 2n$. This finishes the proof. □

5.6. **Explicit construction of the elements in $\mathcal{R}_n$.** For simplicity we assume throughout that σ has infinite order.

In the discussion below we have to compute the cohomology of a line object.

Lemma 5.6.1. *Let $\mathcal{S} = \pi(A/uA)$ be a line object on $\mathbb{P}^2_q$. Let $m \leq -1$. Then $H^1(\mathbb{P}^2_q,\mathcal{S}(m)) \cong (A/Au)^*_{-m-2}$ and $H^i(\mathbb{P}^2_q,\mathcal{S}(m)) = 0$ for $i \neq 1$. Furthermore if*

$\eta \in A_1$ then the induced linear map $H^1(\mathbb{P}^2_q, \mathcal{S}(m)) \xrightarrow{\cdot\eta} H^1(\mathbb{P}^2_q, \mathcal{S}(m+1))$ corresponds to $(\eta\cdot)^*$ on $(A/Au)^*$.

Proof. That $H^0(\mathbb{P}^2_q, \mathcal{S}(m)) = 0$ follows by writing $\mathcal{S}$ as the cokernel of a map $\mathcal{O}(-1) \to \mathcal{O}$ and invoking Theorem 2.6.2. That $H^2(\mathbb{P}^2_q, \mathcal{S}(m)) = 0$ follows by Serre duality (Theorem 2.9.1).

Using Theorem 2.6.2 we find $H^1(\mathbb{P}^2_q, \mathcal{S}(m)) = \ker(\operatorname{Ext}^2(\mathcal{O}(-m), \mathcal{O}(-1)) \xrightarrow{(-,u\cdot)} \operatorname{Ext}^2(\mathcal{O}(-m), \mathcal{O}))$. Using Serre duality (Theorem 2.9.1) this translates into $H^1(\mathbb{P}^2_q, \mathcal{S}(m)) = \operatorname{coker}(\operatorname{Hom}(\mathcal{O}(-1), \mathcal{O}(-m-3))^* \xrightarrow{(u\cdot,-)^*} \operatorname{Hom}(\mathcal{O}, \mathcal{O}(-m-3))^*)$. Dualizing yields that indeed $H^1(\mathbb{P}^2_q, \mathcal{S}(m)) \cong (A/Au)^*_{-m-2}$. That η acts in the indicated way follows by inspecting the appropriate commutative diagram. □

Corollary 5.6.2. *Let $\mathcal{S} = \pi(A/uA)$ be a line object on $\mathbb{P}^2_q$. Then $\mathcal{S}(-1)$ corresponds to $S[-1]$ where S is the representation of Δ given by*

$$(A/Au)^*_1 \;\; \begin{array}{c} \xrightarrow{(x\cdot)^*} \\ \xrightarrow{(y\cdot)^*} \\ \xrightarrow{(z\cdot)^*} \end{array} \;\; k \;\; \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \;\; 0 \tag{5.10}$$

Since line objects on $\mathbb{P}^2_q$ are of the form $\pi(A/uA)$ they are naturally parametrized by points in $\mathbb{P}(A_1)$.

Proposition 5.6.3. *Let $\mathcal{I}$ be a normalized line bundle on $\mathbb{P}^2_q$ with invariant $n > 0$. Then the set of line objects $\mathcal{S}$ such that $\operatorname{Hom}(\mathcal{I}, \mathcal{S}(-1)) \neq 0$ is a curve of degree n in $\mathbb{P}(A_1)$. In particular this set is non-empty.*

Proof. Let $\mathcal{S} = \pi(A/uA)$ with $u = \alpha x + \beta y + \gamma z$. Put $I = \operatorname{Ext}^1(\mathcal{E}, \mathcal{I})$, $S = \operatorname{Ext}^1(\mathcal{E}, \mathcal{S}(-1))$. Then $\operatorname{Hom}(\mathcal{I}, \mathcal{S}(-1)) = \operatorname{Hom}_\Delta(I, S) = \operatorname{Hom}_{\Delta^0}(\operatorname{Ind} I^0, S) = \operatorname{Hom}_{\Delta^0}(I^0, S^0)$ where I^0, S^0 are the restrictions of I and S to Δ^0.

