Non-commutative quasi-Hamiltonian spaces

Michel Van den Bergh

ABSTRACT. In this paper we introduce non-commutative analogues for the
quasi-Hamiltonian G-spaces introduced by Alekseev, Malkin and Meinrenken.
We outline the connection with the non-commutative analogues of quasi-Poisson
algebras which the author had introduced earlier.

CONTENTS

1. Introduction

There has been recent interest in developing a non-commutative version of differential geometry based on Kontsevich's philosophy [**10, 12**] that for a property of a non-commutative k-algebra A to have geometric meaning it should induce standard geometric properties on all representation spaces $\mathrm{Rep}(A, N) = \mathrm{Hom}(A, M_N(k))$. Non-commutative symplectic geometry was developed in [**3, 6, 8, 11**] and a non-commutative version of (quasi-)Poisson geometry was introduced in [**14**].

In this paper we introduce so-called *quasi-bisymplectic algebras* (see §6). These are a multiplicative analogue of algebras equipped with a bisymplectic form (see [**6**]). Our definition is such that the representation spaces of quasi-bisymplectic algebras are quasi-Hamiltonian G-spaces, as introduced in [**2**]. We develop non-commutative analogues for some aspects of the commutative theory [**1, 2**]. In particular we show that there is a one-one correspondence between quasi-bisymplectic algebras and

---

Hamiltonian double quasi-Poisson algebras (introduced in [**14**]) which satisfy a suitable non-degeneracy condition (see Theorem 7.1 below).

As a side result we show that double quasi-Poisson algebras give rise to something we call a "double Lie algebroid" (see Theorem 5.3 below). This is a non-commutative version of [**4**, Thm. 2.5].

In the final section of the paper we show that the Hamiltonian double quasi-Poisson algebras derived from quivers which were introduced in [**14**] are non-degenerate. Hence these algebras are also in a natural way quasi-bisymplectic.

The main result of this paper was announced in [**14**, App A] where we discussed the relation between *ordinary* double Poisson brackets and bisymplectic forms (i.e. the "non-quasi"case). It should be said however that our proof for the equivalence between integrability of double quasi-Poisson brackets and quasi-bisymplectic forms is based on a brute force computation and is less satisfactory than the corresponding proof in [**14**, App A].

This paper depends rather heavily on [**6, 14**]. For the convenience of the reader we have included some preliminary sections explaining the relevant concepts and results. To simplify the exposition we have chosen to write out all our computations over a base ring which is a field, although that is not sufficient for the application to quivers. Therefore in the short section §8.1 we outline the modifications necessary to handle more general situations.

A change in presentation with respect to [**14**] is that throughout the paper we have emphasized a certain functor

$$(-)_N : \mathrm{Bimod}(A) \to \mathrm{Mod}(\mathcal{O}(\mathrm{Rep}(A, N)))$$

which connect an algebra with its N'th representation space. When applied to a non-commutative object this functor yields the corresponding classical object. For example if L is a double Lie algebroid over A (see above) then L_N is a classical Lie algebroid on $\mathrm{Rep}(A, N)$.

The author wishes to thank Victor Ginzburg for explaining some aspects of [**7**].

2. Preliminaries

2.1. Representation spaces. We assume that k is a field of characteristic zero although this hypotheses is often too strong. Throughout A is a finitely generated k-algebra. For $N \in \mathbb{N}$ the associated representation space of A is defined as

$$\mathrm{Rep}(A, N) = \mathrm{Hom}(A, M_N(k))$$

The group Gl_N acts on $\mathrm{Rep}(A, N)$ by conjugation on $M_N(k)$.

A natural point of view in non-commutative algebraic geometry is that for a property of a non-commutative ring A to have geometric meaning it should induce standard geometric properties on all $\mathrm{Rep}(A, N)$.

It is easy to see that $\mathrm{Rep}(A, N)$ is an affine variety and its coordinate ring has a very convenient description. The ring $A_N \overset{\mathrm{def}}{=} \mathcal{O}(\mathrm{Rep}(A, N))$ is generated by symbols $(a_{ij})_{ij=1,\ldots,N}$ for all $a \in A$, subject to the relations

$$(ab)_{ij} = a_{il}b_{lj}$$

together with additivity in a and $1_{ij} = \delta_{ij}$. Here and below we sum over repeated indices. While being convenient this description is of course also very uneconomical. For example if $A = k\langle (x_l)_{l=1,\ldots,n} \rangle$ then $\mathcal{O}(\mathrm{Rep}(A, N)) = k[(x_{l,ij})_{l=1,\ldots,n,i,j=1,\ldots,N}]$.

It is easy to verify that our assumption that A is finitely generated implies that $\mathcal{O}(\mathrm{Rep}(A,N))$ is finitely generated.

If $a \in A$ then $(a_{ij})_{ij}$ defines a matrix values function on $\mathrm{Rep}(A,N)$ which we sometimes denote by $X(a)$. Concretely this is the function which associates to every $\phi : A \to M_N(k) \in \mathrm{Rep}(A,N)$ the matrix $(\phi(a))_{ij}$.

We define the trace $\mathrm{tr}(a)$ of $a \in A$ as a_{ii}. This defines a Gl_n-invariant function on $\mathrm{Rep}(A,N)$. As $\mathrm{tr}([A,A]) = 0$ we see that elements of $A/[A,A]$ correspond to invariant functions on $\mathrm{Rep}(A,N)$. In fact this will be true in all cases we consider below: the non-commutative version of working with Gl_N invariants objects is working modulo commutators.

2.2. Differential forms.

We now consider differential forms. The bimodule Ω_A of differentials is generated as an A-bimodule by the symbols da subject to the relations $d(ab) = a(db) + (da)b$ and linearity. As usual one puts $\Omega A = T_A \Omega_A$. Defining $d(da) = 0$, $d(a) = da$ makes Ω_A into a differential graded algebra. Every homogeneous element ω in ΩA has a representation $a_0 da_1 \cdots da_n$. To such an element one associates a matrix valued differential form $(\omega_{ij})_{ij}$ on $\mathrm{Rep}(Q,\alpha)$:

$$(2.1) \qquad \omega_{ij} = a_{1,ii_1} da_{2,i_1 i_2} \cdots da_{n,i_{n-1}j}$$

If we write it as $X(\omega)$ then (2.1) may be rewritten as

$$X(\omega) = X(a_1) dX(a_2) \cdots dX(a_n)$$

It would be tempting to define the non-commutative de Rham complex of A as ΩA. However, quite remarkably (see [**6**, §2.5]), ΩA is acyclic. That is

$$H^m(\Omega A) = \begin{cases} k & m = 0 \\ 0 & \text{otherwise} \end{cases}$$

Nevertheless ΩA can be used as the basis for a new construction of the cyclic homology of A (see [**7**]).

A different non-commutative analogue of the de Rham complex is the *Karoubi-de Rham complex* which is defined by

$$\mathrm{DR}(A) = \Omega A/[\Omega A, \Omega A]$$

The de Rham cohomology of A is defined as the cohomology of the Karoubi complex. It is closely related to cyclic homology and to equivariant de Rham cohomology of representation spaces (see [**7**, Thm 4.2.3]). According to [**13**, 2.6.7] we have a short exact sequence of reduced (co)homology

$$0 \to \bar{H}^n(\mathrm{DR}(A)) \to \overline{\mathrm{HC}}_n(A) \to \overline{\mathrm{HH}}_{n+1}(A) \to 0$$

Below we will mostly deal with *smooth algebras*, i.e. algebras whose category of bimodules has homological dimension one. In that case $\overline{\mathrm{HH}}_n(A) = 0$ for $n > 1$ and hence $\bar{H}^n(\mathrm{DR}(A)) = \overline{\mathrm{HC}}_n(A)$ for $n \geq 1$.

To any element ω we associate a Gl_n invariant differential form $\mathrm{tr}(\omega) = \omega_{ii}$ on $\mathrm{Rep}(A,N)$. This yields a map

$$\mathrm{tr} : DR(A) \to \Omega(\mathrm{Rep}(A,N))^{\mathrm{Gl}(n)}$$

which is compatible with differentials and hence descends to cohomology.

EXAMPLE 2.1. Let $C = k[t, t^{-1}]$. Then

$$H^i(\mathrm{DR}(C)) = \begin{cases} k & \text{if } i = 0 \\ k(t^{-1}dt)^i & \text{if } i \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

That the cohomology groups have the indicated dimension follows from [**13**, Cor 3.4.15]. It is easy to see that the generators for the cohomology groups are as claimed.

We have $\mathrm{Rep}(A, N) = \mathrm{Gl}_N$. The elements $\mathrm{tr}(t^{-1}dt)^{2i+1}$ are precisely the generators of the de Rham cohomology of Gl_N.

2.3. Vector fields. Now we discuss vector fields. Again there are two possible points of view.

If we insist that a vector field on A induces vector fields on all $\mathrm{Rep}(A, N)$ then a vector field on A should simply be a derivation $\Delta : A \to A$. The induced derivation δ on $\mathcal{O}(\mathrm{Rep}(A, N))$ is then given by

$$\delta(a_{ij}) = \Delta(a)_{ij}$$

A second point of view is that a vector field Δ on A should induce *matrix valued vector fields* $(\Delta_{ij})_{i,j=1,\ldots,n}$ on all $\mathrm{Rep}(A, N)$. Since now $\Delta_{ij}(a_{uv})$ depends on four indices $\Delta(a)$ should be an element of $A \otimes A$. It was first of observed by Crawley-Boevey that the second point of view is often more useful. Put

$$D_A \overset{\text{def}}{=} \mathrm{Der}(A, A \otimes A) = \mathrm{Hom}_{A \otimes A^\circ}(\Omega_A, A \otimes A)$$

where as usual we put the outer bimodule structure on $A \otimes A$. For $\Delta \in D_A$ the corresponding matrix valued vector field on $\mathrm{Rep}(A, N)$ is then given by

$$(2.2) \qquad\qquad \Delta_{ij}(a_{uv}) = \Delta(a)'_{uj}\Delta(a)''_{iv}$$

where by convention we write an element x of $A \otimes A$ as $x' \otimes x''$ (i.e. we drop the summation sign). We will call the peculiar arrangement of indices in (2.2) the *standard index convention*. It will reappear often below.

Starting with D_A we define *the algebra of poly-vector fields DA* on A as the tensor algebra $T_A D_A$ of D_A where we make D_A into an A-bimodule by using the inner bimodule structure on $A \otimes A$. Any homogeneous element δ of DA induces polyvector fields $X(\delta)$ on all representation spaces using a formula similar to (2.1).

The counterpart to the differential on ΩA is the "double Schouten Nijenhuys" bracket on DA which was defined in [**14**]. A double bracket on an ordinary algebra A is a bilinear map

$$\{\!\{-,-\}\!\} : A \times A \to A \otimes A$$

which is a derivation in its second argument (for the outer bimodule structure on A) and which satisfies

$$\{\!\{a, b\}\!\} = -\{\!\{b, a\}\!\}^\circ$$

where $(u \otimes v)^\circ = v \otimes u$. If $\{\!\{-,-\}\!\}$ satisfies the following analogue of the Jacobi identity

$$0 = \{\!\{a, b, c\}\!\} \overset{\text{def}}{=} \{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L + \tau_{(123)}\{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L + \tau_{(132)}\{\!\{c, \{\!\{a, b\}\!\}\}\!\}_L$$

where for $\tau \in S_n$ we define

$$(2.3) \qquad\qquad \tau(a_1 \otimes \cdots \otimes a_n) = a_{\tau^{-1}(1)} \otimes \cdots \otimes a_{\tau^{-1}(n)}$$

(with an appropriate sign in the graded case) then we call $\{\!\{-,-\}\!\}$ a double Poisson bracket. A double Poisson bracket induces a Lie bracket $\{-,-\}$ on $A/[A,A]$ via the formula

$$\{a,b\} = \{\!\{a,b\}\!\}' \{\!\{a,b\}\!\}''$$

The motivation for introducing double Poisson brackets is that they induce ordinary Poisson brackets on $\mathrm{Rep}(A,N)$ via the standard index convention. The precise formula is

$$\{a_{ij}, b_{uv}\} = \{\!\{a,b\}\!\}'_{uj} \{\!\{a,b\}\!\}''_{iv}$$

One of the main results of [**14**] is the following.

