GENERATORS AND REPRESENTABILITY OF FUNCTORS IN COMMUTATIVE AND NONCOMMUTATIVE GEOMETRY

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ABSTRACT. We give a sufficient condition for an Ext-finite triangulated category to be saturated. Saturatedness means that every contravariant cohomological functor of finite type to vector spaces is representable. The condition consists in existence of a strong generator. We prove that the bounded derived categories of coherent sheaves on smooth proper commutative and noncommutative varieties have strong generators, hence saturated. In contrast the similar category for a smooth compact analytic surface with no curves is *not saturated*.

1. INTRODUCTION AND MOTIVATION

In this paper k will be a field. Unless otherwise specified all categories will be k-linear. If $\mathcal{D}$ is a triangulated category then a cohomological functor $H : \mathcal{D} \to \mathrm{Vect}(k)$ is of finite type if for all $A \in \mathcal{D}$, $\sum_i \dim H(A[i]) < \infty$.

This paper is inspired by the following result.

Theorem 1.1. *Assume that X is a regular projective variety over a field k. Let $\mathcal{D}$ be the derived category of bounded coherent complexes on X. Then every contravariant cohomological functor of finite type on $\mathcal{D}$ is representable.*

This theorem was first announced in [8], but the proof in *loc.cit* works only for functors which are homologically bounded with respect to the standard t-structure. In the appendix to this paper we will give a short proof of a generalization of Theorem 1.1 which states that every contravariant cohomological functor of finite type on the derived category of perfect complexes on a (possibly singular) projective variety over a field is representable by a bounded complex of coherent sheaves.

Theorem 1.1 is a motivation for the following definition [8].

Definition 1.2. Assume that $\mathcal{D}$ is Ext-finite, i.e. $\sum_n \dim \mathrm{Hom}(A, B[n]) < \infty$ for all $A, B \in \mathcal{D}$. Then $\mathcal{D}$ is *(right) saturated* if every contravariant cohomological functor of finite type $H : \mathcal{D} \to \mathrm{Vect}(k)$ is representable.

Saturated triangulated categories are significant for non-commutative algebraic geometry. It can be argued that any definition of a non-commutative proper scheme should give rise to an associated saturated triangulated category which is the analogue of the

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bounded derived category of coherent sheaves in the commutative case. Some evidence for this point of view is given by [10].

One of the aims of this paper is to give an intrinsic criterion for $\mathcal{D}$ to be saturated. The central observation is that $\mathcal{D}$ should be finitely generated in a suitable sense. If $E \in \mathcal{D}$ then we say that E is a *classical generator* for $\mathcal{D}$ if $\mathcal{D}$ is the smallest triangulated subcategory of $\mathcal{D}$ containing E which is closed under summands.

If we define $\langle E\rangle_n$ to be the full subcategory of objects in $\mathcal{D}$ which can be obtained From E by taking finite direct sums, summand, shifts and **at most** $n-1$ **cones** then E is a classical generator if and only if $\langle E\rangle \overset{\text{def}}{=} \bigcup_n \langle E\rangle_n = \mathcal{D}$. We say that E is a *strong generator* for $\mathcal{D}$ if for some n we have $\langle E\rangle_n = \mathcal{D}$.

One of our main results is the following.

Theorem 1.3. *Assume that $\mathcal{D}$ is* Ext*-finite and has a strong generator. Assume in addition that $\mathcal{D}$ is Karoubian (i.e. every projector splits). Then $\mathcal{D}$ is saturated.*

Let us give an idea of the proof of this theorem. If E is a classical generator then using a method similar to the one used for the Brown representability theorem one proves (see lemma 2.4.1) that if $\mathcal{D}$ is Ext-finite and has a classical generator and $H : \mathcal{D} \to \mathrm{Vect}(k)$ is a contravariant cohomological functor of finite type then there exists a directed system $(A_i)_{i\in\mathbb{N}} \in \mathcal{D}$ such that $H = \varinjlim \mathrm{Hom}(-, A_i)$. The final step in the traditional proof of the Brown representability theorem consists in taking the homotopy limit $\tilde{A}$ of the directed system and proving that it represents H.

Unfortunately in our setting $\tilde{A}$ is not defined, because the definition of the homotopy limit depends on an infinite summation. To handle this problem, we introduce *n-resolutions of H with respect to a subcategory $\mathcal{E}$* in $\mathcal{D}$ (see §2.3). Such a resolution is a directed system which gives a good approximation for H on the subcategory $\mathcal{E}$. At the price of increasing n, it continues to be a resolution with respect to $\mathcal{E}$ enlarged by cones and direct summands.

We prove several results related to existence of generators and (non)saturatedness for some types of categories of geometric and noncommutative geometric origin.

We discuss the existence of generators and strong generators for schemes. In particular we prove that every quasi-compact, quasi-separated scheme has a classical generator. In combination with a recent result of Keller [19] (see Theorem 3.1.7) this shows that quasi-compact, quasi-separated schemes are affine in a DG- or A_∞-sense. We also prove that on a smooth scheme every classical generator is a strong generator. It follows that the bounded derived category of coherent sheaves on a smooth proper scheme is saturated.

We apply Theorem 1.3 to prove a result which generalizes Theorem 1.1 to the non-commutative case. If R is a (non-commutative) graded left coherent ring then there is a natural category $\mathrm{qgr}(R)$ which is an analogue for coherent sheaves on the projective scheme associated to a commutative graded ring. More precisely $\mathrm{qgr}(R)$ is the category of finitely presented graded left R-modules modulo finite length modules. In Theorem 4.3.4 we show that under appropriate homological conditions on R (which are analogous to those satisfied by smooth projective varieties) the bounded derived category of $\mathrm{qgr}(R)$ is saturated. This application represents our original motivation for studying this subject.

In contrast with the case of algebraic varieties, we prove that the bounded derived category of coherent sheaves (or, equivalently, of complexes of sheaves with coherent cohomology, see corollary 5.2.2) on a smooth compact analytic surface with no curves is not saturated. The proof uses perverse coherent sheaves and a result From [31].

Throughout the paper, if $\mathcal{E}$ is an abelian category then $D^b(\mathcal{E})$ and $D(\mathcal{E})$ denote respectively the bounded and unbounded derived category of $\mathcal{E}$. If Λ is a ring or a DG-algebra then $D(\Lambda)$ denotes its unbounded derived category.

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2. Generators and resolutions in triangulated categories

2.1. **Generators.** In this section we temporarily drop the assumption that our triangulated categories are k-linear. In this section and the next one we define various notions of generators for triangulated categories.

If $\mathcal{D}$ is a triangulated category then a triangulated subcategory $\mathcal{B}$ of $\mathcal{D}$ is called epaisse (thick) if it is closed under isomorphisms and direct summands. As was shown by Rickard [32] this is equivalent to Verdier's original definition.

If $\mathcal{E} = (E_i)_{i\in I}$ is a set of objects then we say that $\mathcal{E}$ *classically generates* $\mathcal{D}$ if the smallest epaisse triangulated subcategory of $\mathcal{D}$ containing $\mathcal{E}$ (called the *epaisse envelope* of $\mathcal{E}$ in $\mathcal{D}$) is equal to $\mathcal{D}$ itself. We say that $\mathcal{D}$ is *finitely generated* if it is classically generated by one object.

By the right orthogonal $\mathcal{E}^\perp$ in $\mathcal{D}$ we denote the full subcategory of $\mathcal{D}$ whose objects A have the property that $\operatorname{Hom}(E_i[n], A) = 0$ for all i and all n. $\mathcal{E}^\perp$ is an epaisse subcategory of $\mathcal{D}$. We say that $\mathcal{E}$ *generates* $\mathcal{D}$ if $\mathcal{E}^\perp = 0$. Clearly if $\mathcal{E}$ classically generates $\mathcal{D}$ then it generates $\mathcal{D}$, but the converse is false.

Assume now that $\mathcal{C}$ is a triangulated category admitting arbitrary direct sums. An object B in $\mathcal{C}$ is *compact* if $\operatorname{Hom}(B, -)$ commutes with direct sums. Let $\mathcal{C}^c$ be the full subcategory of $\mathcal{C}$ consisting of compact objects. The following is proved in [7].

Proposition 2.1.1. *$\mathcal{C}^c$ is Karoubian.*

Proof. (Sketch) Using the standard limit argument one first proves that $\mathcal{C}$ is Karoubian. Since a direct summand of a compact object is compact, this implies that $\mathcal{C}^c$ is Karoubian. □

Then we have the following result by Ravenel and Neeman [25].

Theorem 2.1.2. *A set of objects $\mathcal{E} \subset \mathcal{C}^c$ classically generates $\mathcal{C}^c$ if and only if it generates $\mathcal{C}$.*

2.2. **Strong generators.** In what follows objects and subcategories will be considered in a fixed triangulated category $\mathcal{D}$.

If $\mathcal{E}$ is a subcategory (or simply a set of objects), then we denote by $\operatorname{add}(\mathcal{E})$ the minimal strictly full subcategory in $\mathcal{D}$ which contains $\mathcal{E}$ and is closed under taking finite direct sums and shifts. We denote by $\operatorname{smd}(\mathcal{E})$ the minimal strictly full subcategory which contains $\mathcal{E}$ and is closed under taking (possible) direct summands.

Following [6], one introduces a multiplication on the set of strictly full subcategories. If $\mathcal{A}$ and $\mathcal{B}$ are two such subcategories, let $\mathcal{A} \star \mathcal{B}$ be the strictly full subcategory whose objects X occur in a triangle $A \to X \to B$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$. This multiplication is associative in view of the octahedron axiom. If $\mathcal{A}$ and $\mathcal{B}$ are closed under direct sums and/or shifts, then so is $\mathcal{A} \star \mathcal{B}$.

Lemma 2.2.1. *If $\mathcal{A}$ and $\mathcal{B}$ are closed under finite direct sums then:*

i) $\operatorname{smd}(\mathcal{A}) \star \mathcal{B} \subset \operatorname{smd}(\mathcal{A} \star \mathcal{B})$, $\mathcal{A} \star \operatorname{smd}(\mathcal{B}) \subset \operatorname{smd}(\mathcal{A} \star \mathcal{B})$;
ii) $\operatorname{smd}(\operatorname{smd}(\mathcal{A}) \star \mathcal{B}) = \operatorname{smd}(\mathcal{A} \star \operatorname{smd}(\mathcal{B})) = \operatorname{smd}(\mathcal{A} \star \mathcal{B})$.