Assume that I^0 is given by matrices $X, Y, Z \in M_n(k)$. Then an easy verification shows that $\operatorname{Hom}_{\Delta^0}(I^0, S^0) \neq 0$ if and only if $\det(\alpha X + \beta Y + \gamma Z) = 0$. This is a homogeneous equation in (α, β, γ) and we have to show that it is not identically zero, i.e. we have to show that there is at least one $\mathcal{S}$ such that $\operatorname{Hom}(\mathcal{I}, \mathcal{S}(-1)) = 0$. This follows from lemma 5.6.4 below. □

Lemma 5.6.4. *Let $\mathcal{I}$ be a normalized line bundle on $\mathbb{P}^2_q$ with invariant n and let $\mathcal{P}$ be a point object on $\mathbb{P}^2_q$. Then, modulo zero dimensional objects, there exist at most n different line objects $\mathcal{S}$ such that $\operatorname{Hom}(\mathcal{I}, \mathcal{S}(-1)) \neq 0$ and such that $\operatorname{Hom}(\mathcal{S}, \mathcal{P}) \neq 0$.*

Proof. We use induction on n. Writing $\mathcal{S}$ as the cokernel of a map $\mathcal{O}(-1) \to \mathcal{O}$ we deduce by Theorem 2.6.2 that $\operatorname{Hom}(\mathcal{O}, \mathcal{S}(-1)) = 0$. So the case $n = 0$ is clear by Corollary 3.6. Assume $n > 0$. Let $(\mathcal{S}_i)_{i=1,\ldots,m}$ be the different line objects (modulo zero dimensional objects) satisfying $\operatorname{Hom}(\mathcal{I}, \mathcal{S}_i(-1)) \neq 0$ and $\operatorname{Hom}(\mathcal{S}_i, \mathcal{P}) \neq 0$. If $m = 0$ then we are done. So assume $m > 0$. Let $\mathcal{S}'_i(-1)$ be the kernel of a non-trivial map $\mathcal{S}_i \to \mathcal{P}$. It is proved in [3] that there is some different point object $\mathcal{P}'$ such that for all i: $\operatorname{Hom}(\mathcal{S}_i, \mathcal{P}') \neq 0$. Let $\mathcal{I}'(-1)$ be the kernel of a non-trivial map $\mathcal{I} \to \mathcal{S}_1(-1)$. The subobjects of line objects are shifted line objects and hence the image of $\mathcal{I}$ in $\mathcal{S}_1(-1)$ is a shifted line object. We find by (3.1): $[\mathcal{I}'] = [\mathcal{O}] - (n-b)[\mathcal{P}]$ with $b \geq 1$. From this we deduce that the invariant of $\mathcal{I}'$ is $\leq n-1$.

Since $\mathcal{P} = i_*\mathcal{O}_p$ for some point $p \in E$ it follows by adjointness and by Proposition 4.3 that $\dim \operatorname{Hom}(\mathcal{I}, \mathcal{P}(-1)) = 1$. Hence the composition $\mathcal{I}'(-1) \to \mathcal{I} \to \mathcal{S}_i(-1)$

maps $\mathcal{I}'(-1)$ to $\mathcal{S}'_i(-2)$. We claim that for $i > 1$ this map must be nonzero. If not then there is a non-trivial map $\mathcal{I}/\mathcal{I}'(-1) \to \mathcal{S}_i(-1)$ and since $\mathcal{I}/\mathcal{I}'(-1)$ is also subobject of $\mathcal{S}_1(-1)$ it follows that $\mathcal{S}_1$ and $\mathcal{S}_i$ have a common subobject. But this is impossible since $\mathcal{S}_1$ and $\mathcal{S}_i$ are different modulo zero dimensional objects.

Hence $\operatorname{Hom}(\mathcal{I}', \mathcal{S}'_i(-1)) \neq 0$ for $i = 2, \dots, m$. Since the $\mathcal{S}'_i$ are still different modulo zero dimensional objects, we obtain $m - 1 \leq n - 1$ and hence $m \leq n$. □

The following lemma shows how to reduce the invariant of a line bundle.