PROPOSITION 2.2. *The graded algebra DA has the structure of a* double Gerstenhaber algebra *i.e. a (super) double Poisson algebra with a double Poisson bracket $\{\!\{-,-\}\!\}$ of degree -1.*

For the convenience of the reader we give the construction of the double Schouten-Nijenhuys bracket on D_A.

If $\delta, \Delta \in D_A$ then it is easy to see that

$$\{\!\{\delta, \Delta\}\!\}_{\tilde{l}} = (\delta \otimes 1)\Delta - (1 \otimes \Delta)\delta$$
$$\{\!\{\delta, \Delta\}\!\}_{\tilde{r}} = (1 \otimes \delta)\Delta - (\Delta \otimes 1)\delta = -\{\!\{\Delta, \delta\}\!\}_{\tilde{l}}$$

define derivations $A \to A^{\otimes 3}$ for the outer bimodule structure on $A^{\otimes 3}$. Since Ω_A is finitely generated we obtain

$$\mathrm{Der}_B(A, A^{\otimes 3}) \cong \mathrm{Hom}_{A^e}(\Omega_{A/B}, A \otimes A) \otimes A$$

We view $\{\!\{\delta, \Delta\}\!\}_{\tilde{l}}$ and $\{\!\{\delta, \Delta\}\!\}_{\tilde{r}}$ as elements of $D_A \otimes_k A$ and $A \otimes_k D_A$ respectively. To this end we define

$$\{\!\{\delta, \Delta\}\!\}_l = \tau_{(23)} \circ \{\!\{\delta, \Delta\}\!\}_{\tilde{l}}$$
$$\{\!\{\delta, \Delta\}\!\}_r = \tau_{(12)} \circ \{\!\{\delta, \Delta\}\!\}_{\tilde{r}}$$

and we write

$$\{\!\{\delta, \Delta\}\!\}_l = \{\!\{\delta, \Delta\}\!\}'_l \otimes \{\!\{\delta, \Delta\}\!\}''_l$$
$$\{\!\{\delta, \Delta\}\!\}_r = \{\!\{\delta, \Delta\}\!\}'_r \otimes \{\!\{\delta, \Delta\}\!\}''_r$$

with $\{\!\{\delta, \Delta\}\!\}''_l, \{\!\{\delta, \Delta\}\!\}'_r \in A$, $\{\!\{\delta, \Delta\}\!\}'_l, \{\!\{\delta, \Delta\}\!\}''_r$ in D_A.

An easy verification shows that

$$\{\!\{\delta, \Delta\}\!\}_r = -\{\!\{\Delta, \delta\}\!\}_l^\circ$$

The double Schouten-Nijenhuys bracket is defined on generators by

$$\{\!\{a, b\}\!\} = 0$$
$$\{\!\{\delta, a\}\!\} = \delta(a)$$
$$\{\!\{\delta, \Delta\}\!\} = \{\!\{\delta, \Delta\}\!\}_l + \{\!\{\delta, \Delta\}\!\}_r$$

for $a, b \in A$, $\delta, \Delta \in D_A$, where we regard the righthand sides in the previous display as elements of $DA \otimes DA$.

One may show again that the double Schouten-Nijenhuys bracket on D_A induces the standard Schouten-Nijenhuys bracket on the algebra of polyvector fields on $\mathrm{Rep}(A,N)$ using the standard index convention.

In [**14**] it was shown how to associate a double bracket $\{\!\{-,-\}\!\}_P$ to an element $P \in D^2 A$. The formula is obtained by linear extension from the following formula with $\delta, \Delta \in D_A$:

$$(2.4) \qquad \{\!\{a,b\}\!\}_{\delta\Delta} = \Delta(b)'\delta(a)'' \otimes \delta(a)'\Delta(b)'' - \delta(b)'\Delta(a)'' \otimes \Delta(a)'\delta(b)''$$

If A is smooth then the double Jacobi identity for $\{\!\{-,-\}\!\}$ is equivalent to $\{P,P\} = 0$.

The algebra DA has a remarkable element E which has no commutative counter part. It is the double derivation which sends a to $a \otimes 1 - 1 \otimes a$. It is intimately connected to the Gl_N action on $\mathrm{Rep}(A,N)$. More precisely the following result was proved in [**6, 14**]

PROPOSITION 2.3.1. *Let* $f_{ij} \in M_\alpha = \mathrm{Lie}(\mathrm{Gl}_N)$ *be the elementary matrix which is* 1 *in the* (i,j)-*entry and zero everywhere else. Then* $(E_p)_{ij}$ *acts as* f_{ji} *on* $\mathcal{O}(\mathrm{Rep}(A,N))$.

The element E appears in several pleasing formulas. For example for any polyvector field Δ in DA we have

$$\{\!\{E,\Delta\}\!\} = \Delta \otimes 1 - 1 \otimes \Delta$$

A vector field of the form

$$H_a = \{\!\{a,-\}\!\}$$

is called *Hamiltonian*. An element $\Phi \in A$ is a *moment map* if it realizes E as a Hamiltonian vector field, i.e. if $E = H_\Phi$. We should think of Φ as defining a matrix valued map

$$\Phi_{ij} : \mathrm{Rep}(A,N) \to \mathrm{Rep}(k[t],N) = M_N(k)$$

If Φ is a moment map for $\{\!\{-,-\}\!\}$ then Φ_{ij} is a moment map for the induced Poisson bracket $\{-,-\}$.

2.4. Bisymplectic geometry. In the commutative case symplectic geometry is a special case of Poisson geometry. A first version of non-commutative symplectic geometry was introduced by Kontsevich in [**11**]. See also [**3, 8**]. A version of Poisson geometry following a similar philosophy was introduced by Crawley-Boevey in [**5**].

A different version of non-commutative symplectic geometry which follows a similar philosophy as the Poisson geometry outlined in the previous sections was introduced in [**6**] and baptized "bisymplectic geometry". We recall the definition. If $\delta \in D_A$ then we may define a double derivation

$$i_\delta : \Omega A \to \Omega A \otimes \Omega A$$

in the usual way. For $a \in A$ put

$$i_\delta(a) = 0 \qquad i_\delta(da) = \delta(a)$$

If C is a graded k-algebra and $c = c_1 \otimes \cdots \otimes c_n$ then we put

$$(2.5) \qquad {}^\circ c = \tau_{(1\cdots n)}(c) = (-1)^{|c_n|(|c_1|+\cdots+|c_{n-1}|)} c_n c_1 \cdots c_{n-1}$$

and if $\phi : C \to C^{\otimes 2}$ is a linear map then we define

$$ {}^\circ\phi : C/[C,C] \to C : c \mapsto {}^\circ(\phi(c))$$

We apply this with $C = \Omega A$. Following [**6**] we put

$$\imath_\delta = {}^\circ i_\delta$$

Also following [**6**] we say that an element $\omega \in \mathrm{DR}^2(A)$ is a *bi-non-degenerate* if the map of A-bimodules

$$\imath(\omega) : D_A \rightarrow \Omega_A : \delta \mapsto \imath_\delta \omega$$

is an isomorphism (in fact it is sufficient to assume surjectivity, see Corollary 3.1.3 below). If in addition ω is closed in $\mathrm{DR}(A)$ then we say that ω is *bisymplectic*.

It was shown in [**6**] that if ω is bisymplectic the $\mathrm{tr}(\omega)$ defines a symplectic form on representation spaces.

3. Generalities about bimodules

In the framework of this paper the non-commutative version of a coherent sheaf is a bimodule. In this short section we discuss some elementary aspects of bimodules.

3.1. Pairings. Let A be an arbitrary k-algebra. A pairing (or bilinear map) between $A - A$ bimodules P, Q is a map

$$\langle -, - \rangle : P \times Q \rightarrow A \otimes A$$

such that $\langle p, - \rangle$ is linear for the outer bimodule structure on $A \otimes A$ and $\langle -, q \rangle$ is linear for the inner bimodule structure on $A \otimes A$. The obvious example is of course $P = Q^*$ and $\langle -, - \rangle$ is the evaluation pairing. We say that the pairing is non-degenerate if P, Q are finitely generated projective bimodules and the pairing induces an isomorphism $Q \cong P^*$.

If $\langle -, - \rangle$ is a pairing between P and Q then the opposite pairing between Q and P is given by

$$\langle q, p \rangle^\circ = \langle p, q \rangle'' \otimes \langle p, q \rangle'$$

If we have pairings between P and Q and between P' and Q' the morphisms $\alpha : P \rightarrow P'$, $\beta : Q' \rightarrow Q$ are said to be *adjoint* if

$$\langle \alpha(p), q' \rangle = \langle p, \beta(q') \rangle$$

We say that α is left adjoint to β and that β is a right adjoint of α. Note that if we change the pairings into the opposite ones then a left adjoint becomes a right adjoint and vice versa. Therefore we usually drop the left/right adjectives if what is meant is clear from the context. If the pairing are non-degenerate then we denote an adjoint often by $(-)^*$.

If we have a morphism $\alpha : P \rightarrow Q$ then then we say that α is anti-symmetric (or anti-self adjoint) if $-\alpha$ is left adjoint to α for the appropriate pairings, i.e. if

$$\langle p, \alpha(p') \rangle = -\langle \alpha(p), p' \rangle^\circ$$

for $p, p' \in P$.

Or written out explicitly

(3.1) $$\langle p, \alpha(p') \rangle' \otimes \langle p, \alpha(p') \rangle'' = -\langle p', \alpha(p) \rangle'' \otimes \langle p', \alpha(p) \rangle'$$

If we have a pairing between P and Q as above then $p \in P$ defines a double derivation of degree -1

$$i_p : T_A(Q) \rightarrow T_A(Q) \otimes T_A(Q)$$

such that $i_p(q) = \langle p, q \rangle$. Since Q and P are related by the opposite pairing we may define i_q for $q \in Q$ in the same way. I.e.

$$i_q : T_A(P) \rightarrow T_A(P) \otimes T_A(P)$$

is the double derivation of degree -1 such that $i_q(p) = \langle q, p \rangle^\circ$.

We define $\imath_p = {}^\circ i_p$ and $\imath_q = {}^\circ i_q$ as in §2.4. If $\omega \in T_A^2 Q$ then we put

$$\imath(\omega) : P \to Q : p \mapsto \imath_p(\omega)$$

This is a bimodule morphism between P and Q. The following property of $\imath(\omega)$ will be used below.

