PRESENTATION OF CRITICAL MODULES OF GK-DIMENSION 2 OVER ELLIPTIC ALGEBRAS

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ABSTRACT. We show that critical modules of Gelfand-Kirillov dimension 2 and multiplicity d over an elliptic algebra have (up to modules of lower GK-dimension and shifting) a presentation by $d \times d$-matrices of linear forms. In the language of non-commutative algebraic geometry this amounts to a generic description of "curves" of degree d in a projective quantum plane

1. INTRODUCTION

Below k will be an algebraically closed field and A will be a three dimensional three generator Artin-Schelter regular k-algebra [6]. Thus A is a graded k-algebra having the Hilbert series of a three generator polynomial algebra together with some other desirable homological properties (for this reason one sometimes refers to A as a projective quantum plane).

In [4] the first author initiated the study of graded modules of GK-dimension 2 over A, the motivation being that these somehow correspond to "curves" in the corresponding quantum plane. Although a fairly complete description was obtained, some questions remained open. In the following theorem we answer one of these open questions.

Theorem 1.1. *Assume that in the triple $(E, \sigma, \mathcal{L})$ associated to A [7] E is a smooth elliptic curve and σ is a translation of infinite order. Then every critical graded A-module of GK-dimension 2 and multiplicity d has, up to modules of lower GK-dimension and shifting, a resolution of the form*

$$0 \to A^d(-1) \to A^d \to M \to 0 \tag{1.1}$$

We conjecture that the hypotheses on the triple $(E, \sigma, \mathcal{L})$ are in fact unnecessary for the previous theorem to hold.

It is was shown by the first author in [2] that Theorem 1.1 is true provided every critical normal 2-dimensional A-module maps surjectively onto a point module (see below for definitions). Unfortunately it is not known if this is true. However in this note we succeed in using the idea from [2] while sidestepping this open problem.

Thus, a generic way to describe "curves in a quantum plane" is by $d \times d$ matrices with linear entries. Comparing with the commutative case one could also hope to be able to to show that, up to modules of lower GK-dimension and shifting, there is a resolution

$$0 \to A(-d) \to A \to M \to 0$$

1991 *Mathematics Subject Classification.* Primary 16D50, 16E10, 18G10.

Key words and phrases. Elliptic Algebras, quantum planes, regular algebra. critical modules, Cohen-Macaulay modules.

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This would mean that we could describe the curve corresponding to M by one element of degree d. However it is shown in [1] that this is in general not possible.

In this note all modules are tacitly assumed to be graded and homomorphisms between them degree-preserving. Unless stated otherwise, the term A-module will mean a finitely generated graded right A-module.

2. Preliminaries on A-modules of GK-dimension 2

In this section we make no hypotheses on the triple associated to A.

Let $\mathrm{gr}-A$ be the category of all finitely generated graded right A-modules, and let $(\mathrm{gr}-A)_i$ be the full subcategory of $\mathrm{gr}-A$ consisting of modules of GK-dimension at most i. It is clear that $(\mathrm{gr}-A)_i$ is a Serre subcategory of $\mathrm{gr}-A$.

Definition 2.1. Two A-modules are called *equivalent* if they have the same GK-dimension, say i, and their images in the quotient category $\mathrm{gr}-A/(\mathrm{gr}-A)_{i-1}$ are isomorphic.

For nice rings, the study of A-modules up to equivalence amounts to studying the indecomposable injectives over A. See [3, 11] for applications to the residue-complex. In this paper our main interest will be the study of critical A-modules of GK-dimension 2, up to equivalence.

It is well known that A is Cohen-Macaulay and Auslander-Regular (see [10] for a definition of these notions, and [10, 8] for proofs). This implies that the homological behaviour of A-modules is very close to the commutative case. Thus we will leave the routine proofs of such homological properties to the reader.

It is in particular elementary to prove that any object in $(\mathrm{gr}-A)_2/(\mathrm{gr}-A)_0$ is uniquely(!) represented by a (2-dimensional) *Cohen-Macaulay* module M. This means by definition that $\mathrm{Ext}^i_A(M,A)=0$ unless $i=1$. It is convenient to say that a module of GK-dimension 2 is *normal* if it is Cohen-Macaulay and if $M_n=0$ if and only if $n<0$.

A 2-dimensional Cohen-Macaulay module M has a minimal graded resolution of the form

$$0\to A(-j_1)\oplus\ldots\oplus A(-j_r)\to A(-i_1)\oplus\ldots\oplus A(-i_r)\to M\to 0 \tag{2.1}$$

where the indices i_ν, j_ν indicate the shifts of the module.