Lemma 5.6.5. *Let $\mathcal{I}$ be a normalized line bundle on $\mathbb{P}^2_q$ with invariant $n > 0$. Then there exists a line object $\mathcal{S}$ on $\mathbb{P}^2_q$ such that $\operatorname{Ext}^1(\mathcal{S}(1), \mathcal{I}(-1)) \neq 0$. If $\mathcal{J} = \pi J$ is the middle term of a corresponding non-trivial extension and $\mathcal{J}^{**} = \pi J^{**}$ then $\mathcal{J}^{**}$ is a normalized line bundle with invariant $\leq n - 1$. Furthermore $\mathcal{J}^{**}/\mathcal{I}(-1)$ is a shifted line object.*

Proof. Using Serre duality we have $\operatorname{Ext}^1(\mathcal{S}(1), \mathcal{I}(-1)) = \operatorname{Ext}^1(\mathcal{I}(-1), \mathcal{S}(-2))^* = \operatorname{Ext}^1(\mathcal{I}, \mathcal{S}(-1))^*$. Also using Serre duality we deduce $\operatorname{Ext}^2(\mathcal{I}, \mathcal{S}(-1)) = 0$. Then a simple computation using the Euler form shows that $\dim \operatorname{Hom}(\mathcal{I}, \mathcal{S}(-1)) = \dim \operatorname{Ext}^1(\mathcal{I}, \mathcal{S}(-1))$. Hence it follows from Proposition 5.6.3 that there exist $\mathcal{S}$ such that $\operatorname{Ext}^1(\mathcal{S}(1), \mathcal{I}(-1)) \neq 0$.

Now let $\mathcal{J} = \pi J$ be the middle term of a non-trivial extension of $\mathcal{I}(-1)$ by $\mathcal{S}(1)$. Then we have $[\mathcal{J}] = [\mathcal{O}] - [\mathcal{S}] - n[\mathcal{P}] + [\mathcal{S}] + [\mathcal{P}] = [\mathcal{O}] - (n-1)[\mathcal{P}]$.

We claim that $\mathcal{J}$ is torsion-free. Assume this is not the case and let $\mathcal{F} \subset \mathcal{J}$ be a maximal subobject of $\mathcal{J}$ of dimension ≤ 1. So $\mathcal{F} \neq 0$. Since $\mathcal{I}$ is torsion-free we have $\mathcal{F} \cap \mathcal{I}(-1) = 0$. So we may consider $\mathcal{F}$ as a subobject of $\mathcal{S}(1)$. Hence we obtain an extention

$$0 \to \mathcal{I}(-1) \to \mathcal{J}/\mathcal{F} \to \mathcal{S}(1)/\mathcal{F} \to 0 \tag{5.11}$$

According to lemma 3.4 this extension is split. But this means that $\mathcal{S}(1)/\mathcal{F}$ is a subobject of $\mathcal{J}/\mathcal{F}$ of dimension ≤ 1, contradicting the maximality of $\mathcal{F}$.

It follows from [3] that $\operatorname{GKdim} J^{**}/J \leq 1$. Thus $\mathcal{J}^{**}/\mathcal{J} = b[\mathcal{P}]$ for some $b \geq 0$ by Proposition 3.2. Hence $[\mathcal{J}^{**}] = [\mathcal{O}] - (n - 1 - b)[\mathcal{P}]$.

Let $\mathcal{S}' = \mathcal{J}^{**}/\mathcal{I}(-1)$. Then by lemma 3.4 $\mathcal{S}'$ is pure and furthermore we have $e(\mathcal{S}') = 1$. It then follows easily using the methods of [3] that $\mathcal{S}'$ is a shifted line object. This finishes the proof. □

We can now prove another main result.