Proof. ii) obviously follows From i). If $X \in \operatorname{smd}(\mathcal{A}) \star \mathcal{B}$, then X fits in a triangle $A_0 \to X \to B$ with $B \in \mathcal{B}$ and $A_0 \oplus A_1 = A$ for some $A \in \mathcal{A}$. If we add to this triangle the triangle $A_1 \xrightarrow{\mathrm{id}} A_1 \to 0$, we get the triangle $A \to X \oplus A_1 \to B$, which shows that $X \in \operatorname{smd}(\mathcal{A} \star \mathcal{B})$. This proves the first inclusion in i). The other inclusion is similar. □

Lemma 2.2.2. *The epaisse envelope of a strictly full triangulated subcategory $\mathcal{A} \subset \mathcal{B}$ consists of summands of objects in $\mathcal{A}$.*

Proof. By lemma 2.2.1 we have

$$\operatorname{smd}\mathcal{A} \star \operatorname{smd}\mathcal{A} \subset \operatorname{smd}(\mathcal{A} \star \mathcal{A}) = \operatorname{smd}\mathcal{A}$$

This proves the lemma. □

Now we define a new multiplication on the set of strictly full subcategories closed under finite direct sums by the formula:

$$\mathcal{A} \diamond \mathcal{B} \ = \operatorname{smd}(\mathcal{A} \star \mathcal{B}).$$

This multiplication is associative in view of lemma 2.2.1 and the associativity of $\star$:

$$(\mathcal{A} \diamond \mathcal{B}) \diamond \mathcal{C} = \operatorname{smd}(\operatorname{smd}(\mathcal{A} \star \mathcal{B}) \star \mathcal{C}) = \operatorname{smd}(\mathcal{A} \star \mathcal{B} \star \mathcal{C}) = \operatorname{smd}(\mathcal{A} \star \operatorname{smd}(\mathcal{B} \star \mathcal{C})) = \mathcal{A} \diamond (\mathcal{B} \diamond \mathcal{C}).$$

Moreover the formula holds:

$$\mathcal{A}_1 \diamond \mathcal{A}_2 \diamond \cdots \diamond \mathcal{A}_n = \operatorname{smd}(\mathcal{A}_1 \star \cdots \star \mathcal{A}_n). \tag{2.1}$$

Denote

$$\langle \mathcal{E} \rangle_1 = \operatorname{smd}(\operatorname{add}(\mathcal{E}))$$
$$\langle \mathcal{E} \rangle_k = \langle \mathcal{E} \rangle_{k-1} \diamond \langle \mathcal{E} \rangle_1 = \operatorname{smd}(\langle \mathcal{E} \rangle_1 \star \cdots \star \langle \mathcal{E} \rangle_1) \ (k \text{ factors}).$$
$$\langle \mathcal{E} \rangle = \bigcup_k \langle \mathcal{E} \rangle_k$$

Thus $\langle \mathcal{E} \rangle$ is the epaisse envelope of $\mathcal{E}$ in $\mathcal{D}$. So $\mathcal{E}$ classically generates $\mathcal{D}$ in the sense of §2.1 if and only if $\langle \mathcal{E} \rangle = \mathcal{D}$.

Definition 2.2.3. We say that $\mathcal{E}$ *strongly generates* $\mathcal{D}$ if $\mathcal{D} = \langle \mathcal{E} \rangle_k$, for some k. We say that $\mathcal{D}$ is *strongly finitely generated* if it is strongly generated by one object.

In other words $\mathcal{E}$ strongly generates $\mathcal{D}$ if we can get to any object in $\mathcal{D}$ From objects in $\mathcal{E}$ by a universally bounded number of cones.

Assume now that $\mathcal{C}$ is a triangulated category admitting arbitrary direct sums and let $\mathcal{E}$ be a set of objects in $\mathcal{C}$. We denote by $\overline{\operatorname{add}}(\mathcal{E})$ the minimal strictly full subcategory in $\mathcal{C}$ which contains $\mathcal{E}$ and is closed under taking arbitrary direct sums and shifts. We define $\overline{\langle \mathcal{E} \rangle}_k$ in the same way as $\langle \mathcal{E} \rangle_k$, but replacing add by $\overline{\operatorname{add}}$.

Analyzing the proof of Theorem 2.1.2 one obtains the following statement:

Proposition 2.2.4. *Assume that $\mathcal{E}$ consists of compact objects. Then $\overline{\langle \mathcal{E} \rangle}_k \cap \mathcal{C}^c = \langle \mathcal{E} \rangle_k$.*

Proof. The following is taken From Keller's writeup of the proof of Theorem 2.1.2 (see [20, §5.3]). Let $M \in \overline{\langle \mathcal{E} \rangle}_k \cap \mathcal{C}^c$. Thus M is a summand of an object $Z \in \overline{\langle \mathcal{E} \rangle}_{k-1} \star \overline{\operatorname{add}}(\mathcal{E})$.

We now have a commutative diagram

$$\begin{tikzcd} & M \arrow[d] & & \\ Z_{k-1} \arrow[r] & Z \arrow[r] & Z' \arrow[r] & {} \end{tikzcd}$$

where the lower row is a triangle with $Z_{k-1} \in \overline{\langle \mathcal{E} \rangle}_{k-1}$, $Z' \in \overline{\operatorname{add}}(\mathcal{E})$. Since M is compact the composition $M \to Z \to Z'$ factors through an object M' in $\operatorname{add}(\mathcal{E})$. From this we may construct a morphism of triangles

$$\begin{array}{ccccccc} M_{k-1} & \longrightarrow & M & \longrightarrow & M' & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ Z_{k-1} & \longrightarrow & Z & \longrightarrow & Z' & \longrightarrow \end{array}$$

Repeating this construction we obtain a commutative diagram

$$\begin{array}{ccccccccc} M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & \longrightarrow & M_{k-1} & \longrightarrow & M \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z_1 & \longrightarrow & \cdots & \longrightarrow & Z_{k-1} & \longrightarrow & Z \end{array}$$

By construction the cone of each of the upper maps lies in $\operatorname{add}(\mathcal{E})$. Hence by the octahedral axiom the cone M'' of the composition $M_0 \xrightarrow{\alpha} M$ lies in $\operatorname{add}(\mathcal{E}) \star \cdots \star \operatorname{add}(\mathcal{E})$ (k times).

Now consider the resulting commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{\alpha} & M \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

The right vertical map is split and hence monic. It follows that α is zero and hence M is a summand of M''. This finishes the proof. $\square$

2.3. **Resolutions.** As above $\mathcal{D}$ is a triangulated category and $\mathcal{E}$ is a subcategory of $\mathcal{D}$. If $A \in \mathcal{D}$ then we write h_A for the representable functor $\operatorname{Hom}(-, A)$.

Below we say that a directed system of abelian groups $(G_i, d_i)_{i>0}$ is of order n if the compositions of any n consecutive transition maps is zero (following [17] we could also say that $(G_i)_i$ is a complex of order n).

If $(F_i)_i$ and $(E_i)_i$ are of order a and b respectively and $(F_i)_i \to (G_i)_i \to (E_i)_i$ is exact, then $(G_i)_i$ is easily seen to be of order $a + b$.

Definition 2.3.1. Assume that $H : \mathcal{D} \to \mathrm{Ab}$ is a contravariant cohomological functor. Then an *n-resolution* of H with respect to $\mathcal{E}$ is a directed system of objects $(A_i)_{i>0}$ together with compatible natural transformations $\zeta_i : h_{A_i} \to H$ such that for any $E \in \mathcal{E}$, $p \in \mathbb{Z}$, $\zeta_i(E[p])$ is surjective and $\ker(\zeta_i(E[p]))_i$ is of order n. A *resolution* of H is a 1-resolution.

Lemma 2.3.2. *If $(A_i)_i$ is an n-resolution of H with respect to $\mathcal{E}$ then it is also an n-resolution with respect to $\langle \mathcal{E} \rangle_1$.*

The following key lemma is perhaps less obvious.

Lemma 2.3.3. *Assume that $(A_i)_i$ is an a-resolution of H with respect to $\mathcal{E} \subset \mathcal{D}$ and a b-resolution with respect to $\mathcal{F} \subset \mathcal{D}$. Then $(A_{i+b})_i$ is an $a + b$-resolution with respect to $\langle \mathcal{E} \rangle_1 \diamond \langle \mathcal{F} \rangle_1$.*

Proof. We have $\langle \mathcal{E} \rangle_1 \diamond \langle \mathcal{F} \rangle_1 = \operatorname{smd}(\operatorname{smd} \operatorname{add} \mathcal{E} \star \operatorname{smd} \operatorname{add} \mathcal{F}) = \operatorname{smd}(\operatorname{add} \mathcal{E} \star \operatorname{add} \mathcal{F})$. In view of lemma 2.3.2 we may without loss of generality replace $\operatorname{add}(\mathcal{E})$, $\operatorname{add}(\mathcal{F})$ by $\mathcal{E}, \mathcal{F}$ and then again using lemma 2.3.2 it suffices to show that we have an $a + b$-resolution with respect to $\mathcal{E} \star \mathcal{F}$.

Let $G \in \mathcal{E} \star \mathcal{F}$. Then G fits into a triangle $E \to G \to F$ with $E \in \mathcal{E}$, $F \in \mathcal{F}$. Define $K(U)_i$ and $C(U)_i$ as the directed systems given by the kernel and cokernel of

$\operatorname{Hom}(U, A_i) \to H(U)$. We now look at the following diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
K(E[p+1])_i \arrow[r] \arrow[d] & K(F[p])_i \arrow[r] \arrow[d] & K(G[p])_i \arrow[r] \arrow[d] & K(E[p])_i \arrow[r] \arrow[d] & K(F[p-1])_i \arrow[d] \\
\operatorname{Hom}(E[p+1], A_i) \arrow[r] \arrow[d] & \operatorname{Hom}(F[p], A_i) \arrow[r] \arrow[d] & \operatorname{Hom}(G[p], A_i) \arrow[r] \arrow[d] & \operatorname{Hom}(E[p], A_i) \arrow[r] \arrow[d] & \operatorname{Hom}(F[p-1], A_i) \arrow[d] \\
H(E[p+1]) \arrow[r] \arrow[d] & H(F[p]) \arrow[r] \arrow[d] & H(G[p]) \arrow[r] \arrow[d] & H(E[p]) \arrow[r] \arrow[d] & H(F[p-1]) \arrow[d] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & C(G[p])_i \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

If we think of the spectral sequence associated to the acyclic double complex formed by the two middle rows then we quickly obtain the following:

(1) $C(G[p])_i$ is a subquotient of $K(F[p-1])_i$. It follows that the order of $C(G[p])_i$ is less than or equal to b. Since the transition maps in $C(G[p])_i$ are obviously surjective it follows that $C(G[p])_i = 0$ for $i > b$.

(2) There is an exact sequence:

$$K(F[p])_i \to K(G[p])_i \to K(E[p])_i$$

whence $K(G[p])_i$ has order $a + b$. □

This lemma yields our main result.

Proposition 2.3.4. *Assume that $(A_i)_i$ is a resolution of H with respect to $\mathcal{E} \subset \mathcal{D}$. Take $a \geq 1$. Then:*

(1) *$(A_{ai})_i$ is a resolution of H with respect to $\langle \mathcal{E} \rangle_a$.*

(2) *H is a direct summand of the representable functor $h_{A_{2a}}$ when restricted to $\langle \mathcal{E} \rangle_a$.*

Proof. (1) From lemma 2.3.3 we obtain by induction that $(A_{i+a-1})_i$ is an a-resolution with respect to $\langle \mathcal{E} \rangle_a$. The first element of this a-resolution is A_a. Hence $(A_{ai})_i$ is an honest resolution with respect to $\langle \mathcal{E} \rangle_a$.

(2) For $h_{A_a}(Z)$ we have an exact sequence:

$$0 \to \ker \zeta_a(Z) \to h_{A_a}(Z) \xrightarrow{\zeta_a} H(Z) \to 0$$

The transition map $h_{A_a}(Z) \to h_{A_{2a}}(Z)$ kills $\ker \zeta_a(Z)$. Therefore we obtain a map $\theta(Z) : H(Z) \mapsto h_{A_{2a}}(Z)$, which is natural in Z. It is easily seen that the composition $\zeta_{2a}(Z) \circ \theta(Z)$ is the identity on $H(Z)$. Therefore H is a summand of $h_{A_{2a}}$ when restricted to $\langle \mathcal{E} \rangle_a$. □

2.4. **Construction of resolutions.** In this section $\mathcal{D}$ is an Ext-finite k-linear triangulated category.

Lemma 2.4.1. *Let $E \in \mathcal{D}$ and let $H : \mathcal{D} \to \operatorname{Vect}(k)$ be a contravariant cohomological functor of finite type. Then H has a resolution with respect to E.*

Proof. This is proved in the same way as the Brown representability theorem [20, 27]. For completeness let us repeat the construction of the resolution.