As a minimal projective resolution is unique up to (non-unique) isomorphism, the sequence of indices $\tau=(i_1,\ldots,i_r;j_1,\ldots,j_r)$ appearing in (2.1) is unique if we arrange the sequences $i_1,\ldots,i_r$ and $j_1,\ldots,j_r$ in non-decreasing order. In [4] the sequence τ was called the *type* of M.

From (2.1) it easily follows that if M is an arbitrary A-module of GK-dimension 2 then $\dim M_n = dn+c$ for large n. We call d the multiplicity of M.

A simple object in $(\mathrm{gr}-A)_2/(\mathrm{gr}-A)_1$ is represented by a graded Cohen-Macaulay module M which in addition has no proper quotient of GK-dimension 2. We call such a module *critical*.

Recall the following finiteness theorem [4, Corollary 3.8].

Theorem 2.2. *There are only finitely many types of critical normal Cohen-Macaulay A-modules of GK-dimension 2 with a given multiplicity.*

It is a combinatorial problem to write down all possible types for a given multiplicity: in multiplicity 1 there is only possible type, namely (0;1) corresponding to *line modules*; in multiplicity 2, there are two possible types: (0;2) and (0,0;1,1); in multiplicity 3, there are four: (0;3), (0,0:1,2), (0,1;2,2) and (0,0,0;1,1,1), etc ...

Below we improve certain results from [8, 4] on Hilbert functions of normal modules of GK-dimension 2.

Lemma 2.3. *Let M be a critical normal A-module of GK-dimension 2 and multiplicity d. Then M has a minimal projective resolution of the form*

$$0 \to A^d(-1) \to A^d \to M \to 0 \tag{2.2}$$

if and only if the Hilbert function of M is $h_M(n) = dn + d$ for all $n \geq 0$.

Proof. It is simple to compute the Hilbert function of a module M with presentation (2.2) : $h_M(n) = d[h_A(n) - h_A(n-1)]$, and $h_A(n) - h_A(n-1) = n+1$ for all $n \geq 0$. Conversely, suppose that M has the Hilbert function as mentioned, and let M have a minimal projective resolution given by (2.1). Then $d = h_M(0) = \#\{i_\nu | i_\nu = 0\} \leq r$. But, $r \leq \sum_{\nu=1}^r (j_\nu - i_\nu) = d$ [8, 2.41]. Thus $r = d$ and then the condition $\sum_{\nu=1}^d (j_\nu - i_\nu) = d$ implies, using the fact $j_\nu > i_\nu$ [4, 3.2], that $i_\nu = 0$ and $j_\nu = 1$ for all ν. □

Proposition 2.4. *Let M be a normal A-module of GK-dimension 2 and multiplicity d, with Hilbert function h_M. Then h_M is a non-decreasing. Further, if M is critical then $h_M(n) = dn + c$ for $n \geq d$ (where c is a constant depending on the type of M).*

Proof. Let $s_M(n)$ be the "slope" of the Hilbert function, defined as $s_M(n) = h_M(n) - h_M(n-1)$ for all $n \in \mathbb{Z}$. Since the Hilbert function is additive on exact sequences, so is the slope function. Now let M have a minimal projective resolution as in (2.1). Then,

$$s_M(n) = \sum_{\nu=1}^r s_{A(-i_\nu)}(n) \ - \ \sum_{\nu=1}^r s_{A(-j_\nu)}(n). \tag{2.3}$$

Since $h_A(n) = (n+1)(n+2)/2$ for $n \geq 0$, and 0 for $n < 0$, it follows that $s_{A(-i)}(n) = n - i + 1$ for $n \geq i$, and 0 for $n < i$. Observe that if $i \leq j$ then $s_{A(-i)}(n) \geq s_{A(-j)}(n)$ for all n. Minimality of the resolution (2.1) implies that $i_\nu < j_\nu$ for all ν [4, 3.2]. Now assembling the above facts in (2.3) shows that $s_M(n) \geq 0$ for all n, so $h_M(n)$ is a non-decreasing function. Now suppose M is critical. Then $i_\nu, j_\nu \leq d$ for all ν [4, Proof of 3.8]. Thus, the formula (2.3), together with the remarks made above about the slope function, shows that for $n \geq d$

$$s_M(n) = \sum_{\nu=1}^r (n - i_\nu + 1) \ - \ \sum_{\nu=1}^r (n - j_\nu + 1) = \sum_{\nu=1}^r (j_\nu - i_\nu) = d. \tag{2.4}$$

where the last equality is also a consequence of the minimality of the resolution [8, 2.41]; [4, 3.2]. Now the fact $s_M(n) = d$ for all $n \geq d$ clearly implies that $h_M(n) = dn + c$ for $n \geq d$ with an appropriate choice of the constant c. □

3. The divisor of a module

From here on we assume that A satisfies the hypotheses from Theorem 1.1.