Theorem 5.6.6. *Let $\mathcal{I}$ be a normalized line bundle on $\mathbb{P}^2_q$. Then there exists an $m \in \mathbb{N}$ together with a monomorphism $\mathcal{I}(-m) \hookrightarrow \mathcal{O}$ such that there exists a filtration of line bundles $\mathcal{O} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_u = \mathcal{I}(-m)$ on $\mathbb{P}^2_q$ with the property that the $\mathcal{M}_i/\mathcal{M}_{i+1}$ are shifted line objects.*

Proof. This follows easily from the lemma 5.6.5 and Corollary 3.6. □

Remark 5.6.7. There is some freedom in choosing the line objects occurring in Theorem 5.6.6. We may assume for example that they all map to the same point object.

Appendix A. Serre duality for graded rings

In this section we prove that (a generalization of) BKS-duality holds for graded rings. For the convenience of the reader we restate some definitions so that this appendix can be read independently of the rest of this paper.

Let $\mathcal{A}$ be a k-linear Ext finite triangulated category. By this we mean that for all $\mathcal{M},\mathcal{N} \in \mathcal{A}$ we have $\sum_n \dim_k \operatorname{Hom}(\mathcal{M},\mathcal{N}[n]) < \infty$. The category $\mathcal{A}$ is said to satisfy Bondal-Kapranov-Serre (BKS) duality if there is an autoequivalence $F : \mathcal{A} \to \mathcal{A}$ together with for all $A, B \in \mathcal{A}$ natural isomorphisms

$$\operatorname{Hom}(A,B) \to \operatorname{Hom}(B,FA)'$$

(where $(-)'$ denotes the k-dual).

Let $\mathcal{C}$ be an abelian category. An object O in $D^b(\mathcal{C})$ is said to have finite projective (injective) dimension if $\operatorname{Ext}^i(O,\mathcal{C}) = 0$ ($\operatorname{Ext}^i(\mathcal{C},O) = 0$) for $|i| > u$ for some $u \geq 0$. The minimal such u we call the projective (injective) dimension of O.

In this appendix we assume that A is a connected graded noetherian ring over a k. By $(-)'$ we denote the functor on graded vectorspaces which sends M to $\oplus_n M^*_{-n}$. If we use notations which refer to the left structure of A then we adorn them with a superscript "$\circ$".

We make the following additional assumptions on A:

1. A satisfies χ and the functor τ has finite cohomogical dimension.
2. A satisfies χ° and the functor τ° has finite cohomogical dimension.

These conditions imply that A has a *balanced dualizing complex* [37] given by $R = R\tau(A)' = R\tau^\circ(A)'$ [32, 37]. Below we freely use the properties of such dualizing complexes.

We let $D(A)$ be the derived category of graded right A-modules. $D^b_f(A)$ will be the full subcategory of objects in $D^b(A)$ with finitely generated homology. The category $D^b_f(\operatorname{Tails}(A))$ is the full subcategory of $D^b(\operatorname{Tails}(A))$ consisting of complexes with homology in $\operatorname{tails}(A)$.

We let $D^b_f(A)_{\text{fpd}}$ ($D^b_f(A)_{\text{fid}}$) be the full category of $D^b_f(A)$ consisting of objects of finite projective (injective) dimension. The categories $D^b_f(\operatorname{Tails}(A))_{\text{fpd}}$ and $D^b_f(\operatorname{Tails}(A))_{\text{fid}}$ are defined in a similar way. The fact that τ has finite cohomological dimension implies $\pi A(n) \in D^b_f(\operatorname{Tails}(A))_{\text{fpd}}$.

We will denote the functors $R\underline{\operatorname{Hom}}_A(-,R)$ and $R\underline{\operatorname{Hom}}_{A^\circ}(-,R)$ by D. Since they define a duality between $D^b_f(A)$ and $D^b_f(A^\circ)$ it is clear that they define a duality between $D^b_f(A)_{\text{fid}}$ and $D^b_f(A^\circ)_{\text{fpd}}$ and between $D^b_f(A)_{\text{fpd}}$ and $D^b_f(A^\circ)_{\text{fid}}$.