We start by taking $A_1 = \oplus_n E[n] \otimes_k H(E[n])$. There is an obvious canonical map $\zeta_1 : h_{A_1} \to H$ which is surjective when evaluated on $(E[n])_n$. Let $G = \ker \zeta_1$ and put

$B_1 = \oplus_n E[n] \otimes_k G(E[n])$. Then the composition $h_{B_1} \to G \to h_{A_1}$ is by Yoneda's lemma given by a map $\psi_1 : B_1 \to A_1$. We now have a complex of functors

$$h_{B_1} \xrightarrow{h_{\psi_1}} h_{A_1} \xrightarrow{\zeta_1} H \to 0 \tag{2.2}$$

which is exact when evaluated on $(E[n])_n$.

Let A_2 be the cone of $B_1 \xrightarrow{\psi_1} A_1$. Since H is a cohomological functor we have an exact sequence

$$H(A_2) \to H(A_1) \to H(B_1)$$

which by Yoneda's lemma translates into an exact sequence

$$\operatorname{Hom}(h_{A_2}, H) \to \operatorname{Hom}(h_{A_1}, H) \to \operatorname{Hom}(h_{B_1}, H)$$

From (2.2) it follows that ζ_1 is mapped to zero in $\operatorname{Hom}(h_{B_1}, H)$. Whence ζ_1 lifts to a map $\zeta_2 : h_{A_2} \to H$. The fact that the composition

$$h_{B_1} \to h_{A_1} \to h_{A_2}$$

is zero combined with the exactness of (2.2) on $(E[n])_n$ implies that $\ker \zeta_1(E[n])$ is killed in $h_{A_2}(E[n])$. Thus it is clear that if we repeat this construction we obtain a resolution $(A_i, \zeta_i)_i$ of H with respect to E. □

Lemma 2.4.2. *Assume that $\mathcal{D}$ is* Ext*-finite. Let $H : \mathcal{D} \to \operatorname{Vect}(k)$ be a contravariant cohomological functor of finite type and let E be an arbitrary object in $\mathcal{D}$. Then for all n there exists an object Q_n such that H restricted to $\langle E \rangle_n$ is a direct summand of the representable functor* $\operatorname{Hom}(-, Q_n)$.

Proof. By lemma 2.4.1 H as a resolution with respect to E. Then in the notation of Proposition 2.3.4 we may take $Q_n = A_{2n}$. □

Proof of Theorem 1.3. Let $E \in \mathcal{D}$ be a strong generator and let $H : \mathcal{D} \to \operatorname{Vect}(k)$ be a contravariant cohomological functor of finite type. Then $\mathcal{D} = \langle E \rangle_n$ for some n and according to lemma 2.4.2 H will be a direct summand of $\operatorname{Hom}(-, Q_n)$. This direct summand corresponds to a projector in the endomorphism ring of the functor $\operatorname{Hom}(-, Q_n)$. By Yoneda's lemma we obtain a corresponding projector in $\operatorname{End}(Q_n)$. By the assumption that $\mathcal{D}$ is Karoubian, this projector corresponds to a summand of Q_n. It is easy to see that this summand represents H. □

2.5. **A counter example.** In this section we show with a simple counter example that Theorem 1.3 is false if we only assume the existence of a generator (and not of a strong generator).

Let $R = k[[x]]$ where k is a field and let $\mathcal{E}$ be the category of torsion R-modules. Let S be the simple R-module. Then S is a generator for $\mathcal{D} = D(\mathcal{E})$. To see this, note that $\mathcal{E}$ is hereditary and has enough injectives. So every object in $\mathcal{D}$ is the direct sum of its cohomology objects (see lemma 4.2.8 below for a more general statement). Hence we have to show that the right orthogonal of S in $\mathcal{E}$ is zero. Since $\mathcal{E}$ is closed under injective hulls in $\operatorname{Mod}(R)$ we have $\operatorname{Ext}^*_{\mathcal{E}}(S, M) = \operatorname{Ext}^*_R(S, M)$. If $\operatorname{Ext}^*_R(S, M)$ is zero then M is both x-torsion and uniquely divisible by x. Hence $M = 0$.

It is easy to see that the compact objects in $\mathcal{D}$ are finite direct sums of shifts of $S_n = R/x^n R$. From this it is clear that S is not a strong generator (the number of cones we need to reach S_n depends on n) and neither is any other object in $\mathcal{D}^c$.

$\mathcal{D}$ is also not saturated. Indeed if E is the injective hull of S then $\operatorname{Hom}(-, E)$ defines a functor of finite type which is not representable. This is a special case of the following more general result proved in [31].

Lemma 2.5.1. *Assume that $\mathcal{E}$ is an* Ext*-finite abelian category of finite homological dimension in which every object has finite length. Then $D^b(\mathcal{E})$ is saturated if and only if $\mathcal{E} \cong \operatorname{mod}(\Lambda)$ where Λ is a finite dimensional algebra of finite global dimension and $\operatorname{mod}(\Lambda)$ is the category of finite dimensional Λ-modules.*

In particular, the category $\mathcal{D}$ we considered above cannot be saturated since then it would have enough projectives, which is clearly not the case.

3. Generators and strong generators for schemes.

In this section we consider generators and strong generators for certain types of schemes.

3.1. **Statement of results.** If X is a scheme then by $\operatorname{Qch}(X)$ we will denote the category of quasi-coherent $\mathcal{O}_X$-modules. If X is noetherian then $\operatorname{coh}(X)$ is the category of coherent $\mathcal{O}_X$-modules. If X is a ringed space then $D(X)$ is the derived category of modules of $\mathcal{O}_X$-modules and if X is a scheme then $D_{\mathrm{Qch}}(X)$ will be the derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology. It is clear that $D(X)$ and $D_{\mathrm{Qch}}(X)$ admit arbitrary direct sums.

Quasi-coherent sheaves are well-behaved on *quasi-compact, quasi-separated* schemes. Recall that a quasi-compact scheme is a scheme that has a finite covering by affine open subschemes and a quasi-separated scheme is a scheme such that the intersection of any two affine open subschemes is quasi-compact. Actually it is sufficient to check this last condition on the affine opens of an arbitrary finite affine covering.

A noetherian scheme is quasi-compact and quasi-separated. If X is quasi-compact quasi-separated then $\operatorname{Qch}(X)$ is a Grothendieck category [37].

Our aim is to describe the category of compact objects in $D_{\mathrm{Qch}}(X)$ for a quasi-compact, quasi-separated scheme. Recall that a complex on a scheme is said to be *perfect* if it is locally quasi-isomorphic to a bounded complex of vector bundles. In particular a perfect complex is in $D_{\mathrm{Qch}}(X)$ and if X is quasi-compact then it is in $D^b_{\mathrm{Qch}}(X)$.

We will prove the following theorem.

Theorem 3.1.1. *Assume that X is a quasi-compact, quasi-separated scheme. Then*

(1) *The compact objects in $D_{\mathrm{Qch}}(X)$ are precisely the perfect complexes.*
(2) *$D_{\mathrm{Qch}}(X)$ is generated by a single perfect complex.*

Denote by $D_{\mathrm{perf}}(X)$ the category of perfect complexes on X.

Corollary 3.1.2. *If X is a quasi-compact, quasi-separated then $D_{\mathrm{perf}}(X)$ is finitely generated.*

Proof. This follows From Theorem 3.1.1 and Theorem 2.1.2. □

We recall the following result for *separated schemes.*

Theorem 3.1.3. [1, 7, 21] *If X is quasi-compact and* separated *then the canonical functor $D(\operatorname{Qch}(X)) \to D_{\mathrm{Qch}}(X)$ is an equivalence.*

This result is false (even on the bounded derived categories) if we only assume X to be quasi-compact quasi-separated. A counter example by Verdier is given in [15, App I].

If X is smooth over a field (in particular separated) then using Theorem 3.1.3 or directly it is easy to see that $D^b(\operatorname{coh}(X)) \cong D_{\mathrm{perf}}(X)$. For smooth schemes we will prove the following result:

Theorem 3.1.4. *Assume that X is smooth over a field (in particular separated). Then $D^b(\operatorname{coh}(X))$ is strongly finitely generated.*

Presumably the last theorem is true under the weaker hypothesis that X is noetherian and regular.

Corollary 3.1.5. *Assume that X is smooth and proper over a field. Then $D^b(\mathrm{coh}(X))$ is saturated.*

Proof. This follows From Theorem 1.3 and Proposition 2.1.1. □

Remark 3.1.6. In characteristic zero one may give a different proof of Corollary 3.1.5 as follows. By Chow's lemma and Hironaka's theorem there is a birational dominant map $f : Y \to X$ such that Y is projective and smooth. Since X is smooth it has rational singularites and hence $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. Then f^* makes $D^b(\mathrm{coh}(X))$ into an admissible subcategory [9] in $D^b(\mathrm{coh}(Y))$. In this situation, saturatedness of Y (which follows From Theorem 1.1) implies saturatedness of X.

It is not clear to the authors if this proof can be generalized to characteristic p.

Recently Bernhard Keller has proved the following result [19]

Theorem 3.1.7. *Let $\mathcal{E}$ be a Grothendieck category and assume that $\mathcal{A} = D(\mathcal{E})$ is generated by a compact object E. Then $\mathcal{A} = D(\Lambda)$ where Λ is a DG-algebra whose cohomology is given by* $\mathrm{Ext}^*(E, E)$.

Combining this theorem with Theorem 3.1.1 we find the following corollary to our results

Corollary 3.1.8. *Assume that X is a quasi-compact quasi-separated scheme. Then $D_{\mathrm{Qch}}(X)$ is equivalent to $D(\Lambda)$ for a suitable DG-algebra Λ with bounded cohomology.*

Proof. The fact that Λ has bounded cohomology follows From lemma 3.3.8 below. □

Informally we may say that quasi-compact, quasi-separated schemes are affine in a "derived sense".

3.2. **Extension of compact objects.** First recall the following.

Theorem 3.2.1. [26, Thm 2.1] *Let $\mathcal{D}$ be compactly generated triangulated category admitting arbitrary direct sums and let $\mathcal{K}$ be a triangulated subcategory which is closed under direct sums and which is in addition generated by objects which are compact in $\mathcal{D}$. Put $\mathcal{C} = \mathcal{D}/\mathcal{K}$. Then*

(1) *$\mathcal{C}$ admits arbitrary direct sums;*
(2) *$\mathcal{C}$ is compactly generated;*
(3) *$\mathcal{D}^c$ maps to $\mathcal{C}^c$ under the quotient functor;*
(4) *the induced functor $\mathcal{D}^c/\mathcal{K}^c \to \mathcal{C}^c$ is fully faithful;*
(5) *$\mathcal{C}^c$ is the epaisse envelope of $\mathcal{D}^c/\mathcal{K}^c$.*

Assume that we are in the situation of the previous theorem and put $\mathcal{B} = \mathcal{C}^c$ and let $\mathcal{A}$ be the strict closure of $\mathcal{D}^c/\mathcal{K}^c$. Then $\mathcal{B}$ is the epaisse envelope of $\mathcal{A}$. In this situation there is a simple criterion to decide if an object in $\mathcal{B}$ lies in $\mathcal{A}$. This is contained in the following proposition.

Proposition 3.2.2. *Let $\mathcal{A}$ be a strictly full triangulated subcategory in a triangulated category $\mathcal{B}$ such that the epaisse envelope of $\mathcal{A}$ is $\mathcal{B}$. Then an object X in $\mathcal{B}$ is in $\mathcal{A}$ iff its representative $[X] \in K_0(\mathcal{B})$ belongs to the image of $K_0(\mathcal{A})$.*

We will give the proof below. In the situation of Theorem 3.2.1 this was proved in [26]. In the case of schemes it is [37, Prop. 5.5.4].