PROPOSITION 3.1.1. *The bimodule morphism $\imath(\omega)$ is anti-symmetric.*

PROOF. We need to prove that $\langle p, \imath(\omega)(p') \rangle = \langle p, \imath_{p'}\omega \rangle$ satisfies (3.1). We write ω as $\omega' \otimes \omega''$ with $\omega', \omega'' \in Q$. Then we have

$$i_{p'}(\omega) = i_{p'}(\omega')\omega'' - \omega' i_{p'}(\omega'')$$

so that we get

$$\begin{aligned}
\imath_{p'}(\omega) &= i_{p'}(\omega')''\omega'' i_{p'}(\omega')' - i_{p'}(\omega'')''\omega' i_{p'}(\omega'')' \\
&= \langle p', \omega' \rangle'' \omega'' \langle p', \omega' \rangle' - \langle p', \omega'' \rangle'' \omega' \langle p', \omega'' \rangle'
\end{aligned}$$

and hence

$$\begin{aligned}
\langle p, \imath_{p'}\omega \rangle &= \langle p', \omega' \rangle'' \langle p, \omega'' \rangle \langle p', \omega' \rangle' - \langle p', \omega'' \rangle'' \langle p, \omega' \rangle \langle p', \omega'' \rangle' \\
&= \langle p', \omega' \rangle'' \langle p, \omega'' \rangle' \otimes \langle p, \omega'' \rangle'' \langle p', \omega' \rangle' - \langle p', \omega'' \rangle'' \langle p, \omega' \rangle' \otimes \langle p, \omega' \rangle'' \langle p', \omega'' \rangle'
\end{aligned}$$

which is clearly satisfies (3.1). □

COROLLARY 3.1.2. *Assume that the pairing $\langle -, - \rangle$ is non-degenerate. If $\imath(\omega)$ is surjective then it is an isomorphism.*

PROOF. Let $\alpha = \imath(\omega)$. If α is surjective then its left adjoint $\alpha^* : Q^* \to P^*$ is injective. Since the pairing is non-degenerate we have $Q^* = P$ and $P^* = Q$. Hence we may identify α^* with $-\alpha$. So α is both injective and surjective and hence it is an isomorphism. □

We can now state the following corollary which was asserted in §2.4.

COROLLARY 3.1.3. *Assume that $\omega \in \mathrm{DR}^2(A)$ is such that the map of bimodules*

$$\imath(\omega) : D_A \to \Omega_A : \delta \mapsto \imath_\delta \omega$$

is surjective. Then ω is bi-non-degenerate.

If $\langle -, - \rangle : P \times Q \to A \otimes A$ is non-degenerate then we call a *dual bases* sets of elements $p_\alpha \in P$, $q_\alpha \in Q$ such that $\langle p_\alpha, - \rangle' q_\alpha \langle p_\alpha, - \rangle''$ is the identity map on Q. It is easy to see that this equivalent to $\langle -, q_\alpha \rangle'' p_\alpha \langle -, q_\alpha \rangle'$ being the identity map on P.

PROPOSITION 3.1.4. *Assume that the pairing $\langle -, - \rangle : P \times Q \to A \otimes A$ is non-degenerate. Then the map*

$$\imath : T_A^2 Q \to \mathrm{Hom}_{A^e}(P, Q)$$

defines an isomorphism between $(T_A Q / [T_A Q, T_A Q])_2$ and the anti-symmetric elements of $\mathrm{Hom}_{A^e}(P, Q)$.

PROOF. It is easy to write down an explicit inverse to $\imath$. Let $p_\alpha \in P$, $q_\alpha \in Q$ be dual bases. Then the element of $T_A^2 Q$ corresponding to $\alpha \in \mathrm{Hom}_{A^e}(P, Q)^{\mathrm{asym}}$ is given by $-\frac{1}{2}\alpha(p_\alpha)q_\alpha$. □

We will also have occasion to use the following result

PROPOSITION 3.1.5. *Assume $\langle -, - \rangle$ is a non-degenerate pairing between P and Q.*

(1) *Let $\omega \in (T_A Q / [T_A Q, T_A Q])_n$. If $\imath_p(\omega) = 0$ for all $p \in P$ then $\omega = 0$.*
(2) *Let $\eta \in T^n Q$. If the projection of $i_p(\eta)$ on $A \otimes T^{n-2} Q$ is zero for al $p \in P$ then $\eta = 0$.*

PROOF. We select dual bases p_α, q_α. The following formulas are easily verified for $\omega \in T_A^n Q$

$$q_\alpha \imath_{p_\alpha}(\omega) = n\omega \qquad \mathrm{mod}[-,-]$$

$$(\mathrm{pr}_1(i_{p_\alpha}(\eta)))' q_\alpha (\mathrm{pr}_1(i_{p_\alpha}(\eta)))'' = \eta$$

From this the stated results follow. $\qquad\square$

3.2. Double Lie algebroids. It will be convenient to make the following definition.

DEFINITION 3.2.1. A *double Lie algebroid* over A is an A-bimodule L together with a (graded) double Poisson bracket of degree -1 on $T_A L$.

The archetypical example of a double Lie algebroid is D_A where we equip $T_A D_A = DA$ with its double Schouten bracket (see §2.3).

Assume that L is a double Lie algebroid with double bracket $\{\!\{-,-\}\!\}$. As for DA we have associated operations $\{\!\{-,-\}\!\}_l$, $\{\!\{-,-\}\!\}_r$ which are homomorphisms $L \times L \to L \otimes A$ and $L \times L \to A \otimes L$ respectively that are defined by

$$\{\!\{l_1, l_2\}\!\} = \{\!\{l_1, l_2\}\!\}_l + \{\!\{l_1, l_2\}\!\}_r$$

$\{\!\{-,-\}\!\}_l$ and $\{\!\{-,-\}\!\}_r$ determine each other via

$$\{\!\{l_1, l_2\}\!\}_r = -\{\!\{l_2, l_1\}\!\}_l^\circ$$

It follows that the minimal data necessary to specify a double Lie algebroid is given by

$$\{\!\{-,-\}\!\}_l : L \times L \to L \otimes A$$

$$\{\!\{-,-\}\!\} : L \times A \to A$$

One can write down a minimal set of axioms these operations have to satisfy (see [**14**, (3.4-1)-(3.8-1)]). However it is often more straightforward to use Definition 3.2.1 directly.

3.3. Representation spaces. Let's now discuss how this plays out with representation spaces. if P is an A-bimodule then we define P_N as the A_N (see §2.1)-module generated by symbols p_{ij} which are linear in $p \in P$ and which satisfy for $a \in A$:

$$(ap)_{ij} = a_{iu} p_{uj}$$

$$(pa)_{ij} = a_{uj} p_{iu}$$

In this way we obtain an additive functor

$$(-)_N : \mathrm{Mod}(A^e) \to \mathrm{Mod}(A_N)$$

which sends finitely generated bimodules to finitely generated modules.

This functor has a more intrinsic description as follows.

LEMMA 3.3.1. [**6**] *Consider $M_N(A_N)$ as an A-bimodule via the k-algebra morphism $A \to M_N(A_N) : a \mapsto: a_{ij}$. Consider $M_N(A_N)$ in addition as an A_N-module via the diagonal embedding $A_N \to M_N(A_N)$. Then there is a natural isomorphism*

$$P_N \cong P \otimes_{A^e} M_N(A_N)$$

In particular $(-)_N$ is right exact and sends projective bimodules to projective modules.

PROOF. We will content ourselves by giving the maps. Let f_{ij} be the usual elementary matrix in $M_N(A_N)$. Then one isomorphism is given by

$$P_N \to P \otimes_{A^e} M_N(A_N) : p_{ij} \to p \otimes f_{ji}$$

with the obvious definition for the inverse isomorphism. $\hfill\square$

LEMMA 3.3.2. *We have $(T_A P)_N = S_{A_N} P_N$ (where one the left hand side the symbol $(-)_N$ should be interpreted as acting on the* ring $T_A P$ *and not on the A-bimodule $T_A P$).*

PROOF. It is easy to see that both sides have the same generators and relations. $\hfill\square$

LEMMA 3.3.3. *If $\langle -, - \rangle : P \times Q \to A \otimes A$ is non-degenerate then the corresponding pairing between P_N and Q_N (obtained by applying the standard index convention from §2.3)*

$$\langle -, - \rangle : P_N \otimes Q_N \to A : (p_{ij}, q_{uv}) \mapsto \langle p, q \rangle'_{uj} \langle p, q \rangle''_{iv}$$

is non-degenerate as well.

PROOF. It follows from non-degeneracy that we may select $p_\alpha \in P$, $q_\alpha \in Q$ such that

$$q = \langle p_\alpha, q \rangle' q_\alpha \langle p_\alpha, q \rangle''$$

for all $q \in Q$. It follows

$$
\begin{aligned}
\langle p_{\alpha,ij}, q_{uv} \rangle q_{\alpha,ji} &= \langle p_\alpha, q \rangle'_{uj} \langle p, q_\alpha \rangle''_{iv} q_{\alpha,ji} \\
&= (\langle p_\alpha, q \rangle' q_\alpha \langle p_\alpha, q \rangle'')_{uv} \\
&= q_{uv}
\end{aligned}
$$

and hence the induced map $Q_N \to P_N^* : q_{uv} \mapsto \langle -, q_{uv} \rangle$ is injective and split by $\phi \mapsto q_{\alpha,ji}\phi(p_{\alpha,ij})$. Employing the dual argument we find that it is an isomorphism. $\hfill\square$

PROPOSITION 3.3.4. *Assume that A is smooth. Then the map $D_A \to \operatorname{Der}(A)$ defined by (2.2) yields an isomorphism $(D_A)_N \to \operatorname{Der}(A)$. In particular the tangent space to $\operatorname{Rep}(A, N)$ is generated by the vector fields δ_{ij} for $\delta \in \operatorname{Der}(A, N)$.*

PROOF. It is easy to see that $(\Omega_A)_N = \Omega_{A_N}$. By the previous lemma we find

$$\operatorname{Der}(A_N) \cong \Omega_{A_N}^* \cong ((\Omega_A)_N)^* \cong (D_A)_N$$

It is easy to check that that the actual isomorphism is the asserted one. $\hfill\square$

COROLLARY 3.3.5. *We have $(\Omega A)_N = \bigwedge_{A_N} \Omega_{A_N}$ (where Ω_{A_N} refers to the ordinary commutative differentials) and if A is smooth then $(DA)_N = \bigwedge_{A_N}^* \operatorname{Der}(A_N)$.*

PROOF. The statement about ΩA is easy. The statement about DA follows from Proposition 3.3.4 and Lemma 3.3.2, where we interprete the latter in a graded way, taking into account that D_A and Ω_A have odd degree. $\hfill\square$

Below we sometimes use the map

$$X : P \to M_N(P_N) : p \mapsto (p_{ij})_{ij}$$

For later use we mention some additional result.

LEMMA 3.3.6. *Let* $P, Q, \langle -, - \rangle$ *be as above and let* $p \in P$. *Then the standard index convention (see §2.3) applies to the operator* $\imath_p$. *I.e. if* $\omega \in T_A^2 Q$ *then we have*

$$i_{p_{ij}}(\omega_{uv}) = i_p(\omega)'_{uj} i_p(\omega)''_{iv}$$

In particular we obtain $i_{p_{ij}}(\mathrm{tr}(\omega)) = (\imath_p \omega)_{ij}$ *or in more suggestive notation*

(3.2) $$\mathrm{tr}(\omega)(X(p)) = X(\imath(\omega)(p))$$

where we view $\mathrm{tr}(\omega)$ *as map from* P_N *to* Q_N.