Recall that A contains a certain canonical central element g in degree three such that $B = A/gA$ is isomorphic to the twisted homogeneous coordinate ring

$B(E,\sigma,\mathcal{L})$ [7, 9]. Hence there is a category equivalence [9]

$$(\mathrm{gr}-B)/(\mathrm{gr}-B)_0 \to \mathrm{coh}(E) : M \mapsto \tilde{M} \tag{3.1}$$

reducing the GK-dimension by one. Here $\mathrm{coh}(E)$ denotes the category of coherent $\mathcal{O}_E$-modules. The inverse to $(\tilde{-})$ is given by a functor Γ_* which is defined by

$$\Gamma_*(\mathcal{M}) = \oplus_{n\in\mathbb{Z}}\Gamma(E, s^n\mathcal{M}) \tag{3.2}$$

where $s\mathcal{M} = \sigma_*(\mathcal{M}\otimes_{\mathcal{O}_E}\mathcal{L})$. Note that, even if $\mathcal{M}$ is coherent, $\Gamma_*(\mathcal{M})$ will not always be finitely generated. However all $\Gamma_*(\mathcal{M})_{\geq n}$ are finitely generated. In particular $\Gamma_*(\mathcal{M})$ is finitely generated when considered modulo $(\mathrm{gr}-B)_0$.

Using the functor Γ_* one obtains

$$M(1)^{\sim} = \sigma_*(\tilde{M}\otimes_{\mathcal{O}_E}\mathcal{L}) \tag{3.3}$$

A cyclic graded A-module N generated in degree 0 such that $\dim N_i = 1$ for all i is called a *point module.* It is shown in [7, 8] that in the elliptic case all point modules are annihilated by g. In fact one has [7, 8]:

Proposition 3.1. *The map $p \mapsto \Gamma_*(k(p))_{\geq 0}$ defines a bijection between the points of E and the point modules of A.*

In the sequel we will denote the point module corresponding to $p \in E$ by N_p. Thus $\tilde{N_p} = k(p)$. From (3.3) it follows that

$$N_p(1)_{\geq 1} = N_{\sigma p} \tag{3.4}$$

We associate a divisor $\mathrm{Div}(M)$ on E to a g-torsion-free A-module M of GK-dimension 2. For such a module M/Mg is a B-module of GK-dimension 1, so $(M/Mg)^{\sim}$ is a finite dimensional $\mathcal{O}_E$-module which corresponds to a divisor on E. We denote this divisor by $\mathrm{Div}(M)$. It is easy to check that $\mathrm{Div}(M)$ coincides with $\mathrm{div}(M)$ of [4, §3].

If M has multiplicity d then the degree of $\mathrm{Div}(M)$ is $3d$. In particular $\mathrm{Div}(M) \neq 0$ iff $M \neq 0$.

From (3.3) it follows that

$$\mathrm{Div}(M(1)) = \sigma(\mathrm{Div}(M)) \tag{3.5}$$

The next result is an improvement of [4, 3.35] for g-torsion-free modules.

Theorem 3.2. *Let M be a g-torsion-free A-module of GK-dimension 2, $M \to N_p$ a non-zero map to a point module N_p, and K the kernel of this map. Then*

$$\mathrm{Div}(K) = \mathrm{Div}(M) - (p) + (\sigma^{-3}p) \tag{3.6}$$

Proof. Modulo $(\mathrm{gr}-A)_0$ we have an exact sequence

$$0 \to K \to M \to N_p \to 0$$

Tensoring this sequence with B we obtain an exact sequence modulo $(\mathrm{gr}-A)_0$.

$$0 \to \mathrm{Tor}_1^B(N_p, B) \to K/Kg \to M/Mg \to N_p \to 0$$

(to see this observe that $\mathrm{Tor}_i^A(-,B)$ preserves $(\mathrm{gr}-A)_0$).

Using the standard resolution for B as A-module we find

$$\mathrm{Tor}_1^B(N_p, B) = N_p(-3)$$

Applying $(\tilde{-})$ and using (3.4) yields

$$0 \to k(\sigma^{-3}p) \to (K/Kg)^{\sim} \to (M/Mg)^{\sim} \to k(p) \to 0$$

which gives equation (3.6). □

Lemma 3.3. *Let M be a g-torsion-free A-module of GK-dimension 2.*

1. *If $M_{<0} = 0$ and if p is in the support of* $\operatorname{Div}(M)$ *then there is a non-zero map $M \to N_p$.*
2. *If M is normal and* $\operatorname{Div}(M)$ *has no multiple points, then there is a surjective map from M to a point module.*

Proof. 1. Let $\mathcal{M} = (M/Mg)^{\sim}$. There is a non-zero map $\mathcal{M} \to k(p)$, hence a map

$$M \to M/Mg = (M/Mg)_{\geq 0} \to \Gamma_*(\mathcal{M})_{\geq 0} \to \Gamma_*(k(p))_{\geq 0} = N_p$$

which is surjective in high degree.