We immediately obtain the following corollary.

Corollary 3.2.3. *In the situation of Proposition 3.2.2 if $X \in \mathcal{B}$ then $X \oplus X[1] \in \mathcal{A}$.*

The rest of this subsection is devoted to proving Proposition 3.2.2.

For an abelian monoid M with an operation $\oplus$, denote by $F(M)$ the free abelian group generated by elements of M and by $G(M)$ the quotient of $F(M)$ by the subgroup $E(M)$ generated by elements $[X \oplus Y] - [X] - [Y]$ taken for all pairs of elements $X, Y \in M$.

For an additive category $\mathcal{A}$ denote by $G_+(\mathcal{A})$ the abelian monoid with elements the isomorphy classes of objects in $\mathcal{A}$ and with operation $\oplus$. We also use the notation $F(\mathcal{A})$, $G(\mathcal{A})$, $E(\mathcal{A})$ for the corresponding groups $F(G_+(\mathcal{A}))$, $G(G_+(\mathcal{A}))$, $E(G_+(\mathcal{A}))$.

The following lemma is classical and easy to prove.

Lemma 3.2.4. *For two objects X and Y in an additive category $\mathcal{A}$, $[X] = [Y]$ in $G(\mathcal{A})$ iff there exists $Z \in \mathcal{A}$ such that $X \oplus Z \cong Y \oplus Z$.*

If $\mathcal{A}$ is a strictly full additive subcategory in $\mathcal{B}$ then the natural morphism $F(\mathcal{A}) \to F(\mathcal{B})$ is obviously an embedding, which takes $E(\mathcal{A})$ to $E(\mathcal{B})$. Thus, we may regard $F(\mathcal{A})$, $E(\mathcal{A})$ as subgroups of $F(\mathcal{B})$, $E(\mathcal{B})$.

Lemma 3.2.5. *Let $\mathcal{A}$ be a strictly full additive subcategory in an additive category $\mathcal{B}$ such that any object in $\mathcal{B}$ is a direct summand of an object in $\mathcal{A}$. Then $E(\mathcal{B}) \cap F(\mathcal{A}) = E(\mathcal{A})$.*

Proof. Any element in $G(\mathcal{A})$ can be presented in the form $[X] - [Y]$ with $X, Y \in \mathcal{A}$. Hence any element in $F(\mathcal{A})$ has the form $[X] - [Y] + v$ with $X, Y \in \mathcal{A}$ and $v \in E(\mathcal{A})$. Suppose this element is in $E(\mathcal{B})$. Since $E(\mathcal{A}) \subset E(\mathcal{B})$, then $[X] - [Y] \in E(\mathcal{B})$. Then by lemma 3.2.4 there exists $Z \in \mathcal{B}$ such that $X \oplus Z \cong Y \oplus Z$. By the assumption we can find Z' such that $Z \oplus Z'$ is in $\mathcal{A}$. Then $X \oplus (Z \oplus Z') \cong Y \oplus (Z \oplus Z')$. It follows that $[X] - [Y] \in E(\mathcal{A})$. □

The Grothendieck group $K_0(\mathcal{A})$ of a triangulated category $\mathcal{A}$ is the free abelian group generated by the isomorphy classes of objects modulo the relations $[Y] = [X] + [Z]$ taken for all exact triangles $X \to Y \to Z \to \cdots$. Denote by $I(\mathcal{A})$ the kernel of the natural homomorphism $G(\mathcal{A}) \to K_0(\mathcal{A})$.

Proposition 3.2.6. *Let $\mathcal{A}$ be a strictly full triangulated subcategory in a triangulated category $\mathcal{B}$. Suppose that the epaisse envelope of $\mathcal{A}$ coincides with $\mathcal{B}$. Then:*

(i) *The induced homomorphism $G(\mathcal{A}) \to G(\mathcal{B})$ is monic.*
(ii) *The induced homomorphism $I(\mathcal{A}) \to I(\mathcal{B})$ is an isomorphism.*
(iii) *The induced homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ is monic.*

Proof. It is clear that $\mathcal{A}$ satisfies the conditions of the last lemma. A lifting to $F(\mathcal{A})$ of an element x From the kernel of $G(\mathcal{A}) \to G(\mathcal{B})$ belongs to $E(\mathcal{B})$. Hence by the last lemma it is in $E(\mathcal{A})$. Then x is zero, and (i) is checked.

It follows From (i) that $I(\mathcal{A}) \to I(\mathcal{B})$ is monic. Let us show it is epic. The group $I(\mathcal{B})$ is the subgroup in $G(\mathcal{B})$ generated by elements $[Y] - [X] - [Z]$ where $X \to Y \to Z$ is a triangle in $\mathcal{B}$. Find elements X' and Z' in $\mathcal{B}$ such that $X' \oplus X$ and $Z \oplus Z'$ are in $\mathcal{A}$. Add the trivial triangles $X' \to X' \to 0$ and $0 \to Z' \to Z'$ to the primary triangle. Then we get the triangle:

$$X' \oplus X \to X' \oplus Y \oplus Z' \to Z \oplus Z'.$$

As the two extreme elements of the triangle are in $\mathcal{A}$ then so is the middle one. Hence $[X' \oplus Y \oplus Z'] - [X' \oplus X] - [Z \oplus Z']$ is an element in $I(\mathcal{A})$. Its image in $I(\mathcal{B})$ coincides with $[Y] - [X] - [Z]$ modulo relations in $G(\mathcal{B})$. This proves (ii).

An element From the kernel of $K_0(\mathcal{A}) \to K_0(\mathcal{B})$, once lifted to $G(\mathcal{A}) \subset G(\mathcal{B})$, is in $I(\mathcal{B})$. Hence by (ii) it is in $I(\mathcal{A})$. Then (iii) follows. □

Proof of Proposition 3.2.2. In view of proposition 3.2.6 we may regard $G(\mathcal{A})$ as a subgroup of $G(\mathcal{B})$. Let $i : \mathcal{A} \to \mathcal{B}$ be the embedding functor. Denote by $\mathrm{K}_0(i)$, $G(i)$ the corresponding homomorphisms of groups. ¿From the snake lemma and Proposition 3.2.6.(ii) it follows that the induced homomorphism on cokernels $\mathrm{Coker}\, G(i) \to \mathrm{Coker}\, \mathrm{K}_0(i)$ is monic. Hence the image in $\mathrm{K}_0(\mathcal{B})$ of an element x in $G(\mathcal{B})$ is in $\mathrm{K}_0(\mathcal{A})$ iff $x \in G(\mathcal{A})$.

Let us prove the following criterion for an object $X \in \mathcal{B}$ to yield an element in $G(\mathcal{A})$:

$$[X] \in G(\mathcal{A}) \Longleftrightarrow X \oplus A_1 = A_2, \tag{3.1}$$

for some A_1, A_2 in $\mathcal{A}$.

Indeed, if $[X] \in G(\mathcal{A})$ then $[X] = [Y]-[Z]$ for some $Y, Z \in \mathcal{A}$. Therefore, $[X \oplus Z] = [Y]$. By lemma 3.2.4 there exists $W \in \mathcal{B}$, such that $X \oplus Z \oplus W \cong Y \oplus W$. By hypotheses we can find $V \in \mathcal{B}$, such that $U = W \oplus V \in \mathcal{A}$. Then $X \oplus (Z \oplus U) \cong Y \oplus U$. This proves (3.1).

But the right hand side of (3.1) yields a (split) exact triangle of the form $A_1 \to A_2 \to X$, i.e. $X \in \mathcal{A}$. □

3.3. **Compact generators for derived categories of quasi-coherent sheaves.** Recall that an object in the homotopy category of complexes is K-injective if it is right orthogonal to the acyclic complexes. Spaltenstein [35] has proved that every complex of $\mathcal{O}_X$-modules on a ringed space X has a K-injective resolution. Right derived functors are computed by evaluating the original functor on a K-injective resolution.

Most of the arguments below are based on Mayer-Vietoris type triangles. Let us indicate how these are constructed. Assume $X = U_1 \cup U_2$ with U_1, U_2 open and put $U_{12} = U_1 \cap U_2$. Let j_1, j_2 and j_{12} be the inclusions of U_1, U_2 and U_{12} into X. By looking at stalks we see that we have a short exact sequence in $\mathrm{Mod}(\mathcal{O}_X)$:

$$0 \to j_{12!}\mathcal{O}_{U_{12}} \to j_{1!}\mathcal{O}_{U_1} \oplus j_{2!}\mathcal{O}_{U_2} \to \mathcal{O}_X \to 0$$

If $A \in D(X)$ then we obtain a triangle

$$\mathrm{R}\mathcal{H}om(\mathcal{O}_X, A) \to \mathrm{R}\mathcal{H}om(j_{1!}\mathcal{O}_{U_1}, A) \oplus \mathrm{R}\mathcal{H}om(j_{2!}\mathcal{O}_{U_2}, A) \to \mathrm{R}\mathcal{H}om(j_{12!}\mathcal{O}_{U_{12}}, A) \to$$

For the definition of $\mathrm{R}\mathcal{H}om$ see [35, Prop. 6.1]

If $A^{\cdot}$ is a K-injective complex on X and $j : U \to X$ is an open embedding then $j^*A^{\cdot}$ is K-injective on U. This follows From the existence of the exact left adjoint $j_!$. From this we easily obtain $\mathrm{R}\mathcal{H}om(j_!\mathcal{O}_U, A) = Rj_*(j^*A)$. Hence we obtain a triangle

$$A \to Rj_{1*}(j_1^*(A)) \oplus Rj_{2*}(j_2^*(A)) \to Rj_{12*}(j_{12}^*(A)) \to \tag{3.2}$$

¿From this triangle we may derive other Mayer-Vietoris type triangles by applying suitable functors. If f is a map $X \to Y$ and the restrictions of f to U_1, U_2, U_{12} are denoted by f_1, f_2, f_{12} respectively then applying Rf_* we obtain a triangle

$$Rf_*A \to Rf_{1*}(j_1^*(A)) \oplus Rf_{2*}(j_2^*(A)) \to Rf_{12*}(j_{12}^*(A)) \to \tag{3.3}$$

Let E be another object in $D(X)$. Applying $\mathrm{RHom}(E, -)$ to (3.2) we find a triangle

$$\mathrm{RHom}(E, A) \to \mathrm{RHom}(j_1^*E, j_1^*A) \oplus \mathrm{RHom}(j_2^*E, j_2^*A) \to \mathrm{RHom}(j_{12}^*E, j_{12}^*A)) \to \tag{3.4}$$

The Mayer-Vietoris triangles may be used in connection with the following principle:

Proposition 3.3.1. *(Reduction principle) Let P be a property satisfied by some schemes. Assume in addition the following.*

1. *P is true for affine schemes.*
2. *If P holds for U_1, U_2, U_{12} as above then it holds for X.*

Then P holds for all quasi-compact quasi-separated schemes.

Proof. (See the proof of [22, Lemma 3.9.2.4]) Let X be quasi-compact, quasi-separated. Since X has a finite affine cover it has a finite cover by quasi-compact separated schemes $X_1, \dots, X_n$. We use induction on n. Put $U_1 = X_1 \cap \cdots \cap X_{n-1}$, $U_2 = X_n$. Being open subsets of X, U_1, U_2, U_{12} are quasi-separated and by looking at affine covers of the X_i we easily see that these subsets are also quasi-compact. Furthermore since $X_i \cap X_n$ is a subscheme of a separated scheme, it is itself separated. Hence U_1 and U_{12} have coverings by $n-1$ quasi-compact separated schemes. By induction we may assume now $n = 1$, in other words, X is separated.