LEMMA 3.3.7. *If* L *is a double Lie algebroid over* A *then* L_N *is a Lie algebroid over* A_N *with Lie bracket and anchor map given by the standard index convention. I.e. for* $l, l_1, l_2 \in L$, $a \in A$

$$[l_{1,ij}, l_{2,uv}] = \{\!\{l_1, l_2\}\!\}'_{uj} \{\!\{l_1, l_2\}\!\}''_{iv}$$

$$\rho(l_{ij})(a_{uv}) = \{\!\{l, a\}\!\}'_{uj} \{\!\{l, a\}\!\}''_{iv}$$

4. Quasi-Poisson and quasi-Hamiltonian G-spaces

Here we summarize some definitions from [**1**, **2**]. Let G be a linear algebraic group and put $\mathfrak{g} = \mathrm{Lie}(G)$. We assume that $\mathfrak{g}$ carries G-invariant symmetric bilinear form $(-, -)$.

Let $(f_a)_a$, $(f^a)_a$ be dual bases of $\mathfrak{g}$. Then there is a canonical invariant element $\phi \in \wedge^3 \mathfrak{g}$ given by

$$\phi = \frac{1}{12} c^{abc} f_a \wedge f_b \wedge f_c$$

where

$$c^{abc} = (f^a, [f^b, f^c])$$

If $\xi \in \mathfrak{g}$ we have left and right invariant vector fields on G defined by

$$(\xi^L(f))(x) = \frac{d}{dt} f(x e^{\xi t})$$

$$(\xi^R(f))(x) = \frac{d}{dt} f(e^{\xi t} x)$$

Assume now that G acts on a smooth affine variety X. If $\xi \in \mathfrak{g}$ then $(\xi_x)_{x \in X}$ is the vector field on X defined by

$$(v_{\xi_x}(f))(x) = \frac{d}{dt} f(e^{-t\xi} x)$$

This convection is such that $\mathfrak{g} \to TX : \xi \mapsto (\xi_x)_x$ is a morphism of Lie algebras.

The element $\phi \in \wedge^3 \mathfrak{g}$ induces a three vector field ϕ_X on X. Following [**1**] an element $P \in \bigwedge^2_{\mathcal{O}(X)} \mathrm{Der}(\mathcal{O}(X))$ is said to be a *quasi-Poisson bracket* if

$$\{P, P\} = \phi_X$$

A *Hamiltonian quasi-Poisson G-space* is a triple (X, P, Φ) such that (X, P) is quasi-Poisson and such that Φ is a so-called *multiplicative moment map*. I.e.

$$\{h \circ \Phi, -\} = \frac{1}{2} f_X^a \left((f_a^L + f_a^R)(h) \circ \Phi \right)$$

for all $f \in \mathcal{O}(G)$.

A Hamiltonian quasi-Poisson G-space is said to be *non-degenerate* if for $x \in X$ the map

$$T_x^* \oplus \mathfrak{g} \mapsto T_x : (\eta, \xi) \mapsto P_x(\eta) + \xi_x$$

is surjective.

Now we let θ and $\bar{\theta}$ be respectively the left and right invariant $\mathfrak{g}$-valued Maurer-Cartan forms:

$$\theta = g^{-1}dg \qquad \bar{\theta} = dg \cdot g^{-1}$$

We let χ be the canonical G-invariant three form on G:

$$\chi = \frac{1}{12}(\theta, [\theta, \theta])$$

A quasi-Hamiltonian G-variety is a triple (X, ω, Φ) where M is a smooth G-variety, $\omega \in (T^{*,2})^G$ and $\Phi : X \to G$ is a G-equivariant map (for the given action on X and the adjoint action of G on G) such that the following axioms are satisfied:

(B1) $d\omega = \Phi^*\chi$.[1]
(B2) $\forall \xi \in \mathfrak{g} : i_{\xi_X}(\omega) = \frac{1}{2}\Phi^*(\theta + \bar{\theta}, \xi)$.
(B3) For all x in X we have

$$\ker \omega_x = \{\xi_x \mid \xi \in \ker(\mathrm{Ad}_{\Phi(x)} +1) \subset \mathfrak{g}\}$$

Let us explain how to read (B3). By definition ω_x is the map

$$T_X \to T_X^* : \delta \mapsto i_\delta(\omega)$$

evaluated in x. For $\xi \in \mathfrak{g}$, ξ_x is the vector field on X given by ξ evaluated in x. (B3) states that $\ker \omega_x$ is a specific part of the image of the map $\mathfrak{g} \to T_{X,x} : \xi \mapsto \xi_x$. As the form of condition (B3) is not so convenient for us so we state an equivalent version.

(B3') The map

$$T_x \oplus \mathfrak{g} \to T_x^* : (\delta, \xi) \mapsto \omega_x(\delta) + (\xi, \Phi^*(\theta)_x)$$

is surjective.

LEMMA 4.1. *(B3) and (B3') are equivalent.*

PROOF. (Sketch) We first dualize (B3'). Let $\beta : T_x \to \mathfrak{g} : \delta \mapsto i_\delta(\Phi^*(\theta))$. Then (B3') is equivalent to the condition

(B3") The map $(\omega_x, \beta) : T_x \to T_x^* \oplus \mathfrak{g}$ is injective.

It is a straightforward verification using (B2) that the following diagram

(4.1)
$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\alpha} & \mathfrak{g} \\ \beta \uparrow & & \uparrow \gamma \\ T_x & \xrightarrow[\omega_x]{} & \Omega_x \end{array}$$

with $\alpha(\xi) = -\frac{1}{2}(1 + \mathrm{Ad}(\Phi(x)))(\xi)$ and $(\gamma(\eta), \xi) = i_{\xi_x}(\eta)$ is commutative, Furthermore using the properties of θ, $\bar{\theta}$ (see e.g [**9**, Ch II]) we find $\beta(\xi_x) = (1 - \mathrm{Ad}(\Phi(x))^{-1})(\xi)$.

Let us show that (B3) implies (B3"). Assume that δ is such that $i_\delta(\omega_x) = 0$ and $i_\delta(\Phi^*(\theta)) = 0$. By (B3) we have that $\delta = \xi_x$ where $(1 + \mathrm{Ad}(\Phi(x)))(\xi) = 0$.

---

[1]We follow the convention from [**1**, Def 10.1]. In [**2**] this formula is given with a minus sign

Since $\beta(\xi_x) = 0$ we also have $(1 - \mathrm{Ad}(\Phi(x)))(\xi) = 0$ by diagram (4.1). These two facts together imply that $\xi = 0$.

Now we prove the converse. Assume that $\delta \in \ker(\omega_x)$. Then $\xi = \beta(\delta) \in \ker\alpha$. I.e.

$$(4.2) \qquad\qquad (1 + \mathrm{Ad}(\Phi(x)))(\xi) = 0$$

Then

$$\beta(\xi_x) = (1 - \mathrm{Ad}(\Phi(x)^{-1})(\xi) = 2\xi$$

Since $(1 + \mathrm{Ad}(\Phi(x)))(\xi) = 0$ we also have $\omega_x(\xi_x) = 0$.

It follows that $= \delta - \frac{1}{2}\xi_x$ is both in the kernel of β an ω_x. Hence by (B3") $\delta = \frac{1}{2}\xi_x$. (B3) now follows from (4.2). $\qquad\square$

We now state the main theorem of [**1**, §10]. Write $\theta = f^a\theta_a$ where $\theta_a \in T_X^*$ and similarly for $\bar\theta$.

THEOREM 4.2. [**1**, Thm 10.3] *Every non-degenerate Hamiltonian quasi-Poisson space (X, P, Φ) carries a unique 2-form ω such that (X, ω, Φ) is a quasi-Hamiltonian G-space and such that ω and P satisfy the following compatibility condition*

$$(4.3) \qquad\qquad P \circ \omega = 1 - \frac{1}{4}f_X^a \otimes \Phi^*(\theta_a - \bar\theta_a)$$

(as maps $T_X^* \to T_X$). Conversely on every quasi-Hamiltonian G-space (X, ω, Φ) there is a unique bivector field P such that (M, X, Φ) is a non-degenerate quasi-Hamiltonian G-manifold and (4.3) is satisfied.

We will say that (P, ω, Φ) are compatible if (4.3) holds.

5. Hamiltonian double quasi-Poisson algebras

The non-commutative version of quasi-Poisson algebras was worked out in [**14**].

CONVENTION 5.1. *From now on our non-commutative algebras are always smooth.*

Let $P \in (DA/[DA, DA])_2$. We say that P is a *double quasi-Poisson bracket* if the following condition holds

(ℙ1) $\{P, P\} = \frac{1}{12}E^3 \qquad \mathrm{mod}[-, -]$.

In addition an invertible element $\Phi \in A$ is said to be a *multiplicative moment map* if the following condition holds.

(ℙ2) $\{\Phi, -\} = \frac{1}{2}(\Phi E + E\Phi)$.

Finally we say that P is *non-degenerate* if the following condition holds

(ℙ3) The map $\Omega_A \oplus AEA \to D_A : (\eta, \delta) \mapsto \imath(P)(\eta) + \delta$ is surjective.

It was proved in [**14**] that (ℙ1) and (ℙ2) imply the corresponding properties on representation spaces. The same is also true for (ℙ3) as the following proposition shows.

PROPOSITION 5.2. *Assume that (ℙ1)(ℙ3) hold. Then $\mathrm{tr}(P)$ defines a non-degenerate quasi-Poisson bracket on $\mathrm{Rep}(A, N)$*

PROOF. Applying the right exact functor $(-)_N$ (lemma 3.3.1) and using Proposition 2.3.1 together with Corollary 3.3.5 we find that the map

$$\Omega_{A_N} \oplus A_N \otimes \mathfrak{gl}_N \to \mathrm{Der}(A_N)$$

is surjective. This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following result is a non-commutative version of [**4**, Thm. 2.5].

THEOREM 5.3. *Assume that (A, P) is a double quasi-Poisson algebra. Then $\tilde{\Omega}_A = \Omega_A \oplus AEA$ has the structure of a double Lie algebroid where the double bracket is defined as follows.*

$$\{\!\!\{da, b\}\!\!\}_{\tilde{\Omega}_A} = \{\!\!\{a, b\}\!\!\}$$

$$\{\!\!\{da, db\}\!\!\}_{\tilde{\Omega}_A} = d\{\!\!\{a, b\}\!\!\} + \frac{1}{4}[b, [a, E \otimes 1 - 1 \otimes E]_*]$$

$$\{\!\!\{E, X\}\!\!\}_{\tilde{\Omega}_A} = X \otimes 1 - 1 \otimes X$$

for $a, b \in A$, $X \in T_A \tilde{\Omega}_A$ and where $[-, -]_*$ denotes the commutator for the inner A-bimodule structure on $AEA \otimes AEA$. Furthermore the map

$$\Omega_A \oplus AEA \to D_A$$

defined in (ℙ3) is a morphism of double Lie algebroids.

Our proof of this theorem is a rather painful direct computation. We omit the details. The fact that (ℙ3) defines a morphism of double Lie algebroids translates into the following proposition which is an analogue of [**14**, Prop. 3.5.1].