It is also clear that these functors induce a duality between $D^b_f(\operatorname{Tails}(A))$ and $D^b_f(\operatorname{Tails}(A^{\text{opp}}))$. We denote these induced functors also by D. Again they define a duality between $D^b_f(\operatorname{Tails}(A))_{\text{fid}}$ and $D^b_f(\operatorname{Tails}(A^\circ))_{\text{fpd}}$ and between $D^b_f(\operatorname{Tails}(A))_{\text{fpd}}$ and $D^b_f(\operatorname{Tails}(A^\circ))_{\text{fid}}$. Recall the following:

Lemma A.1. *Let $\mathcal{P} \in D^b_f(\operatorname{Tails}(A))_{\text{fpd}}$. Then there exists an object $P \in D^b_f(A)_{\text{fpd}}$ such that $\mathcal{P}$ is a direct summand of πP.*

Proof. This can be deduced from general results about compact objects in triangulated categories. For simplicity we give a direct proof based on a trick which the authors learned from Maxim Kontsevich. Take M arbitrary such that $\pi M = \mathcal{P}$.

Take a quasi-isomorphism $Q \to M$ where Q is a right bounded complex of finitely generated projective modules. This yields a triangle:

$$(\pi Z)[a] \to \sigma_{\geq -a}\pi Q \to \mathcal{P}$$

where $Z = \ker(Q_{-a} \to Q_{-a+1})$. This triangle corresponds to an element of $\mathrm{Ext}^{a+1}(\mathcal{P}, \pi Z)$ which must be zero for large a. Hence $\sigma_{\geq a}\pi Q = \mathcal{P} \oplus (\pi Z)[a]$. This proves the lemma. □

We recall the following fact.

Proposition A.2. *The functors* $- \overset{\mathbf{L}}{\otimes}_A R$ *and* $R\underline{\mathrm{Hom}}_A(R, -)$ *induce inverse equivalences between* $D^b_f(A)_{\mathrm{fpd}}$ *and* $D^b_f(A)_{\mathrm{fid}}$.

Proof. If $P \in D^b_f(A)_{\mathrm{fpd}}$ then it is quasi-isomorphic to a bounded complex of finitely generated projective A-modules. For such such a complex it is clear that $P \otimes_A R$ has finite injective dimension. There is a canonical map $P \to R\underline{\mathrm{Hom}}(R, P \otimes_A R)$ which is an isomorphism for $P = A$. By induction over triangles one shows that it is an isomorphism for all P.

Conversely assume $I \in D^b_f(A)_{\mathrm{fid}}$. Then by duality $R\underline{\mathrm{Hom}}(R, I) = R\underline{\mathrm{Hom}}(DI, A)$. By the above discussion $DI \in D^b_f(A^\circ)_{\mathrm{fpd}}$. Hence $R\underline{\mathrm{Hom}}_A(DI, A) \in D^b_f(A)_{\mathrm{fpd}}$. We also find $R\underline{\mathrm{Hom}}_A(DI, A) \otimes_A R = R\underline{\mathrm{Hom}}_A(DI, R) = I$. □

The functor $- \otimes_A R$ induces a functor $D^-(\mathrm{Tails}(A)) \to D^-(\mathrm{Tails}(A))$ which we denote by $- \otimes \mathcal{R}$. Similarly the functor $\mathbf{R}\mathrm{Hom}_A(R, -)$ induces a functor $D^+(\mathrm{Tails}(A)) \to D^+(\mathrm{Tails}(A))$ which we denote by $\mathrm{R}\mathcal{H}om(\mathcal{R}, -)$.

Proposition A.3. *The functors* $- \otimes \mathcal{R}$ *and* $\mathrm{R}\mathcal{H}om(\mathcal{R}, -)$ *induces inverse equivalences between* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$ *and* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$.

Proof. If $\mathcal{P} \in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$ then by lemma A.1 $\mathcal{P}$ is direct summand of some πP with $P \in D^b_f(A)_{\mathrm{fpd}}$.

Using the proof of the previous proposition this easily implies that $\mathcal{P} \otimes \mathcal{R} \in D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$ and $\mathrm{R}\mathcal{H}om(\mathcal{R}, \mathcal{P} \otimes \mathcal{R}) = \mathcal{P}$ (essentially because we may reduce to $\mathcal{P} = \pi A(n)$ for some n).