We now repeat the same argument with an affine cover $X = X_1 \cup \cdots \cup X_n$. Since X is separated $X_i \cap X_n$ is affine and induction on n reduces us to the case X affine and we are done. □

Remark 3.3.2. It is easy to see that the class of quasi-compact, quasi-separated schemes is the biggest class of schemes to which the reduction principle is applicable (for all properties P).

A map $f : X \to Y$ between schemes is said to be quasi-compact, resp. quasi-separated if for every affine open $U \subset Y$ the inverse image of U is quasi-compact, resp. quasi-separated. Quasi-compact and quasi-separated morphisms are stable under composition and pullback.

Theorem 3.3.3. [22, Prop. 3.9.2] *If $f : X \to Y$ is quasi-compact, quasi-separated then*

(1) *Rf_* maps $D_{\mathrm{Qch}}(X)$ into $D_{\mathrm{Qch}}(Y)$.*
(2) *If Y is quasi-compact then the image of $D_{\mathrm{Qch}}(X)^{\le 0}$ lies in $D_{\mathrm{Qch}}(Y)^{\le N}$ for some N.*

Proof. Since this statement is crucial for what follows we sketch the proof. We may clearly assume that Y is affine. Then by the Mayer-Vietoris triangle (3.3) and the reduction principle we may assume that X is also affine.

To prove (2) it is now clearly sufficient to prove that the image of $D_{\mathrm{Qch}}(X)^{\le 0}$ lies in $D_{\mathrm{Qch}}(Y)^{\le 0}$. Let $A \in D_{\mathrm{Qch}}(X)^{\le 0}$. According to [35, Prop 3.13] A has a so-called "special" K-injective resolution I. By construction I is an inverse limit $\varprojlim_n I_n$ of left bounded injective resolutions of $\tau_{\ge -n} A$ such that $I_n \to I_{n-1}$ is split epi in every degree.

Now $f_* I$ is the "sheaffication" of $U \mapsto \Gamma(f^{-1}(U), I)$ where U runs through the affine opens of Y. Note that $f^{-1}(U)$ is also affine. Hence it is sufficient to show for all $V \subset X$ affine open that $\Gamma(V, I) = \varprojlim_n \Gamma(V, I_n)$ is acyclic in degrees > 0. This is clearly true for $\Gamma(V, I_n)$. Furthermore the map $\Gamma(V, I_n) \to \Gamma(V, I_{n-1})$ is surjective, and a quasi-isomorphism in degree $\ge -n+1$. We can now conclude by [35, Lemma 0.11] which guarantees under these conditions that $H^i(\varprojlim \Gamma(V, I_n)) = \varprojlim H^i(\Gamma(V, I_n))$ for all i.

Now we prove (1). Since we have an affine map it is clear that Rf_* maps $\mathrm{Qch}(X)$ to $\mathrm{Qch}(Y)$. Hence to conclude it is sufficient to prove that for $A \in D_{\mathrm{Qch}}(X)$ we have $H^i(Rf_*(A)) = f_*(H^i(A))$. If $A \in D^+_{\mathrm{Qch}}(X)$ then this is clear by devissage. The case of arbitrary A is handled by writing it as an extension $\tau_{<-N} A \to A \to \tau_{\ge -N} A \to$ for $N \gg 0$. □

Corollary 3.3.4. *Assume that $f : X \to Y$ is quasi-compact, quasi-separated. Then Rf_* commutes with arbitrary direct sums on $D_{\mathrm{Qch}}(X)$.*

Proof. This question is local on Y so we may assume that Y is affine. Since a direct sums of injective resolutions is a complex of flabby sheaves, which are acyclic for f_*, and since in addition f_* commutes with direct sums it is clear that Rf_* commutes with arbitrary direct sums in $D(X)^{\geq -N}$ for all N.

Let $(A_i)_{i\in I}$ be a family of objects in $D_{\mathrm{Qch}}(X)$. Then according to Theorem 3.3.3(2) for N large compared to j we have the following sequence of equalities: $H^j(Rf_*(\oplus_i A_i)) = H^j(Rf_*(\tau_{\geq -N}(\oplus_i A_i))) = H^j(Rf_*(\oplus_i(\tau_{\geq -N}A_i))) = \oplus_i H^j(Rf_*(\tau_{\geq -N}A_i)) = \oplus_i H^j(Rf_*A_i)$. Thus we obtain that the canonical map $\oplus_i H^j(Rf_*A_i) \to H^j(Rf_*(\oplus_i A_i))$ is a quasi-isomorphism. □

The following analogue of Serre's theorem is a special case of Theorem 3.1.3.

Corollary 3.3.5. [1, 7, 21] *Assume that $X = \operatorname{Spec} R$ is affine. Then the obvious functor $D(R) = D(\mathrm{Qch}(X)) \to D_{\mathrm{Qch}}(X)$ has a quasi-inverse given by $R\Gamma(X,-)$.*

Proof. It is easy to see that this amounts to showing that if $A \in D_{\mathrm{Qch}}(X)$ then $H^i(R\Gamma(X,A)) = \Gamma(X, H^i(A))$. For X left bounded this is clear and we may reduce the general case to this using (the analogue for $R\Gamma$) of Theorem 3.3.3(2) (in the same way as in the previous corollary). □

Recall the following result [7].

Lemma 3.3.6. *If R is a ring then the compact objects in $D(R)$ are precisely the perfect complexes (bounded complexes of finitely generated projective modules).*

Lemma 3.3.7. *If X is quasi-compact quasi-separated and $E \in D_{\mathrm{Qch}}(X)$ is perfect then E is compact in $D_{\mathrm{Qch}}(X)$.*

Proof. In the notation of (3.4) it follows From the five-lemma that if j_1^*E, j_2^*E and j_{12}^*E are compact then so is E. By the reduction principle it is then sufficient to consider the affine case but this follows From lemma 3.3.6 and corollary 3.3.5. □

The following lemma was needed for Corollary 3.1.8.

Lemma 3.3.8. *If X is quasi-compact, quasi-separated, $E \in D_{\mathrm{perf}}(X)$ and $F \in D^b_{\mathrm{Qch}}(X)$ then $\mathrm{RHom}(E,F)$ is bounded.*

Proof. This follows From (3.4) and the reduction principle. □

Proof of Theorem 3.1.1. Our proof that $D_{\mathrm{Qch}}(X)$ is generated by a single perfect complex is a modification of the proof of [27, Prop. 2.5]. We proceed by induction on the number of elements in an affine covering of X. The case where X itself is affine is obvious by Corollary 3.3.5: the generating object is $\mathcal{O}_X$. To perform the induction step we consider the situation where X has an open covering $U \cup Y$ with Y quasi-compact and $D_{\mathrm{Qch}}(Y)$ having a perfect generator E and $U = \operatorname{Spec} R$ being affine. Put $S = U \cap Y$ and let the inclusion maps be as in the following diagram

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "\beta"'] & U \arrow[d, "\gamma"] \\ Y \arrow[r, "\delta"'] & X \end{tikzcd}$$

Let $V = X \setminus Y = U \setminus S$. Then V is a closed subset of U and X. Since S is quasi-compact and U is affine it follows that V is defined by a finite number of elements $f_1, \ldots, f_n \in R$. Let Q be the object in $D_{\mathrm{Qch}}(U)^c$ associated to the complex of free R-modules $\otimes_i(R \xrightarrow{f_i} R)$. According to [7] Q is a compact generator for the kernel of the restriction map $\alpha^* : D_{\mathrm{Qch}}(U) \to D_{\mathrm{Qch}}(S)$.

Since the homology of Q has support in V it follows that $R\gamma_* Q \mid Y = 0$. Furthermore we have $R\gamma_* Q \mid U = Q$ (this holds for any Q and any open immersion $U \subset X$). It follows that $R\gamma_* Q$ is perfect. Furthermore From the Mayer-Vietoris triangle (3.4) (with $U_1 = Y$, $U_2 = U$, $E = R\gamma_* Q$ and $A = Z$) we obtain

$$\mathrm{Hom}(R\gamma_* Q, Z) = \mathrm{Hom}(Q, Z \mid U) \tag{3.5}$$

for any $Z \in D_{\mathrm{Qch}}(X)$.

Since $D_{\mathrm{Qch}}(U)$ is compactly generated and $\ker \alpha_* : D_{\mathrm{Qch}}(U) \to D_{\mathrm{Qch}}(S)$ is generated by a compact object in $D_{\mathrm{Qch}}(U)$ it follows From Theorem 3.2.1 and Corollary 3.2.3 that there exists $F \in D_{\mathrm{Qch}}(U)^c$ such that $F \mid S = E' \mid S$ with $E' = E \oplus E[1]$. By Corollary 3.3.5 F is a perfect complex. The perfect complexes F on U and E' on Y can be glued yielding a perfect complex on X in the following way. Define $P \in D_{\mathrm{Qch}}(X)$ by the exact triangle

$$P \to R\gamma_* F \oplus R\delta_* E' \to R\delta\beta_*(E' \mid S) \to$$

(the middle arrow is the direct sum of the two obvious morphisms). One can easily check that $\delta^* P = E'$, $\gamma^* P = F$ by applying δ^* and γ^* to this triangle. Thus P is perfect.

We claim that $C = P \oplus R\gamma_* Q$ is a compact generator for $D_{\mathrm{Qch}}(X)$.

Assume that Z is right orthogonal to $R\gamma_* Q$. Using (3.5) we find that $Z \mid U$ is right orthogonal to Q. It follows $Z \mid U \to R\alpha_*(Z \mid S)$ is an isomorphism (cf. [7]) and hence $R\gamma_*(Z \mid U) = R(\delta\beta)_*(Z \mid S)$. We then obtain From the Mayer-Vietoris triangle (3.2) that the map

$$Z \to R\delta_*(Z \mid Y) \tag{3.6}$$

is an isomorphism.

Assume now in addition that Z is right orthogonal to P. Then by the isomorphism (3.6) and adjointness we obtain that $Z \mid Y$ is right orthogonal to $P \mid Y = E \oplus E[1]$. Hence $Z \mid Y = 0$. Again using the isomorphism (3.6) we obtain that $Z = 0$. This finishes the proof of the fact that $D_{\mathrm{Qch}}(X)$ is generated by a single perfect complex.

Now we will prove that all compact objects are perfect. By lemma 3.3.7 and Theorem 2.1.2 it follows that every compact object is a direct summand of a perfect complex. But by looking on an affine cover and invoking Corollary 3.3.5 we see that a direct summand of a perfect complex is perfect. □

3.4. **Strong generators for smooth schemes.** In this section we prove Theorem 3.1.4. The proof uses an extension of Beilinson's "resolution of the diagonal" argument. The idea for this approach is due to Maxim Kontsevich.

Lemma 3.4.1. *Let $f_1 : X \to W$, $f_2 : Y \to W$ be quasi-compact maps of quasi-compact schemes. Assume that E, F are compact generators for $D_{\mathrm{Qch}}(X)$ and $D_{\mathrm{Qch}}(Y)$. Then $E \boxtimes_W F$ is a compact generator for $D_{\mathrm{Qch}}(X \times_W Y)$.*

Proof. The fact that $E \boxtimes_W F$ is compact follows From Theorem 3.1.1. So we only need to show that $E \boxtimes_W F$ is a generator. Assume that Z is right orthogonal to $E \boxtimes_W F$. Let $\mathrm{pr}_{1,2}$ be the projections of $X \times_W Y$ on the first and the second factor. Since

$$\mathrm{Hom}_{X\times_W Y}(E \boxtimes_W F, Z[m+n]) = \mathrm{Hom}_{X\times_W Y}(L\,\mathrm{pr}_1^* E, \mathrm{R}\mathcal{H}om_{X\times Y}(L\,\mathrm{pr}_2^* F, Z[m])[n])$$

we deduce that $R\,\mathrm{pr}_{1*}\,\mathrm{R}\mathcal{H}om_{X\times_W Y}(L\,\mathrm{pr}_2^* F, Z[m]) = 0$ for m arbitrary.