PROPOSITION 5.4. *The following are equivalent*
  (1) $\{\!\!\{-, -\}\!\!\}$ *is a double quasi-Poisson bracket.*
  (2) *The following identity holds for all $a, b \in A$:*

$$\{\!\!\{H_a, H_b\}\!\!\}_l - H'_{\{\!\!\{a,b\}\!\!\}} \otimes \{\!\!\{a, b\}\!\!\}'' = \frac{1}{4}[b, [a, E \otimes 1]_*]$$

  (3) *The following identity holds for all $a, b \in A$:*

$$\{\!\!\{H_a, H_b\}\!\!\}_r - \{\!\!\{a, b\}\!\!\}' \otimes H_{\{\!\!\{a,b\}\!\!\}''} = -\frac{1}{4}[b, [a, 1 \otimes E]_*]$$

  (4) *The following identity holds for all $a, b \in A$:*

$$\{\!\!\{H_a, H_b\}\!\!\} - H_{\{\!\!\{a,b\}\!\!\}} = \frac{1}{4}[b, [a, E \otimes 1 - 1 \otimes E]_*]$$

  where we use the convention $H_{x' \otimes x''} = H_{x'} \otimes x'' + x' \otimes H_{x''}$.

PROOF. Below it will be convenient to use the notation $^\circ(-)$ introduced in (2.5) as well as the convention

$$(a_1 \otimes \cdots \otimes a_m)(b_1 \otimes \cdots \otimes b_n) = a_1 \otimes \cdots \otimes a_m b_1 \otimes \cdots b_n$$

and similarly for longer products. According to the proof of [**14**, Prop. 3.5.1] we have for $a, b, c \in A$

$$\{\!\!\{a, b, c\}\!\!\} = \tau_{23}((\{\!\!\{H_a, H_b\}\!\!\}_l - H_{\{\!\!\{a,b\}\!\!\}'}(-) \otimes \{\!\!\{a, b\}\!\!\}'')(c))$$

Also according to [**14**, §5] a bracket is quasi-double Poisson if it satisfies

$$\{\!\!\{-, -, -\}\!\!\} = \frac{1}{12}\{\!\!\{-, -, -\}\!\!\}_{E^3}$$

By the formulas in [**14**, §4] it follows

$$\{\!\!\{-,-,-\}\!\!\}_{E^3} = 3°(E(a)°E(b)°E(c)°)$$

Hence

(5.1) $$(\{\!\!\{H_a, H_b\}\!\!\}_l - H_{\{\!\!\{a,b\}\!\!\}'}(-) \otimes \{\!\!\{a,b\}\!\!\}'')(c) = \frac{1}{4}\tau_{23}°(E(a)°E(b)°E(c)°)$$

A straightforward verification yields

$$(E(a)°E(b)°E(-)°) = [b, [a, E \otimes 1]_*](c)$$

This proves the equivence between (1) and (2). The equivalence between (2) and (3) follows easily from the fact that $\{\!\!\{a,b\}\!\!\}_r = -\{\!\!\{b,a\}\!\!\}_l°$. The sum of (2) and (3) yields (4). To go back we use projection. □

6. Quasi-bisymplectic algebras

The algebras we will introduce are a non-commutative analogue of quasi-Hamiltonian G-spaces introduced in [**2**] (see §4). By definition a *quasi-bisymplectic algebra* will be a triple (A, ω, Φ) where $\omega \in \mathrm{DR}^2(A)$ and $\Phi \in A^*$ satisfying the following conditions

(𝔹1) $d\omega = \frac{1}{6}(\Phi^{-1}d\Phi)^3 \quad \mathrm{mod}[-,-]$.
(𝔹2) $\imath_E \omega = \frac{1}{2}(\Phi^{-1}d\Phi + d\Phi \cdot \Phi^{-1})$
(𝔹3) The map

$$D_A \oplus Ad\Phi A \to \Omega_A : (\delta, \eta) \mapsto \imath(\omega)(\delta) + \eta$$

is surjective.

PROPOSITION 6.1. *If (A, ω, Φ) is a quasi-bisymplectic algebra then $(\mathrm{Rep}(A, N), \mathrm{tr}(\omega), X(\Phi))$ is a quasi-Hamiltonian Gl_N-space.*

PROOF. It is easy to see that we have

(6.1) $$X(\Phi^{-1}d\Phi) = X(\Phi)^*(\theta) \qquad X(d\Phi \cdot \Phi^{-1}) = X(\Phi)^*(\bar{\theta})$$

and

(6.2) $$X(\Phi)^*(\chi) = \frac{1}{6}\mathrm{tr}(\Phi^{-1}d\Phi)^3$$

This implies (B1). To check (B2) we test it with $\xi = f_{ji}$. Then according to lemma 3.3.6 and Prop. 2.3.1 we have

$$i_{f_{ji}}(\mathrm{tr}(\omega)) = (\imath_E \omega)_{ij}$$

We also have

$$X(\Phi)^*(\theta + \bar{\theta}, f_{ji}) = (\Phi^{-1}d\Phi + d\Phi \cdot \Phi^{-1})_{ij}$$

so that we see that (𝔹2) implies (B2).

Now we consider (B3) (or rather its variant (B3')). Apply the functor $(-)_N$ to (𝔹3). Using Corollary 3.3.5 we find that the map

$$\mathrm{Der}(A_N) \oplus \sum_{ij} A_N(d\Phi)_{ij} \to \Omega_{A_N}$$

is surjective. But the forms $(d\Phi)_{ij}$ generate the same A_N-module as $X(\Phi)^*(\theta)_{ij} = (\Phi^{-1}d\Phi)_{ij}$, finishing the proof. □

REMARK 6.2. Note that the appearance of the element $(\Phi^{-1}d\Phi)^3 \in \mathrm{DR}^3(A)$ is rather natural in view of Example 2.1. This is the only non-zero element in $H^3(\mathrm{DR}(A))$ that can be constructed from the single element Φ.

Let us say that $P \in D^2 A$, $\omega \in \Omega^2 A$, $\Phi \in A^*$ are *compatible* if the following identity holds for all $\delta \in D_A$:

$$(\mathbb{C}) \qquad (\imath(P) \circ i(\omega))(\delta) = \delta - \frac{1}{4}\delta(\Phi)''(E\Phi^{-1} - \Phi^{-1}E)\delta(\Phi)'$$

PROPOSITION 6.3. *If P, ω, Φ are compatible then $(\mathrm{tr}(P), \mathrm{tr}(\omega), X(\Phi))$ are compatible as well (see (4.3)).*

PROOF. We apply the map $X(-)$. Using Cor. 3.3.5, (3.2) and Prop. 2.3.1 we find

$$(\mathrm{tr}(P) \circ \mathrm{tr}(\omega))(\delta_{ij}) = \delta_{ij} - \frac{1}{4}\delta(\Phi)''_{iu}(f_{vu}(\Phi^{-1})_{vw} - (\Phi^{-1})_{uv}f_{wv})\delta(\Phi)'_{wj}$$

$$= \delta_{ij} - \frac{1}{4}(f_{vu}(\Phi^{-1})_{vw} - (\Phi^{-1})_{uv}f_{wv})\delta(\Phi)'_{wj}\delta(\Phi)''_{iu}$$

$$= \delta_{ij} - \frac{1}{4}(f_{vu}(\Phi^{-1})_{vw} - (\Phi^{-1})_{uv}f_{wv})\delta_{ij}(\Phi_{wu})$$

$$= \delta_{ij} - \frac{1}{4}(f_{vu}(\Phi^{-1})_{vw}i_{\delta_{ij}}(d\Phi_{wu}) - i_{\delta_{ij}}(d\Phi_{wu})(\Phi^{-1})_{uv}f_{wv})$$

$$= (\mathrm{id} - \frac{1}{4}f_{vu}((\Phi^{-1}d\Phi)_{vu} - (d\Phi \cdot \Phi^{-1})_{vu}))(\delta_{ij})$$

where we have viewed the f_{ij} as vector fields on $\mathrm{Rep}(A, N)$. We are now done by (6.1). $\qquad\square$

7. Compatibility

Our aim is to prove a non-commutative analogue of Theorem 4.2.

THEOREM 7.1. *Fix $\Phi \in A^*$.*
  (1) *For every $P \in (DA/[DA, DA])_2$ satisfying ($\mathbb{P}$2)($\mathbb{P}$3) there exists a unique $\omega \in (\Omega A/[\Omega A, \Omega A])_2$ satisfying ($\mathbb{B}$2)($\mathbb{B}$3) and ($\mathbb{C}$).*
  (2) *For every $\omega \in (\Omega A/[\Omega A, \Omega A])_2$ satisfying ($\mathbb{B}$2)($\mathbb{B}$3) there exists a unique $P \in (DA/[DA, DA])_2$ satisfying ($\mathbb{P}$2)($\mathbb{P}$3) and ($\mathbb{C}$).*
  (3) *If ω, P correspond to one another as in (1)(2) then the integrability conditions ($\mathbb{B}$1) and ($\mathbb{P}$1) are equivalent.*

We will prove this theorem in this section. Throughout we fix $\Phi \in A^*$ and we let $\omega \in (\Omega A/[\Omega A, \Omega A]))_2$, $P \in (DA/[DA, DA]))_2$ be such that ($\mathbb{B}$2) and ($\mathbb{P}$2) are satisfied. First we consider the following diagram which summarizes a number of relevant maps which we will use below.

$$(7.1)$$

$$
\begin{array}{ccccccc}
\Omega_A & \xrightarrow{e} & AE^*A & \xrightarrow{T^0} & Ad\Phi A & \xrightarrow{c} & \Omega_A \\
{\scriptstyle \imath(P)}\downarrow & & {\scriptstyle \jmath}\downarrow & & {\scriptstyle \imath}\downarrow & & \downarrow{\scriptstyle \imath(P)} \\
D_A & \xrightarrow[e]{} & A(d\Phi)^*A & \xrightarrow[S^0]{} & AEA & \xrightarrow[c]{} & D_A \\
{\scriptstyle \imath(\omega)}\downarrow & & {\scriptstyle \jmath}\downarrow & & {\scriptstyle \imath}\downarrow & & \downarrow{\scriptstyle \imath(\omega)} \\
\Omega_A & \xrightarrow[e]{} & AE^*A & \xrightarrow[T^0]{} & Ad\Phi A & \xrightarrow[c]{} & \Omega_A
\end{array}
$$

The following conventions are used: first of all we view AE^*A, $A(d\Phi)^*A$, AEA and $Ad\Phi A$ as free bimodules with one generator. Furthermore we regard the diagram as being doubly infinite with period $(3,2)$. The bimodules occur in pairs related by an obvious non-degenerate pairing: (D_A, Ω_A), (AE^*A, AEA) and $(A(d\Phi)^*A, A(d\Phi)A)$. We now define the maps.

(1) c stands for "canonical map".

(2) e is adjoint to c. The formulas for the two variants are as follows.

$$e(d\phi) = \phi E^* - E^*\phi$$
$$e(\delta) = \delta(\Phi)''(d\Phi)^*\delta(\Phi)'$$

(3) $\imath$ is the restricted version of $\imath(P)$ and $\imath(\omega)$. The two variants are given by $(\mathbb{P}2)$ and $(\mathbb{B}2)$. I.e. explicitly

$$\imath(d\Phi) = \frac{1}{2}(E\Phi + \Phi E)$$
$$\imath(E) = \frac{1}{2}(\Phi^{-1}d\Phi + d\Phi \cdot \Phi^{-1})$$

(4) $\jmath$ is adjoint to $-\imath$. The two variants are given by the following formulas

$$\jmath(d\Phi^*) = -\frac{1}{2}(\Phi^{-1}E^* + E^*\Phi^{-1})$$
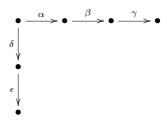$$\jmath(E^*) = -\frac{1}{2}(\Phi(d\Phi)^* + (d\Phi)^*\Phi)$$

(5) The maps S^0 and T^0 are adjoint to one another. They are respectively given by the following formulas.

$$S^0((d\Phi)^*) = E\Phi^{-1} - \Phi^{-1}E$$
$$T^0(E^*) = \Phi^{-1}d\Phi - d\Phi\Phi^{-1}$$

From the stated adjointness properties it follows that the (7.1) is self dual, up to sign. We have the following fact.