Conversely assume $\mathcal{I} = \pi I \in D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$. Then $\mathrm{R}\mathcal{H}om(\mathcal{R}, \mathcal{I}) = \pi R\underline{\mathrm{Hom}}(R, I) = \pi R\underline{\mathrm{Hom}}(DI, A)$.

We have by definition $\pi DI = D\pi I$, and hence $\pi DI \in D^b_f(A)_{\mathrm{fpd}}$. Then it follows from lemma A.1 that πDI is a direct summand of some πQ with $Q \in D^b_f(Q)_{\mathrm{fpd}}$.

We easily deduce from this that $\pi R\underline{\mathrm{Hom}}(DI, A)$ is a direct summand of $\pi R\underline{\mathrm{Hom}}(Q, A)$ and hence $\mathrm{R}\mathcal{H}om(\mathcal{R}, \mathcal{I}) = \pi R\underline{\mathrm{Hom}}(DI, A) \in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$. The proof now continous as the proof of the previous proposition. □

Theorem A.4. *(Serre duality) For all* $\mathcal{M} \in D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$, $\mathcal{N} \in D^b_f(\mathrm{Tails}(A))$ *there are natural isomorphisms*

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}(\mathcal{N}, F\mathcal{M})'$$

where

$$F\mathcal{M} = (\mathcal{M} \overset{\mathbf{L}}{\otimes} \mathcal{R})[-1] \tag{A.1}$$

Furthermore the functor F *defines an equivalence between* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fpd}}$ *and* $D^b_f(\mathrm{Tails}(A))_{\mathrm{fid}}$.

Proof. As in [38] our proof of Serre duality is based on the local duality formula [32, 37]. The formulation of local duality in [32] used the functor $R\tau$ but the same proof works for the functor RQ where $Q = \omega \circ \pi$. Furthermore it is possible to throw an extra perfect complex into the bargain. If we do this we obtain canonical isomorphisms

$$\underline{\mathrm{Hom}}_A(N, P \otimes_A (RQA)') \cong \underline{\mathrm{Hom}}_A(P, RQN)' \tag{A.2}$$

for $N \in D(A)$ and $P \in D^b_f(A)_{\mathrm{fpd}}$. By adjointness $\underline{\mathrm{Hom}}_A(P, RQN)_0 = \mathrm{Hom}_{\mathrm{Tails}(A)}(\pi P, \pi N)$. In addition, if we apply (A.2) with N finite dimensional then we find $\underline{\mathrm{Hom}}_A(N, P \otimes_A (RQA)') = 0$. Thus using lemma A.1 we obtain for $N \in D^b_f(A)$: $\underline{\mathrm{Hom}}_A(N, P \otimes_A (RQA)')_0 = \mathrm{Hom}_{\mathrm{Tails}(A)}(\pi N, \pi(P \otimes_A (RQA)')$. Now the standard triangle for local cohomology yields $RQA = \mathrm{cone}(R\tau A \to A)$ and thus $(RQA)' = \mathrm{cone}(A' \to R)[-1]$. Using the fact that A' is torsion we easily obtain from this: $\pi(P \otimes_A (RQA)') = F(\pi P)$ where F is defined as in the statement of the theorem. So now we have shown

$$\mathrm{Hom}_{\mathrm{Tails}(A)}(\pi N, F(\pi P)) \cong \mathrm{Hom}_{\mathrm{Tails}(A)}(\pi P, \pi N)' \tag{A.3}$$

Now we obtain from lemma A.1 that $\mathcal{M}$ is a direct summand of a complex πP with $P \in D^b_f(A)_{\mathrm{fpd}}$. Thus (A.3) is true for $\mathcal{M}$ and this finishes of the the first part of the theorem. The last part is Proposition A.3. $\square$

Corollary A.5. *If* Tails(A) *has finite global dimension then* $D^b_f(\mathrm{Tails}(A))$ *satisfies BKS-duality.*

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