Now let U, V and T be open affines in X, Y and W such that $f_1(U) \subset T$, $f_2(V) \subset T$. We find

$$0 = \Gamma(U, R\,\mathrm{pr}_{1*}\,\mathrm{R}\mathcal{H}om_{X\times Y}(L\,\mathrm{pr}_2^* F, Z[m+n])) = \mathrm{Hom}_Y(F, R\,\mathrm{pr}_{2*}(Z[m] \mid U \times_W Y)[n])$$

From which we deduce $R\operatorname{pr}_{2*}(Z[m] \mid U \times_W Y) = 0$. Restricting to V yields $\Gamma(U \times_W V, Z[m]) = \Gamma(U \times_T V, Z[m]) = 0$.

Since U, V, T, m are arbitrary and since $X \times_Z Y$ is covered by the affine open sets $U \times_T V$ this implies $Z = 0$ by Corollary 3.3.5. $\square$

Proof of Theorem 3.1.4. Assume that X is smooth over the field k and let E be a compact generator for $D_{\mathrm{Qch}}(X)$. Then $X \times X$ is smooth as well and if $\Delta \subset X \times X$ is the diagonal then $\mathcal{O}_\Delta$ is compact by Theorem 3.1.1. Hence according to Theorem 2.1.2 and the above lemma $\mathcal{O}_\Delta \in \langle E \boxtimes E \rangle_k$ for certain $k \in \mathbb{N}$. Let $Z \in D_{\mathrm{Qch}}(X)$. Then $Z = R\operatorname{pr}_{1*}(\operatorname{pr}_2^* Z \overset{L}{\otimes} \mathcal{O}_\Delta)$ and hence $Z \in \langle R\operatorname{pr}_{1*}(\operatorname{pr}_2^* Z \otimes E \boxtimes E)\rangle_k = \langle E \otimes R\Gamma(E \overset{L}{\otimes} Z)\rangle_k$. Since $R\Gamma(E \overset{L}{\otimes} Z)$ is a complex of vector spaces we find that $Z \in \overline{\langle E \rangle}_k$ and hence by Proposition 2.2.4 $D_{\mathrm{Qch}}(X)^c = \langle E \rangle_k$. Since for smooth varieties we have $D_{\mathrm{Qch}}(X)^c = D^b(\mathrm{coh}(X))$. This finishes the proof of Theorem 3.1.4. $\square$

4. Derived categories for graded rings

In this section we will associate to a graded ring R a category $\mathrm{QGr}(R)$ which is a non-commutative analogue of the category of quasi-coherent sheaves on a projective variety [2, 39]. We will prove that under appropriate homological conditions on R the category of compact objects $D(\mathrm{QGr}(R))^c$ in the derived category of $\mathrm{QGr}(R)$ is strongly finitely generated and hence saturated.

If R is coherent then we may also introduce a category $\mathrm{qgr}(R)$ which is analogous to the category of coherent sheaves on a projective variety. Under the homological conditions alluded to above we have $D(\mathrm{QGr}(R))^c = D^b(\mathrm{qgr}(R))$. Thus in this way we obtain a complete non-commutative analogue to Theorem 1.1.

4.1. **Generalities.** In this section we develop some rudiments of projective geometry for graded rings. We begin with some of the standard material on functors related to the category of graded R-modules. Since we do not assume initially that R is noetherian or coherent we state some of the basic facts and give their proofs.

Below $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ is a connected graded ring over a field k with graded maximal ideal $m = \oplus_{n>0} R_n$. Following [38] we assume throughout that $\dim \operatorname{Ext}^i(k,k) < \infty$ for all $i \geq 0$. In particular R is finitely presented. Note that this condition on R is left-right symmetric.

$\mathrm{Gr}(R)$ denotes the category of graded left R-modules. For $n \in \mathbb{Z}$, $\mathrm{Gr}(R)$ comes equipped with a shift functor $M \mapsto M(n)$ where $M(n)$ is defined by $M(n)_j = M_{n+j}$.

We will write $\operatorname{Ext}^i_{\mathrm{Gr}(R)}(M,N)$ for the Ext-groups in $\mathrm{Gr}(R)$ and $\underline{\operatorname{Ext}}^i_R(M,N)$ for the graded Ext-groups $\oplus_n \operatorname{Ext}^i_{\mathrm{Gr}(R)}(M, N(n))$. Thus $\operatorname{Ext}^i_{\mathrm{Gr}(R)}(M,N) = \underline{\operatorname{Ext}}^i_R(M,N)_0$.

We say that $M \in \mathrm{Gr}(R)$ is torsion if it is locally finite dimensional, or equivalently if for all $a \in M$ there exists n such that $m^n a = 0$. Let $\mathrm{Tors}(R)$ denote the corresponding full subcategory of $\mathrm{Gr}(R)$. Since R is finitely generated, $\mathrm{Tors}(R)$ is a localizing subcategory of $\mathrm{Gr}(R)$. Furthermore finitely generated objects in $\mathrm{Tors}(R)$ are finite dimensional. Let $\mathrm{QGr}(R) = \mathrm{Gr}(R)/\mathrm{Tors}(R)$. We define τ as the functor which assigns to a graded R module its maximal torsion module. By $\pi : \mathrm{Gr}(R) \to \mathrm{QGr}(R)$ we denote the quotient functor. By standard localization theory π is exact and commutes with colimits. We denote the (fully faithful) right adjoint to π by ω and we denote the composition $\omega\pi$ by Q. Since $\pi\omega$ is the identity it follows $Q^2 = Q$.

The shift functors $M \mapsto M(n)$ define shift functors on $\mathrm{QGr}(R)$ for which we will use the same notation. Finally we will write $\mathcal{O} = \pi R$. Note that by adjointness it follows that

$$(R^i\omega M)_0 = \mathrm{Ext}^i_{\mathrm{QGr}(R)}(\mathcal{O}, M) \tag{4.1}$$

for $M \in \mathrm{QGr}(R)$.

Lemma 4.1.1. *For any directed system $(N_i)_{i\in I}$ and for any n we have*

$$\mathrm{Ext}^j_R(R/R_{\geq n}, \operatorname{inj\,lim} N_i) = \operatorname{inj\,lim}_i \mathrm{Ext}^j_R(R/R_{\geq n}, N_i)$$

Proof. The fact that $\dim \mathrm{Ext}^i(k,k) < \infty$ implies that $R/R_{\geq n}$ has a graded resolution consisting of finitely generated free modules. From this fact the lemma is clear. □

Lemma 4.1.2. *$R^i\tau$ commutes with filtered colimits (and hence with direct sums) for all i.*

Proof. This follows From the description [36]

$$R^i\tau = \varinjlim_n \mathrm{Ext}^i_R(R/R_{\geq n}, -) \tag{4.2}$$

together with lemma 4.1.1. □

Lemma 4.1.3. *Assume that T is torsion. Then*

$$R^i\tau T = 0 \qquad \text{for} \qquad i > 0$$

Proof. By lemma 4.1.2 it suffices to prove this in the case that T is finite dimensional. But then it is clear From (4.2) if we look at the degrees of the generators of the modules occuring in a minimal free resolution of $R/R_{\geq n}$. □

Lemma 4.1.4. *Q is given by*

$$QM = \varinjlim_n \mathrm{Hom}_R(R_{\geq n}, M)$$

Proof. Standard localization theory [36] tells us

$$QM = \varinjlim_n \mathrm{Hom}_R(R_{\geq n}, M/\tau M)$$

So we need to show $\varinjlim_n \mathrm{Ext}^i_R(R_{\geq n}, \tau M) = 0$ for $i \leq 1$. The vanishing of $\varinjlim_n \mathrm{Hom}_R(R_{\geq n}, \tau M)$ is obvious and since $\mathrm{Ext}^1_R(R_{\geq n}, \tau M) = \mathrm{Ext}^2_R(R/R_{\geq n}, \tau M)$ the other vanishing follows From lemmas 4.1.3 and (4.2). □

Lemma 4.1.5. *For $M \in \mathrm{Gr}(R)$ there is a long exact sequence*

$$0 \to \tau M \to M \to QM \to R^1\tau M \to 0$$

and isomorphisms $R^iQM = R^{i+1}\tau M$ for $i \geq 1$. In particular R^iQ vanishes on $\mathrm{Tors}(R)$ and commutes with filtered colimits.

Proof. This follows From the long exact sequence obtained by applying $\varinjlim_n \mathrm{Hom}_R(-, M)$ to the system of exact sequences $0 \to R_{\leq n} \to R \to R/R_{\geq n} \to 0$ and then invoking lemma 4.1.4 and (4.2). □

Lemma 4.1.6. *One has $R^iQ = R^i\omega \circ \pi$.*

Proof. One has to show that if $E \in \mathrm{Gr}(R)$ is injective then πE is acyclic for ω. Let

$$0 \to \pi E \to F_0 \to F_1 \to F_2 \to \cdots$$

be an injective resolution of πE. Since $\pi\omega$ is the identity, applying ω to this sequence we get that

$$0 \to QE \to \omega F_0 \to \omega F_1 \to \omega F_2 \to \cdots \tag{4.3}$$

is a complex with homology in $\mathrm{Tors}(R)$. Since $E \to QE$ has torsion kernel and cokernel, then applying R^iQ to this morphism we find (using the vanishing of R^iQ on torsion objects by lemma 4.1.5)

$$R^iQ(QE) = R^iQE = \begin{cases} QE & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, $R^jQ(\omega F_i) = 0$ for $j > 0$, given the fact that ωF_i is injective by adjointness.

Then the spectral sequence for hyper cohomology yields that (4.3) becomes exact if we apply Q. Since $Q^2E = QE$ and $Q\omega F_i = \omega F_i$ it follows that the original sequence was already exact. □

Lemma 4.1.7. *$R^i\omega$ commutes with filtered colimits.*

Proof. Let $(M_j)_j$ be a directed system in $\mathrm{QGr}(R)$. Then we have

$$\begin{aligned} R^i\omega(\varinjlim_j M_j) &= R^i\omega(\varinjlim_j \pi\omega M_j) = (R^i\omega \circ \pi)(\varinjlim_j \omega M_j) = R^iQ(\varinjlim_j \omega M_j) \\ &= \varinjlim_j R^iQ(\omega M_j) = \varinjlim_j (R^i\omega \circ \pi \circ \omega)(M_j) = \varinjlim_j R^i\omega(M_j) \end{aligned}$$ □

In the sequel we will make the following assumption on τ:

Hypothesis. τ has finite cohomological dimension, i.e. $R^n\tau = 0$ for $n \gg 0$.

This hypothesis implies that ω and Q also have finite cohomological dimension by lemmas 4.1.5, 4.1.6. Furthermore using the methods in [14] we may define the unbounded derived functors $R\tau$, $R\omega$, RQ by means of acyclic resolutions. From the definition one easily deduces the following properties:

Properties.
(1) $R\tau$, $R\omega$, $\mathrm{Ext}^i(\mathcal{O}, -)$, RQ, $R\pi = \pi$ commute with direct sums.
(2) $R\omega$ is the right adjoint to π and $\pi \circ R\omega = \mathrm{id}$.
(3) $R\tau$ is the left adjoint to the inclusion functor $D_{\mathrm{Tors}(R)}(\mathrm{Gr}(R)) \to D(\mathrm{Gr}(R))$.
(4) $R\omega \circ R\tau = 0$.
(5) $RQ = R\omega \circ \pi$.
(6) For $M \in D(\mathrm{Gr}(R))$ there is a triangle:

$$R\tau M \to M \to RQM \to \tag{4.4}$$

4.2. **Saturatedness.** In this section we will show that under suitable hypotheses the category $D(\mathrm{QGr}(R))^c$ is Ext-finite and saturated.