LEMMA 7.2. (1) *The diagram* (7.1) *is commutative.*
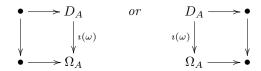(2) *Consider the diagram as doubly infinite. For any configuration of arrows*

such that the vertical arrows do not involve $\imath(\omega), \imath(P)$ *we have*

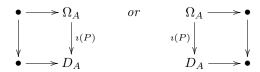(7.2) $$\frac{1}{4}\gamma\beta\alpha + \epsilon\delta = 1$$

(3) *If we contract in* (7.1) *the pairs of horizontal consecutive arrows that have* Ω_A *or* D_A *in the middle then the* 2×2 *subdiagrams in the resulting diagram are bicartesian.*

(4) *If (B3) holds then any subdiagram*

$$
\begin{array}{ccc}
\bullet \longrightarrow D_A & \quad or \quad & D_A \longrightarrow \bullet \\
\downarrow \quad\quad \downarrow {\scriptstyle\imath(\omega)} & & {\scriptstyle\imath(\omega)}\downarrow \quad\quad \downarrow \\
\bullet \longrightarrow \Omega_A & & \Omega_A \longrightarrow \bullet
\end{array}
$$

of (7.1) *is bicartesian.*

(5) *Similarly if (P3) holds then any subdiagram*

$$
\begin{array}{ccc}
\bullet \longrightarrow \Omega_A & \quad or \quad & \Omega_A \longrightarrow \bullet \\
\downarrow \quad\quad \downarrow {\scriptstyle\imath(P)} & & {\scriptstyle\imath(P)}\downarrow \quad\quad \downarrow \\
\bullet \longrightarrow D_A & & D_A \longrightarrow \bullet
\end{array}
$$

of (7.1) *is bicartesian.*

REMARK 7.3. To avoid confusion: statements in the above lemma which do not refer to $\imath(\omega)$ or $\imath(P)$ remain true if these maps are not present in the diagram.

PROOF. We leave the verification of (1) and (2) to the reader. For (3) will only give an example. E.g. we need to check in particular that the following diagram is bicartesian

$$
\begin{array}{ccc}
Ad\Phi A & \xrightarrow{\ ec\ } & AE^*A \\
{\scriptstyle\imath}\downarrow & & \downarrow{\scriptstyle\jmath} \\
AEA & \xrightarrow[\ ec\ ]{} & A(d\Phi)^*A
\end{array}
$$

Put $R = A^e$, $\Phi_1 = \Phi \otimes 1$, $\Phi_2 = 1 \otimes \Phi$. The latter are two commuting invertible elements of R. Removing all confusing decoration the above diagram basically looks like

$$
\begin{array}{ccc}
R & \xrightarrow{\ \Phi_1-\Phi_2\ } & R \\
{\scriptstyle\Phi_1+\Phi_2}\downarrow & & \downarrow{\scriptstyle\Phi_1+\Phi_2} \\
R & \xrightarrow[\ \Phi_1-\Phi_2\ ]{} & R
\end{array}
$$

It is now clear that this diagram is bicartesian (using the fact that Φ_1 and Φ_2 are invertible). The other 2×2-diagrams in (7.1) are similar.

(4) and (5) are similar. To prove (4) we only need to consider the first diagram as the second follows by duality. Assume that there are $\delta \in D_A$, $\eta \in Ad\Phi A$ such that

$$
(7.3) \qquad\qquad\qquad \imath(\omega)(\delta) = c(\eta)
$$

We need to prove that there is a unique $\Delta \in AEA$ such that $c(\Delta) = \delta$, $\imath(\Delta) = \eta$. We need to worry only about existence since (2) implies that

$$
(ec, \imath) : AEA \to A(d\Phi)^*A \oplus Ad\Phi A
$$

is injective and hence $(c, \imath)$ is also injective.

Applying e to (7.3) we find $\jmath e(\delta) = e\imath(\omega)(c) = ec(\eta)$. By (3) there is an element $\delta' \in AEA$ such that $\imath(\delta') = \eta$, $ec(\delta') = e(\delta)$. Replacing η by $\eta - \imath(\delta')$, δ by $\delta - c(\delta')$ we reduce to the case $\eta = 0$, $e(\delta) = 0$, $\imath(\omega)(\delta) = 0$.

Now we note that ($\mathbb{B}3$) implies by duality that the following map is injective

$$(\imath(\omega), e) : D_A \to \Omega_A \oplus A(d\Phi)^* A$$

Hence $\delta = 0$ and we can take $\Delta = 0$. $\square$

Using the maps in (7.1) the compatibility condition ($\mathbb{C}$) can be reformulated as

$$(7.4) \qquad \imath(P)\imath(\omega) = 1 - \frac{1}{4}S$$

where $S = cS^0e$. In fact it will be convenient to give some other equivalent formulations of (7.4). Consider the following matrices

$$\bar{\omega} = \begin{pmatrix} \imath(\omega) & c \\ \frac{1}{4}S^0e & -\imath \end{pmatrix} \qquad \bar{P} = \begin{pmatrix} \imath(P) & c \\ \frac{1}{4}T^0e & -\imath \end{pmatrix}$$

and view them as maps between $D_A \oplus Ad\Phi A$ and $\Omega_A \oplus AEA$.

Furthermore consider the matrices

$$\tilde{\omega} = \begin{pmatrix} \imath(\omega) & 1 \\ \frac{1}{4}S & -\imath(P) \end{pmatrix} \qquad \tilde{P} = \begin{pmatrix} \imath(P) & 1 \\ \frac{1}{4}T & -\imath(\omega) \end{pmatrix}$$

and view them as maps between $D_A \oplus \Omega_A$ and $\Omega_A \oplus D_A$.

PROPOSITION 7.4. *The following conditions are all equivalent.*

$$(7.5) \qquad \imath(P)\imath(\omega) = 1 - \frac{1}{4}S$$

$$(7.6) \qquad \imath(\omega)\imath(P) = 1 - \frac{1}{4}T$$

$$(7.7) \qquad \bar{P}\bar{\omega} = \mathrm{id}$$

$$(7.8) \qquad \bar{\omega}\bar{P} = \mathrm{id}$$

$$(7.9) \qquad \tilde{P}\bar{\omega} = \mathrm{id}$$

$$(7.10) \qquad \tilde{\omega}\bar{P} = \mathrm{id}$$

PROOF. (7.5) and (7.6) are equivalent since they are adjoint. To understand (7.7) we compute the product

$$\bar{P}\bar{\omega} = \begin{pmatrix} \imath(P) & c \\ \frac{1}{4}T^0e & -\imath \end{pmatrix} \begin{pmatrix} \imath(\omega) & c \\ \frac{1}{4}S^0e & -\imath \end{pmatrix} = \begin{pmatrix} \imath(P)\imath(\omega) + \frac{1}{4}S & \imath(P)c - c\imath \\ \frac{1}{4}(T^0e\imath(\omega) - \imath S^0e) & \frac{1}{4}T^0ec + \imath\imath \end{pmatrix} = \begin{pmatrix} \imath(P)\imath(\omega) + \frac{1}{4}S & 0 \\ 0 & 1 \end{pmatrix}$$

where we have used lemma 7.2. It is now clear that (7.5) and (7.7) are equivalent. Similarly (7.6) and (7.8) are equivalent. To understand (7.9) we write the product out again explicitly.

$$\tilde{P}\tilde{\omega} = \begin{pmatrix} \imath(P) & 1 \\ \frac{1}{4}T & -\imath(\omega) \end{pmatrix} \begin{pmatrix} \imath(\omega) & 1 \\ \frac{1}{4}S & -\imath(P) \end{pmatrix} = \begin{pmatrix} \imath(P)\imath(\omega) + \frac{1}{4}S & 0 \\ 0 & \imath(\omega)\imath(P) + \frac{1}{4}T \end{pmatrix}$$

where we have used lemma 7.2 again. It follows that (7.9) is equivalent to (7.5)(7.6) simultaneously. The same holds for (7.9). $\square$

PROOF OF THEOREM 7.1(1). If $\imath(\omega)$ exists it will have the following two properties for $\eta \in \Omega_A$, $\delta \in AEA$

$$(7.11) \qquad \imath(\omega)\imath(P)(\eta) = \eta - \frac{1}{4}T(\eta)$$

$$(7.12) \qquad \imath(\omega)c(\delta) = c\imath(\delta)$$

where we have used lemmas 7.2 and Prop. 7.4. Since we are assuming ($\mathbb{P}3$) every element of D_A can be written as a sum $\imath(P)(\eta) + c(\delta)$ for $\eta \in \Omega_A$ and $\delta \in AEA$. Hence the properties (7.11)(7.12) characterize $\imath(\omega)$ uniquely (and ω as well by Prop. 3.1.4).

We still need to prove two things.

(1) The equations (7.11)(7.12) are non-contradictory.
(2) The resulting $\imath(\omega)$ is actually anti-symmetric (so that it genuinely comes from an element $\omega \in \Omega_A^2$).
(3) $\imath(\omega)$ satisfies ($\mathbb{B}3$).

We discuss (1) first. Suppose that $\imath(P)(\eta) = c(\delta)$. Then by lemma 7.1 there exist $\eta_0 \in Ad\Phi_p A$ such that $\eta = c(\eta_0)$, $\delta = \imath(\eta_0)$. Then

$$\eta - \frac{1}{4}T(\eta) = c(\eta_0) - \frac{1}{4}cT^0ec(\eta_0)$$
$$= c(\eta_0 - \frac{1}{4}T^0ec(\eta_0))$$
$$= c\imath(\eta_0)$$
$$= c\imath(\delta)$$

where we have used lemma 7.1.

Now we prove (2). To avoid confusion we write X for the map $\imath(\omega)$ we have constructed. Thus we have

(7.13)
$$X\imath(P) = 1 - \frac{1}{4}T$$
$$Xc = c\imath$$

Dualizing these equations we get

(7.14)
$$-\imath(P)X^* = 1 - \frac{1}{4}S$$
$$eX^* = -\jmath e$$

We write (7.13) in matrix form.

$$X(\imath(P)\ c) = (1 - \frac{1}{4}T\ c\imath)$$

Left multiplication by $(i(P) - e)^t$ yields

$$\begin{pmatrix} i(P) \\ -e \end{pmatrix} X(\imath(P)\ c) = \begin{pmatrix} \imath(P) \\ -e \end{pmatrix}(1 - \frac{1}{4}T\ c\imath)$$
$$= \begin{pmatrix} 1 - \frac{1}{4}S \\ -\jmath e \end{pmatrix}(\imath(P)\ c)$$

Since $(\imath(P)\ c)$ is injective we conclude

$$\imath(P)(X) = 1 - \frac{1}{4}S$$
$$-eX = -\jmath e$$

Comparing with (7.14), and using that $(\imath(P) - e)$ is surjective we conclude that indeed $X^* = -X$.