2. Assume $\operatorname{Div}(M) = p_1 + \cdots + p_t$. Thus $(M/Mg)^{\sim} = k(p_1) \oplus \cdots k(p_t)$. Hence

$$u : M/Mg \to N_{p_1} \oplus \cdots \oplus N_{p_t} \tag{3.7}$$

has finite dimensional kernel and cokernel. From the Cohen-Macaulayness of M it easily follows that M/Mg has no finite dimensional submodules so the map u is actually injective. Let $0 \neq m \in M_0$. Since u is injective, $u(m)$ has a non-zero component in some $(N_{p_i})_0$. This implies that the composition of u with the projection $N_{p_1} \oplus \cdots \oplus N_{p_t} \to N_{p_i}$ is a surjective map of M/Mg onto N_{p_i}, hence giving a surjective map $M \to N_{p_i}$. □

4. Proof of Theorem 1.1

Since M is critical it is either g-torsion or g-torsion free. It is convenient to consider these cases separately.

Proof of Theorem 1.1. (g-torsion free case) Since $(\tilde{-})$ is insensitive to $(\mathrm{gr}-A)_0$, the same holds for $\operatorname{Div}(-)$. Hence we may assume that M is Cohen-Macaulay. Shifting if necessary we may furthermore assume that $M_{<0} = 0$.

Let N be some positive number and let p be in the support of $\operatorname{Div}(M)$. Put $K = \ker(M \to N_p)$. It is easy to see that K is also Cohen-Macaulay and critical. According to (3.6) p in $\operatorname{Div}(M)$ is replaced by $\sigma^{-3}p$ in $\operatorname{Div}(K)$. It is clear that by replacing M by K and repeating this procedure in an intelligent way we can assure that eventually the following becomes true

(A_N) $\operatorname{Div}(M)$ is reduced and $\sigma^{3i}p \neq q$ $(0 < |i| \leq N)$ for $p, q \in \operatorname{Supp}(\operatorname{Div}(M))$

(where we use of course that σ has infinite order and no fixed points). Now make M normal by shifting. According to (3.5), condition (A_N) is still satisfied.

We construct a descending chain of Cohen-Macaulay submodules of M

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_N$$

such that each M_i satisfies (A_{N-i}) This is done as follows. Suppose we have constructed M_i. Choose n such that $K = M_i(n)$ is normal. By (3.5) $\operatorname{Div}(K)$ is still multiplicity free and hence by lemma 3.3 K maps surjectively to a point module N_q. We put $M_{i+1} = \ker(K \to N_q)(-n)$. According to (3.5)(3.6) M_{i+1} will satisfy (A_{N-i-1})

The Hilbert function of M_i is easy to compute:

$$h_{M_i}(n) = \max(h_M(n) - i, 0). \tag{4.1}$$

According to Proposition $h_M(n)$ is ascending and

$$h_M(n) = dn + c_M \qquad \text{for } n \geq d$$

where c_M is a constant depending only on the type of M. Thus by Theorem 2.2 there are only a finite number of possibilities for c_M. Choose $N = \max_M(d^2 + c_M)$ Applying (4.1) with $i = d^2 + c_M$ we find

$$h_{M_i}(n) = \max(d(n-d), 0)$$

Hence by lemma 2.3 it follows that $M_i(d+1)$ has a resolution of the form (1.1). □

Proof of 1.1. (g-torsion case) From the category equivalence (3.1) it follows immediately that all critical GK-modules of GK-dimension 2 are equivalent. To find one which has a resolution as in (1.1) we have to find one which has the correct Hilbert series (according to lemma 2.3). To do this we use the inverse Γ_* to $\tilde{(-)}$ (3.2).

Just as in the commutative case it is easy to see (using the local cohomology criterion) that if $\mathcal{M}$ is torsion free then $\Gamma^*(\mathcal{M})$ is Cohen-Macaulay. See [5] for foundations.

Let $\mathcal{M}$ be a line bundle of degree 3, not isomorphic to $\sigma_*\mathcal{L}$. By Riemann-Roch it is easy to see that $\Gamma_*(\mathcal{M})$ has the correct Hilbert series. □

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