We recycle the notations and assumptions of the previous section. Recall that if M is a graded R-module then Artin and Zhang [2] say that R satisfies $\chi(M)$ if $\dim \mathrm{Ext}^i_R(k, M) < \infty$ for all i. The significance of this condition is the following

Lemma 4.2.1. [2, Cor. 3.6(3)] *The following are equivalent.*
(1) *R satisfies $\chi(M)$.*
(2) *For all i, $R^i\tau M$ is finite dimensional in every degree and in addition has right bounded grading.*

Below we will say that R satisfies μ if it satisfies $\chi(R)$.

Lemma 4.2.2. (i) *$D(\mathrm{QGr}(R))$ is generated by $\{\mathcal{O}(n)\}_{n\in\mathbb{Z}}$.*
(ii) *One has $D(\mathrm{QGr}(R))^c = \langle \mathcal{O}(n)_{n\in\mathbb{Z}}\rangle$.*

Proof. Assume that $M \in D(\mathrm{QGr}(R))$ is right orthogonal to $\{\mathcal{O}(n)\}_{n\in\mathbb{Z}}$. Using adjointness this implies that $R\omega M$ is right orthogonal to $R(n)$. Hence $R\omega M = 0$, but then $0 = \pi \circ R\omega M = M$.

By property (1) above $\mathcal{O}(n)$ is compact. Hence (ii) follows From (i) together with Theorem 2.1.2. $\square$

Corollary 4.2.3. *Assume that R satisfies μ. Then $D(\mathrm{QGr}(R))^c$ is* Ext*-finite.*

Proof. By lemma 4.2.2 $D(\mathrm{QGr}(R))^c$ is classically generated by $\{\mathcal{O}(n)\}_{n\in\mathbb{Z}}$. Hence it suffices to prove that $\sum_i \dim \mathrm{Ext}^i(\mathcal{O}(m), \mathcal{O}(n))$ is finite. Now we have $\mathrm{Ext}^i(\mathcal{O}(m), \mathcal{O}(n)) = \mathrm{Ext}^i(\mathcal{O}, \mathcal{O}(n-m)) = (R^i\omega\mathcal{O})_{n-m} = (R^iQR)_{n-m}$ by (4.1) and property (5). The corollary now follows From lemma 4.2.1 and the triangle (4.4). $\square$

We will now show that we can do better. In the rest of this section $\sigma_{\le}$, $\sigma_{\ge}$, $\tau_{\le}$ and $\tau_{\ge}$ denote respectively the "stupid" and "smart" truncations of complexes.

Lemma 4.2.4. *Let d be the cohomological dimension of ω. Then there exists a number $l \le 0$ such that $\mathcal{O}(n) \in \langle \mathcal{O}(k)_{l\le k\le 0}\rangle_{d+1}$ for all $n > 0$ (see §2.1.2 for notations).*

Proof. Let $(F_{in})_{i\ge 0}$ be a minimal free resolution of $(R/R_{\ge n})(n)$ (where as usual F_{in} is placed in complex degree $-i$). Clearly $F_{0n} = R(n)$ and the other F_{in} are direct sums of $R(v)$'s with $v \le 0$. Put $Z_i = \ker(F_{in} \to F_{i-1,n})$. Then $\sigma_{\le -1}\sigma_{\ge -d-1}(\pi F_{\cdot n})$ represents an element of $\mathrm{Ext}^{d+1}(\mathcal{O}(n), \pi Z_{d+1})$ which is zero by (4.1). Thus $\mathcal{O}(n)$ is a direct summand of $\sigma_{\le -1}\sigma_{\ge -d-1}(\pi F_{\cdot n})$. This shows that $\mathcal{O}(n) \in \langle \mathcal{O}(k)_{k\le 0}\rangle_{d+1}$.

To obtain the stronger conclusion of the proposition we have to bound above the u such that $R(-u)$ occurs in $\sigma_{\le -1}\sigma_{\ge -d-1}F_{\cdot n}$. That is we have to bound u such that $k(u)$ occurs in $\mathrm{Ext}^i_R((R/R_{\ge n})(n), k)$ for $i \le d+1$. Since $(R/R_{\ge n})(n)$ is an extension of $k(t)$, $0 < t \le n$ we have to bound the $k(u)$ occuring in $\mathrm{Ext}^i_R(k(t), k)$ for $i \le d+1$ and $t > 0$. Since $\mathrm{Ext}^i_R(k(t), k) = \mathrm{Ext}^i_R(k, k)(-t)$ such a bound is given by the maximal v such that $k(v)$ occurs in $\mathrm{Ext}^i_R(k, k)$ for $i \le d+1$. $\square$

Now we discuss the case when $\mathrm{QGr}(R)$ has finite homological dimension. Recall that if $\mathcal{C}$ is an abelian category then the homological dimension of $\mathcal{C}$ is the maximal i such that there exist $M, N \in \mathcal{C}$ with the property that $\mathrm{Ext}^i_{\mathcal{C}}(M, N) \ne 0$.

Lemma 4.2.5. *Assume that $\mathrm{QGr}(R)$ has finite homological dimension. Then the functor τ has finite cohomological dimension.*

Proof. This follows From combining Lemmas 4.1.5,4.1.6 with (4.1). $\square$

Lemma 4.2.6. *Assume that $\mathrm{QGr}(R)$ has homological dimension $h < \infty$. Then for every $M \in \mathrm{QGr}(R)$ one has $M \in \overline{\langle \mathcal{O}(k)_k\rangle}_{h+1}$.*

Proof. This is proved by observing that if $M = \pi N$ then a sufficiently long free resolution of N splits in $\mathrm{QGr}(R)$. The same argument was used in the proof of lemma 4.2.4. $\square$

Lemma 4.2.7. *Assume that $\mathrm{QGr}(R)$ has homological dimension h. Then one has $D(\mathrm{QGr}(R)) = \overline{\langle \mathcal{O}(k)_k\rangle}_{2h}$.*

Proof. Let $U \in D(\mathrm{QGr}(R))$. It is easy to see that we can construct maps $\alpha_i : Q_i \to U$ with the following properties:

(1) Q_i is a complex consisting of (possibly infinite) direct sums of $\mathcal{O}(k)$'s which starts in degree $ih+1$ and ends in degree $(i+1)h-1$.
(2) $H^*(\alpha_i)$ is an isomorphism in homology in degrees $ih+2$ up to $(i+1)h-1$ and surjective in degree $ih+1$.

Now put $Q = \oplus_i Q_i$, $\alpha = \oplus_i \alpha_i : Q \to U$ and let V be the cone of α. We find that $H^p(V) = 0$ except when $h \mid p$. Invoking lemma 4.2.8 below we find that $V = \oplus_i H^{ih}(V)$. By using lemma 4.2.6 each of the $H^{ih}(V)$ can be produced by using at most h cones. So the total number of cones we need is:

$$h-2(\text{to produce } Q) + h(\text{to produce } V) + 1(\text{to produce } U \text{ From } Q, V) = 2h-1 \qquad \square$$

The following lemma was used in the proof.

Lemma 4.2.8. *Assume that $\mathcal{C}$ is an abelian category which satisfies AB4 (exact direct sums) and has enough injectives. Assume that the homological dimension of $\mathcal{C}$ is $h < \infty$ and let $V \in D(\mathcal{C})$ be a complex satisfying $H^p(V) = 0$ unless $h \mid p$. Then $V = \oplus_i H^{ih}(V)$.*

Proof. Write $H(V) = \oplus H^{ih}(V)[-ih]$ (this sum exist since we have AB4 [7]). We want to construct a quasi-isomorphism $H(V) \to V$. To this end it is sufficient to construct maps $H^{ih}(V)[-ih] \to V$ which induce isomorphisms on the ih'th cohomology. Since $\tau_{\leq ih} X \to X$ induces an isomorphism on H^{ih}, it is clearly sufficient to show that the canonical map $\tau_{\leq ih} V \to H^{ih}(V)[-ih]$ splits. From the triangle

$$\tau_{\leq (i-1)h} V \to \tau_{\leq ih} V \to H^{ih}(V)[-ih] \to$$

we find that we have to show that

$$\operatorname{Hom}(H^{ih}(V)[-ih], \tau_{\leq (i-1)h} V[1]) = 0 \tag{4.5}$$

Now according to [14, Thm 5.1, Cor. 5.3], if $\mathcal{C}$ has enough injectives and $\operatorname{Hom}(H^i(V), -)$ has finite cohomological dimension then we can compute $\operatorname{Hom}(H^i(V), -)$ (which is equal to $H^0(\operatorname{RHom}(H^i(V), -))$) by acyclic resolutions. It follows easily that an object in $D(\mathcal{C})^{\leq -N}$ can be represented by an acyclic complex which is non-zero only in degree $\leq -N+h$. This clearly implies (4.5). $\square$

Some of the statements below will refer to the ring R^{opp}. As a rule we will decorate the corresponding notations by a superscript "opp".

Lemma 4.2.9. *Assume that $\operatorname{QGr}(R)$ has homological dimension $h < \infty$ and that R satisfies μ^{opp}. Then for $n > 0$, $\mathcal{O}(-n) \in \langle \mathcal{O}(k)_{k \geq 0} \rangle_{h+1}$.*

Proof. This is proved in a similar way as lemma 4.2.4. We start with a minimal resolution of $(R/R_{\geq n})(n)^{\text{opp}}$, dualizing we obtain a complex starting with $R(-n)$ whose homology is finite dimensional (using the μ^{opp}-condition). Applying π we obtain an exact sequence which start with $\mathcal{O}(-n)$ and consists in higher degrees of direct sums of $\mathcal{O}(k)$, $k \geq 0$. As in lemma 4.2.4 $\mathcal{O}(-n)$ will be a direct summand of a truncation of length $h+1$ of this exact sequence. $\square$

Lemma 4.2.10. *Assume that τ^{opp} has finite cohomological dimension and that in addition R satisfies μ and μ^{opp}. Assume furthermore that $\operatorname{QGr}(R)$ has finite homological dimension. Then there exist numbers $m \leq 0$, $e \geq 1$ such that $\mathcal{O}(n) \in \langle \mathcal{O}(k)_{m \leq k \leq 0} \rangle_e$ for all n.*

Proof. This follows by combining lemma 4.2.4 with lemma 4.2.9. $\square$

Proposition 4.2.11. *Assume that τ^{opp} has finite cohomological dimension and that in addition R satisfies μ and μ^{opp}. Assume furthermore that* $\operatorname{QGr}(R)$ *has finite homological dimension. Then the following holds.*

(1) $D(\operatorname{QGr}(R)) = \overline{\langle \mathcal{O}(k)_{a\le k\le 0}\rangle}_b$ *for some* $a \le 0$, $b \ge 1$.
(2) $D(\operatorname{QGr}(R))^c = \langle \mathcal{O}(k)_{a\le k\le 0}\rangle_b$.

In particular the Ext*-finite triangulated category* $D(\operatorname{QGr}(R))^c$ *is strongly finitely generated.*

Proof. (1) follows by combining lemma 4.2.10 with lemma 4.2.7. (2) follows From Proposition 2.2.4. □

We can now finally prove the following theorem.

Theorem 4.2.12. *Under the hypotheses of the previous proposition* $D(\operatorname{QGr}(R))^c$ *is saturated.*

Proof. This follows Theorem 1.3, Proposition 2.1.1 and the previous proposition. □

4.3. **The case that R is coherent.** Let R satisfy the blanket assumptions made in the beginning of §4.1 and assume that R is left graded coherent. In other words the kernel of a graded map between two free graded R modules of finite rank is finitely generated. Let $\operatorname{gr}(R)$ be the category of finitely presented graded R-modules. Since R is coherent this is an abelian category.