We finish the proof by noting that (3) follows immediately from (7.6). $\square$

REMARK 7.5. Assume that P and ω are compatible. We have unearthed quite a bit of structure in the diagram (7.1). I have not seen this type of structure at other places.

(1) The diagram is $(3,2)$ periodic.
(2) Denote the horizontal and vertical maps by h and v respectively. h and v satisfy the following equations:

$$hv = vh$$

$$v^2 + \frac{1}{4}h^3 = 1$$

(3) Every 2×2 subdiagram is bicartesian.
(4) The diagram is self dual up to a sign which must be applied to the vertical maps.

REMARK 7.6. If we view the maps $\tilde{\omega}$ and $\tilde{P}$ as endomorphisms of $D_A \oplus \Omega_A$ then in matrix form they look like

$$\tilde{\omega} = \begin{pmatrix} \frac{1}{4}S & -\imath(P) \\ \imath(\omega) & 1 \end{pmatrix} \qquad \tilde{P} = \begin{pmatrix} 1 & \imath(P) \\ -\imath(\omega) & \frac{1}{4}T \end{pmatrix}$$

We define a symmetric non-degenerate pairing between $D_A \oplus \Omega_A$ and itself using the formula

$$\langle (\delta, \eta), (\delta', \eta') \rangle = \langle \delta, \eta' \rangle + \langle \delta', \eta \rangle$$

For this pairing it is easy to check that $\tilde{\omega}$ is adjoint to $\tilde{P}$. In other words $\tilde{\omega}$ is a unitary transformation of $D_A \oplus \Omega_A$. This suggest a connection with a non-commutative version of Dirac geometry (yet to be created). In the commutative case this connection is well understood [**4**].

Part (2) or Theorem 7.1 is proved in the same way as (1). It remains to prove (3), i.e. the equivalence between the integrability conditions on P and ω.

To do this we may follow a similar method as in [**14**, Appendix]. We start by expressing the integrability condition in terms of Hamiltonian vector fields (see §2.3). Throughout we assume that P, ω, Φ satisfy $(\mathbb{P}2)(\mathbb{P}3)(\mathbb{B}2)(\mathbb{B}3)$ as well as the compatibility condition $(\mathbb{C})$. We let $\{\!\{-,-\}\!\}$ be the double bracket on A induced by P (see (2.4)). We recall that by Proposition 5.4 P is quasi-Poisson if and only if for all $a, b \in A$ we have the following identity in $D_A \otimes A$

$$(7.15) \qquad \{\!\{H_a, H_b\}\!\}_l - H_{\{\!\{a,b\}\!\}'}(-) \otimes \{\!\{a,b\}\!\}'' = \frac{1}{4}[b, [a, E \otimes 1]_*]$$

In order to work with Hamiltonian vector fields one needs some formulas which are straightforward verifications

LEMMA 7.7. *One has*

$$i_{H_a}(\omega) = da - \frac{1}{4}[a, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}]$$

$$i_{H_a}(d\Phi) = -\frac{1}{2}[a, \Phi \otimes 1 + 1 \otimes \Phi]_*$$

PROOF. The first formula follows from the following computation

$$\imath(\omega)(H_a) = \imath(\omega)\imath(P)(da)$$
$$= (1 - \frac{1}{4}T)(da)$$
$$= da - \frac{1}{4}\sum_p [a, \Phi^{-1}d\Phi - d\Phi\Phi^{-1}]$$

Now we prove the second formula We have

$$i_{H_a}(d\Phi) = H_a(\Phi) = \{\!\{a, \Phi\}\!\} = -\{\!\{\Phi, a\}\!\}^\circ$$

and according to [**14**, Def 5.1.4]

$$-\{\!\{\Phi, a\}\!\}^\circ = -\frac{1}{2}((\Phi E + E\Phi)(a))^\circ$$

We have

$$(uEv)(a) = av \otimes u - v \otimes ua$$

and thus

$$(uEv)(a)^\circ = u \otimes av - ua \otimes v$$

so that we get

$$i_{H_a}(d\Phi) = -\frac{1}{2}(\Phi \otimes a - \Phi a \otimes 1 + 1 \otimes a\Phi - a \otimes \Phi)$$

as asserted. $\qquad\qquad\square$

Using these formulas and a tedious computation one verifies the following lemma

LEMMA 7.8. *The formula* (7.15) *always holds when evaluated on* $d\Phi \otimes 1$.

From this lemma we deduce

LEMMA 7.9. *The element* P *defines a quasi-Poisson algebra if and only if the following identity in* $\Omega_A \otimes A$ *holds*

$$(7.16) \quad \imath_{\{\!\{H_a, H_b\}\!\}'_l}(\omega) \otimes \{\!\{H_a, H_b\}\!\}''_l - \imath_{H_{\{\!\{a,b\}\!\}'}}(\omega) \otimes \{\!\{a, b\}\!\}'' = \frac{1}{4}[b, [a, \imath_E(\omega) \otimes 1]_*]$$

PROOF. By lemma 7.2 we have that $(\imath(\omega), e) : D_A \to \Omega_A \oplus A(d\Phi)^*A$ is injective. Hence to check that $\delta \in D_A$ is zero it is sufficient to verify that $\imath_\delta(\omega) = 0$ and $e(\delta) = \delta(\Phi)''(d\Phi)^*\delta(\Phi)' = 0$. The latter holds if $\delta(\Phi) = 0$. The lemma now follows by writing (7.15) as $\delta' \otimes \delta'' = 0$ with $\delta' \in D_A$ and using Lemma 7.8. $\qquad\square$

To compute the part of (7.16) involving $\{\!\{H_a, H_b\}\!\}_l$ we may use the formula [**14**, (A.6)].

$$i_{H_a}\mathcal{L}_{H_b} - \tau_{12}L_{H_b}i_{H_a} = \imath_{\{\!\{H_a, H_b\}\!\}'_l} \otimes \{\!\{H_a, H_b\}\!\}''_l + \{\!\{H_a, H_b\}\!\}'_r \otimes \imath_{\{\!\{H_a, H_b\}\!\}''_r}$$

Here $L_\delta = di_\delta + i_\delta d$ and $\mathcal{L}_\delta = {}^\circ L_\delta$. Applying the left hand side of this formula to ω we see that everything is computable except the term $i_{H_a}\imath_{H_b}d\omega$ appearing as part of $i_{H_a}\mathcal{L}_{H_b}(\omega)$. After a long and tedious computation we arrive at the following lemma.

LEMMA 7.10. *The formula* (7.16) *holds if and only if*

$$pr_1 \, i_{H_a} i_{H_b} d\omega = \frac{1}{6} \, pr_1 \, i_{H_a} i_{H_b} (\Phi^{-1} d\Phi)^3$$

holds for al a, b.

PROOF OF THEOREM 7.1(3). It follows from lemma 7.10 that if ω is integrable then so is P. To prove the converse we first do some computations

$$\imath_E d\omega = -d(\imath_E \omega) \qquad [\mathbf{14}, (A,7)]$$

$$= -\frac{1}{2} d(\Phi^{-1} d\Phi + d\Phi \cdot \Phi^{-1})$$

$$= \frac{1}{2} (\Phi^{-1} d\Phi \cdot \Phi^{-1} d\Phi - d\Phi \cdot \Phi^{-1} d\Phi \cdot \Phi^{-1})$$

On the other hand

$$\frac{1}{6} \imath_E (\Phi^{-1} d\Phi)^3 = \frac{1}{6} \circ i_E (\Phi^{-1} d\Phi)^3$$

$$= \frac{1}{6} \circ (\Phi^{-1} (\Phi \otimes 1 - 1 \otimes \Phi) \Phi^{-1} d\Phi \cdot \Phi^{-1} d\Phi + \cdots)$$

$$= \frac{1}{2} (\Phi^{-1} d\Phi \cdot \Phi^{-1} d\Phi - d\Phi \cdot \Phi^{-1} d\Phi \cdot \Phi^{-1})$$

So that we have deduced

$$\imath_E d\omega = \frac{1}{6} \imath_E (\Phi^{-1} d\Phi)^3$$

Write

$$\eta = d\omega - \frac{1}{6} (\Phi^{-1} d\Phi)^3$$

Assuming that P is integrable we need to prove that $\eta = 0$ (modulo commutators). We already know by lemma 7.10 that $pr_1 \, i_{H_a} i_{H_b} \eta = 0$ and by the computation above we have $\imath_E \eta = 0$. By [$\mathbf{14}$, Lemma A.5.2] we have $\imath(da) = H_a$. Hence by (P3) we have $\sum_{a \in A} A H_a A + AEA = D_A$. We observe

$$pr_1 \, i_E \imath_{H_b} \eta = - \, pr_1 \, \tau_{12} i_{H_b} \imath_E \eta \qquad [\mathbf{14}, (A.5)]$$

$$= 0$$

Hence by Prop. 3.1.5(2) it follows that $\imath_{H_b} \eta = 0$. Using $\imath_E \eta = 0$ once again we conclude by Prop. 3.1.5(1) that $\eta = 0$ (modulo commutators). $\qquad \square$

8. More general base rings and quivers

8.1. Generalities. In this section we show that the double quasi-Poisson brackets constructed on (localized) path algebras of double quivers are non-degenerate and hence Theorem 7.1 applies to them.

However first we note that for quivers it is more natural to use as base ring not a field but a direct sum of fields indexed by the vertices. In [$\mathbf{6}$] it was shown how to set up the theory over an arbitrary semi-simple base ring B. In [$\mathbf{14}$] we worked over the base ring $B = ke_1 + \cdots + ke_n$ with the e_p's being commuting idempotents. Since we rely on results from [$\mathbf{14}$] we will do the same in this section.

So assume that B is as in the previous paragraph. We use differentials and polyvector fields relative to B (relevant notations: $\Omega_{A/B}$, $\Omega_B A$, $D_{A/B}$, $D_B A$). In this setting the canonical element E is defined as $\sum_p E_p$ where

$$E_p(a) = ae_p \otimes e_p - e_p \otimes e_p a$$

A (multiplicative) moment map Φ is now of the form $\sum_p \Phi_p$ with $\Phi \in e_p A e_p$. With these conventions the definitions and results in this paper go through verbatim. We will accept this without further discussion. For use below we introduce the following convention. If $c = e_p A e_q$ then, changing standard terminology, an *inverse* of c is an element c^{-1} of $e_q A e_p$ such that $c \cdot c^{-1} = e_p$, $c^{-1}c = e_q$. It is easy to see that c^{-1} is uniquely determined.

8.2. Fusion. As in [**14**] our application to quivers depends on a process called fusion (introduced in the commutative case in [**1**]). In the case of quivers fusion amounts to gluing vertices but it is beneficial to work somewhat generally. We first construct and algebra $\bar{A}$ from A by formally adjoining two variables e_{12}, e_{21} satisfying the usual matrix relations $e_{uv}e_{wt} = \delta_{wv}e_{ut}$ (with $e_{ii} = e_i$). The fusion algebra of A along e_1, e_2 is defined as

$$A^f = \epsilon \bar{A} \epsilon$$

where $\epsilon = 1 - e_2$. Clearly $\bar{A}$ is a $\bar{B}$-algebra and A^f is a B^f-algebra. We will identify B^f with $ke_1 + ke_3 + \cdots + ke_n$.

If $a \in A$ then we consider it as an element of $\bar{A}$ and we write a^f for $\epsilon a \epsilon + e_{12} a e_{21}$. We extend this convention to forms and polyvector fields. Note that in [**14**, §5.3] it was shown than the operations $(\bar{-})$ and $(-)^f$ are compatible with the formation of $D_B A$ and its Schouten bracket. A similar result is true for $\Omega_B A$. The following result was proved in [**14**]

THEOREM 8.2.1. [**14**, Thm 5.3.1, 5.3.2] *Assume that (A, P, Φ) is a Hamiltonian double quasi-Poisson algebra (smooth as always). Then the same is true for (A^f, P^{ff}, Φ^{ff}) where*

$$P^{ff} = P^f - \frac{1}{2}E_1^f E_2^f \qquad and \qquad \Phi_i^{ff} = \begin{cases} \Phi_1^f \Phi_2^f & if\ i = 1 \\ \Phi_i^f & if\ i > 2 \end{cases}$$

In this section we prove the following result.

PROPOSITION 8.2.2. *Assume that (A, P, Φ) is a non-degenerate Hamiltonian double quasi-Poisson algebra. Then the same is true for (A^f, P^{ff}, Φ^{ff}).*

PROOF. In order to avoid confusing notations we define $F_p \in D_{B^f}(A^f)$ for $p \neq 2$ by $F_p(a) = ae_p \otimes e_p - e_p \otimes e_p a$. It follows from [**14**, (5.3)] that

(8.1) $$F_p = \begin{cases} E_1^f + E_2^f & \text{if } p = 1 \\ E_p^f & \text{otherwise} \end{cases}$$

Since A is non-degenerate the following map is surjective

$$(\imath(P), c) : \Omega_{A/B} \oplus \sum_p A E_p A \to D_{A/B}$$

From this we deduce (by base extension) that the following map is also surjective.

$$(\imath(\bar{P}), c) : \Omega_{\bar{A}/\bar{B}} \oplus \sum_p \bar{A}\bar{E}_p\bar{A} \to D_{\bar{A}/\bar{B}}$$

By [**14**, Prop 4.2.1, lemma A.5.2] we have

(8.2) $$\imath(P)(db) = -\{P, b\}$$

Compatibility of fusion with Schouten brackets yields that

$$(\imath(P^f), c) : \Omega_{A^f/B^f} \oplus \sum_p \epsilon \bar{A} \bar{E}_p \bar{A} \epsilon \to D_{A^f/B^f}$$

is also surjective. In addition we have $\epsilon \bar{A} \bar{E}_p \bar{A} \epsilon = A^f E_p^f A^f$.

It follows easily that

$$(\imath(P^{ff}), c) : \Omega_{A^f/B^f} \oplus \sum_p A^f E_p^f A^f \to D_{A^f/B^f}$$

is surjective as well. Using (8.1) we see that the current proposition is proved provided we can show that E_1^f is in the image of $\imath(P^{ff})$, modulo $(F_p)_p$. We do this next. We have

$$\imath(P^{ff})(d\Phi_2^f) = -\{P^f - \frac{1}{2}E_1^f E_2^f, \Phi_2^f\}$$

We use some formulas we have already proved. I.e. the formula before [**14**, (5.5)] yields

$$\{P^f, \Phi_2^f\} = -\frac{1}{2}(E_2^f \Phi_2^f + \Phi_2^f E_2^f)$$

$$= -\frac{1}{2}((F_1 - E_1^f)\Phi_2^f + \Phi_2^f(F_1 - E_1^f))$$

$$= -\frac{1}{2}(F_1 \Phi_2^f + \Phi_2^f F_1) + \frac{1}{2}(E_1^f \Phi_2^f + \Phi_2^f E_1^f)$$

We also have (using the formulas after [**14**, (5.5)])

$$\{\!\{E_1^f E_2^f, \Phi_2^f\}\!\} = E_1^f * \{\!\{E_2^f, \Phi_2^f\}\!\} - \{\!\{E_1^f, \Phi_2^f\}\!\} * E_2^f$$

$$= E_1^f * (\Phi_2^f \otimes e_1 - e_1 \otimes \Phi_2^f)$$

$$= \Phi_2^f \otimes E_1^f - E_1^f \otimes \Phi_2^f$$

("$*$" represents the inner bimodule structure) so that we get

$$\{E_1^f E_2^f, \Phi_2^f\} = \Phi_2^f E_1^f - E_1^f \Phi_2^f$$

Hence

$$\{P^f - \frac{1}{2}E_1^f E_2^f, \Phi_2^f\} = -\frac{1}{2}(F_1 \Phi_2^f + \Phi_2^f F_1) + \frac{1}{2}(E_1^f \Phi_2^f + \Phi_2^f E_1^f) - \frac{1}{2}(\Phi_2^f E_1^f - E_1^f \Phi_2^f)$$

$$= -\frac{1}{2}(F_1 \Phi_2^f + \Phi_2^f F_1) + E_1^f \Phi_2^f$$

So we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

8.3. Quivers. Below we assume that Q is a finite quiver whose vertices are indexed from 1 to n. We also use Q to refer to the set of arrows of Q. The head and tail of an arrow a are denoted by $h(a)$, $t(a)$ respectively.

Associated to Q is the double quiver which has the same vertices as Q and arrows $\{a, a^* \mid a \in Q\}$ where $h(a^*) = t(a)$, $t(a^*) = h(a)$. It will be convenient to write $(a^*)^* = a$ and to define for $a \in \bar{Q}$, $\epsilon(a) = 1$ if $a \in Q$ and $\epsilon(a) = -1$ otherwise. We let A be the path algebra of $k\bar{Q}$ to which we adjoin the inverses of $1 + aa^*$ for all $a \in \bar{Q}$.

For $a \in \bar{Q}$ one has a corresponding "partial derivative" in $D_{A/B}$ defined by

$$\frac{\partial b}{\partial a} = \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $D_{A/B}$ is a projective A-bimodule with generators $\partial/\partial a \in e_{h(a)}D_{A/B}e_{t(a)}$. Note that $\partial/\partial a$ goes in the "opposite direction" as a.

According to [**14**, Thm 6.7] A has a Hamiltonian double quasi-Poisson structure given by

$$P = \frac{1}{2} \left(\sum_{a \in \bar{Q}} \left(\epsilon(a)(1 + a^*a) \frac{\partial}{\partial a} \frac{\partial}{\partial a^*} \right) - \sum_{a < b \in \bar{Q}} \left(\frac{\partial}{\partial a^*} a^* - a \frac{\partial}{\partial a} \right) \left(\frac{\partial}{\partial b^*} b^* - b \frac{\partial}{\partial b} \right) \right)$$

(8.3)
$$\Phi = \prod_{a \in \bar{Q}} (1 + aa^*)^{\epsilon(a)}$$

where "$<$" refers to an arbitrary ordering on the edges of Q, which is also used to order the terms in the product (8.3).

PROPOSITION 8.3.1. *This Hamiltonian double quasi-Poisson structure is non-degenerate.*

PROOF. This Hamiltonian double quasi-Poisson structure was constructed in [**14**] using fusion starting from (multiple copies of) the following basic quiver.

(8.4)
$$1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2$$

By Proposition 8.2.2 we may assume that $\bar{Q}$ is equal to (8.4). The formula for P then simplifies to

$$P = \frac{1}{2} \left((1 + a^*a) \frac{\partial}{\partial a} \frac{\partial}{\partial a^*} - (1 + aa^*) \frac{\partial}{\partial a^*} \frac{\partial}{\partial a} \right)$$

and by [**14**, (6.2)]

$$E_1 = \frac{\partial}{\partial a^*} a^* - a \frac{\partial}{\partial a}$$

$$E_2 = \frac{\partial}{\partial a} a - a^* \frac{\partial}{\partial a^*}$$

To check non-degeneracy we compute $\imath(P)(da)$ and $\imath(P)(da^*)$. We find

$$\imath(P)(da) = \frac{1}{2} \circ \left((1 + a^*a) i_{da} \frac{\partial}{\partial a} \frac{\partial}{\partial a^*} + (1 + aa^*) \frac{\partial}{\partial a^*} i_{da} \frac{\partial}{\partial a} \right)$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial a^*} (1 + a^*a) + (1 + aa^*) \frac{\partial}{\partial a^*} \right)$$

A similar computation yields

$$\imath(P)(da^*) = -\frac{1}{2} \left((1 + a^*a) \frac{\partial}{\partial a} + \frac{\partial}{\partial a} (1 + aa^*) \right)$$

We have

$$a^* E_1 + E_2 a^* = -a^* a \frac{\partial}{\partial a} + \frac{\partial}{\partial a} a a^*$$

$$= -(1 + a^* a) \frac{\partial}{\partial a} + \frac{\partial}{\partial a}(1 + a a^*)$$

$$E_1 a + a E_2 = \frac{\partial}{\partial a^*} a^* a + - a a^* \frac{\partial}{\partial a^*}$$

$$= \frac{\partial}{\partial a^*}(1 + a^* a) - (1 + a a^*) \frac{\partial}{\partial a^*}$$

Hence

$$\imath(P)(da) = (1 + a a^*) \frac{\partial}{\partial a^*} \qquad \mathrm{mod}(E_1, E_2)$$

$$\imath(P)(da^*) = -(1 + a^* a) \frac{\partial}{\partial a} \qquad \mathrm{mod}(E_1, E_2)$$

Since both $(1 + a a^*)$ and $(1 + a^* a)$ are invertible we are done. $\qquad\square$

References

[1] A. Alekseev, Y. Kosmann-Schwarzbach, and E. Meinrenken, *Quasi-Poisson manifolds*, Canad. J. Math. **54** (2002), no. 1, 3–29.
[2] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495.
[3] R. Bocklandt and L. Le Bruyn, *Necklace Lie algebras and noncommutative symplectic geometry*, Math. Z. **240** (2002), no. 1, 141–167.
[4] H. Bursztyn and M. Crainic, *Dirac structures, momentum maps, and quasi-Poisson manifolds*, The breadth of symplectic and Poisson geometry (Boston, MA), Progr. Math., vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 1–40.
[5] W. Crawley-Boevey, *A note on non-commutative Poisson structures*, math.QA/0506268.
[6] W. Crawley-Boevey, P. Etingof, and V. Ginzburg, *Noncommutative geometry and quiver algebras*, Adv. Math. **209** (2007), no. 1, 274–336.
[7] V. Ginzburg, *Double derivations and cyclic homology*, math.AG/050230.
[8] V. Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett. **8** (2001), no. 3, 377–400.
[9] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.
[10] M. Kontsevich and A. Rosenberg, *Noncommutative smooth spaces*, The Gelfand Mathematical Seminars, 1996–1999 (Boston, MA), Birkhäuser Boston, Boston, MA, 2000, pp. 85–108.
[11] M. Kontsevich, *Formal (non)commutative symplectic geometry*, The Gel'fand Mathematical Seminars, 1990–1992 (Boston, MA), Birkhäuser Boston, Boston, MA, 1993, pp. 173–187.
[12] L. Le Bruyn, *Noncommutative Geometry and Caley-smooth Orders*, Pure and Applied Mathematics, vol. 250, Chapman & Hall/CRC, 2008.
[13] J.-L. Loday, *Cyclic homology*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, Springer-Verlag, Berlin, 1998, Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.
[14] M. Van den Bergh, *Double Poisson algebras*, to appear in Trans. Amer. Math. Soc.

DEPARTEMENT WINI, UNIVERSITEIT HASSELT, 3090 DIEPENBEEK, BELGIUM
E-mail address: michel.vandenbergh@uhasselt.be