Put $\operatorname{tors}(R) = \operatorname{gr}(R) \cap \operatorname{Tors}(R)$. Then $\operatorname{tors}(R)$ consists of the finite dimensional graded R-modules. We put $\operatorname{qgr}(R) = \operatorname{gr}(R)/\operatorname{tors}(R)$. It is easy to see that the obvious functor $\operatorname{qgr}(R) \to \operatorname{QGr}(R)$ is fully faithful.

Lemma 4.3.1. *Let $M \in \operatorname{qgr}(R)$. Then $\operatorname{Ext}^i_{\operatorname{qgr}(R)}(M,-)$ commutes with filtered colimits.*

Proof. By Lemma 4.1.7 and (4.1) this is clearly true if $M = \mathcal{O}(n)$ and it is a tautology if $i < 0$. To treat the general we construct a short exact sequence
$$0 \to N \to F \to M \to 0$$
where F is a finite sum of shifts of $\mathcal{O}(n)$. Let $(T_j)_j$ be a directed system. We now have the following commutative diagram
$$\begin{tikzcd}
\varinjlim_j \operatorname{Ext}^{i-1}(F,T_j) \arrow[r] \arrow[d, "\alpha"] & \varinjlim_j \operatorname{Ext}^{i-1}(N,T_j) \arrow[r] \arrow[d, "\beta"] & \varinjlim_j \operatorname{Ext}^{i}(M,T_j) \arrow[r] \arrow[d, "\gamma"] & \varinjlim_j \operatorname{Ext}^{i}(F,T_j) \arrow[r] \arrow[d, "\delta"] & \varinjlim_j \operatorname{Ext}^{i}(N,T_j) \arrow[d, "\epsilon"] \\
\operatorname{Ext}^{i-1}(F,\varinjlim_j T_j) \arrow[r] & \operatorname{Ext}^{i-1}(N,\varinjlim_j T_j) \arrow[r] & \operatorname{Ext}^{i}(M,\varinjlim_j T_j) \arrow[r] & \operatorname{Ext}^{i}(F,\varinjlim_j T_j) \arrow[r] & \operatorname{Ext}^{i}(N,\varinjlim_j T_j)
\end{tikzcd}$$
α and δ are isomorphisms by the above discussion. Furthermore we may assume by induction that β is an isomorphism. It now follows by diagram chasing that γ is monic. Then, replacing M by N we find that ϵ is also monic. Performing another diagram chase yields that γ is also epic. □

Lemma 4.3.2. *Assume* $\operatorname{QGr}(R)$ *has finite cohomological dimension. Then* $D(\operatorname{QGr}(R))^c = D^b_{\operatorname{qgr}(R)}(\operatorname{QGr}(R))$.

Proof. By Lemma 4.2.2 $D(\operatorname{QGr}(R))^c$ is classically generated by $\{\mathcal{O}(n)\}_{n\in\mathbb{Z}}$. Since $\mathcal{O}(n) \in \operatorname{qgr}(R)$ this proves one inclusion.

To prove the other inclusion we have to show that every $M \in \operatorname{qgr}(R)$ is compact. This follows easily From the fact that by hypotheses $\operatorname{Ext}^i(M,-)$ has finite cohomological dimension combined with Lemma 4.3.1. □

To conclude we give an alternative description of $D^b_{\operatorname{qgr}(R)}(\operatorname{QGr}(R))$.

Lemma 4.3.3. *The canonical functor $D^b(\mathrm{qgr}(R)) \to D^b_{\mathrm{qgr}(R)}(\mathrm{QGr}(R))$ is an equivalence.*

Proof. According to the dual version of [18, 1.7.11] it is sufficient to prove the following result: if $B \to C$ is an epimorphism in $\mathrm{QGr}(R)$ with $C \in \mathrm{qgr}(R)$ then there exists a map $D \to B$ with $D \in \mathrm{qgr}(R)$ such that the composition $D \to B \to C$ is an epimorphism.

The map $B \to C$ is obtained From a map $\theta : B_0 \to C_0$ in $\mathrm{Gr}(R)$ with $C_0 \in \mathrm{gr}(R)$. But then the cokernel of θ is finite dimensional and hence without loss of generality we may assume that θ is epic. Since C_0 is finitely generated we may select a finitely generated graded submodule D_0 of B_0 which contains inverse images of the generators of C_0. This proves what we want. □

Combining everything we now obtain:

Theorem 4.3.4. *Let R be a graded left coherent ring which satisfies the following hypotheses.*

1. $\dim \mathrm{Ext}^i(k, k)$ *is finite dimensional for all i.*
2. R *satisfies μ and μ^{opp}.*
3. τ^{opp} *has finite cohomological dimension.*
4. $\mathrm{QGr}(R)$ *has finite homological dimension.*

Then $D^b(\mathrm{qgr}(R))$ is Ext*-finite and saturated.*

5. Derived categories of analytic surfaces

We have shown in Corollary 3.1.5 that if X is a smooth proper algebraic variety over a field k then $D^b(\mathrm{coh}(X))$ is saturated. Since smooth proper algebraic varieties and compact analytic manifolds have similar properties it is a natural question to ask if this result remains true if we assume that X is compact analytic. In this section we show that the answer to this question is negative.

5.1. **Serre functors.** Let X be a connected compact complex analytic manifold of dimension n. Write $D^b_{\mathrm{coh}}(X)$ for the bounded derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. We first prove that $D^b_{\mathrm{coh}}(X)$ has a Serre functor [8]. This is presumably well-known.

Proposition 5.1.1. *Let $\mathcal{E}, \mathcal{F} \in D^b_{\mathrm{coh}}(X)$. Then there are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, S\mathcal{E})^* \tag{5.1}$$

where $S\mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_X} \omega_X[n]$.

Proof. We start with classical Serre duality [30]:

- $H^n(X, \omega_X) = \mathbb{C}$.
- Let $\mathcal{F} \in \mathrm{coh}(X)$. The Yoneda pairing
$$H^i(X, \mathcal{F}) \otimes \mathrm{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \to \mathbb{C}$$
is non-degenerate.

Now let $\mathcal{F} \in D^b_{\mathrm{coh}}(\mathcal{O}_X)$. From the pairing

$$R\Gamma(X, \mathcal{F}) \otimes_k R\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \to R\Gamma(X, \omega_X) \to \mathbb{C}[-n]$$

we obtain a map

$$R\Gamma(X, \mathcal{F}) \to R\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X[n])^* \tag{5.2}$$

We claim that this is an isomorphism. By induction over triangles we reduce to the case $\mathcal{F} \in \mathrm{coh}(X)$. Then to show that (5.2) is an isomorphism we have to show that it is an isomorphism on cohomology, which is precisely classical Serre duality.

If $\mathcal{E} \in D^-(X)$, $\mathcal{F} \in D^+(X)$ then we have the usual local-global isomorphism [40]:

$$R\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) = R\Gamma(X, \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}))$$

Now assume $\mathcal{E},\mathcal{F} \in D^b_{\mathrm{coh}}(X)$, $\mathcal{G} \in D^+(X)$. We claim that the following holds.

(a) $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) \in D^b_{\mathrm{coh}}(X)$.

(b) The natural map

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{E} \overset{L}{\otimes} \mathcal{G}) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}),\mathcal{G})$$

is an isomorphism.

Since these statements are local we may assume that $\mathcal{E}$, $\mathcal{F}$ are bounded free complexes. In that case (a) and (b) are obvious.

The proof of the proposition now follows From the following computation:

$$\begin{aligned}
R\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}) &= R\Gamma(X, \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}))\\
&\cong R\operatorname{Hom}_{\mathcal{O}_X}(\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}),\omega_X[n])^*\\
&= R\Gamma(X, \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{F}),\omega_X[n]))^*\\
&= R\Gamma(X, \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\omega_X[n]\otimes\mathcal{E}))^*\\
&= R\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\omega_X[n]\otimes\mathcal{E})^* \quad \square
\end{aligned}$$

5.2. **Comparison of** Ext. If X is algebraic then it is well-known and easy to prove that $D^b(\mathrm{coh}(X))$ and $D^b_{\mathrm{coh}}(X)$ are equivalent. We don't know if the corresponding result is true for the complex analytic case. For surfaces it is implied by the following proposition.

Proposition 5.2.1. *Let X be a smooth compact analytic surface. Then the Yoneda* Ext*-groups in* $\mathrm{coh}(X)$ *coincide with the* Ext*-groups in the category of all $\mathcal{O}_X$-modules.*

Proof. Let us respectively write ${}^I\operatorname{Ext}$ and ${}^{II}\operatorname{Ext}$ for the Yoneda Ext and the Ext in $\mathrm{Mod}(\mathcal{O}_X)$. Both Ext's are δ-functors in their first and second argument and they coincide in degree zero. Hence to show that ${}^I\operatorname{Ext} = {}^{II}\operatorname{Ext}$ it is sufficient to show that ${}^{II}\operatorname{Ext}$ is elementwise effaceable in its first argument [15, Lemma II.2.1.3]. That is if $i > 0$, $\mathcal{E},\mathcal{F} \in \mathrm{coh}(X)$ and $f \in \operatorname{Ext}^i(\mathcal{E},\mathcal{F})$ then we have to show that there exists an epimorphism $\mathcal{E}' \to \mathcal{E}$ in $\mathrm{coh}(X)$ such that the image of f under the induced map ${}^{II}\operatorname{Ext}^i(\mathcal{E},\mathcal{F}) \to {}^{II}\operatorname{Ext}^i(\mathcal{E}',\mathcal{F})$ is zero.

Let $\mathcal{E},\mathcal{F} \in \mathrm{coh}(X)$. We clearly have

$${}^I\operatorname{Ext}^1(\mathcal{E},\mathcal{F}) = {}^{II}\operatorname{Ext}^1(\mathcal{E},\mathcal{F})$$

since the extension of two coherent sheaves is coherent. Since ${}^I\operatorname{Ext}^1$ is effaceable, so is ${}^{II}\operatorname{Ext}^1$.

Furthermore we also have ${}^{II}\operatorname{Ext}^i(\mathcal{E},\mathcal{F}) = 0$ for $i > 2$. This follows for example From (5.1). Hence by [6] we only have to show that ${}^{II}\operatorname{Ext}^2$ is effaceable. To do this we use the following sublemma:

Sublemma. Let $\mathcal{E},\mathcal{F} \in \mathrm{coh}(X)$. Choose $x \in X$ and let m_x be the corresponding maximal ideal in $\mathcal{O}_X$. Then there exists n such that ${}^{II}\operatorname{Ext}^2(m_x^n\mathcal{E},\mathcal{F}) = 0$.

Proof. By (5.1) it suffices to show that for $n \gg 0$ one has $\operatorname{Hom}(\mathcal{G}, m_x^n\mathcal{E}) = 0$ with $\mathcal{G} = \mathcal{F}\otimes\omega_X^{-1}$. Since $\operatorname{Hom}(\mathcal{G}, m_x^n\mathcal{E})$ is finite dimensional it is clearly sufficient to show that for $a \in \mathbb{N}$ there exists $b > a$ such that $\operatorname{Hom}(\mathcal{G}, m_x^b\mathcal{E}) \neq \operatorname{Hom}(\mathcal{G}, m_x^a\mathcal{E})$.

So pick a non-zero $f : \mathcal{G} \to m_x^a\mathcal{E}$. Then there will exist b such that $\operatorname{im} f_x \not\subset m_x^b\mathcal{E}_x$ (look at stalks). Hence $f \notin \operatorname{Hom}(\mathcal{G}, m_x^b\mathcal{E})$. This finishes the proof. $\square$

To complete the proof that ${}^{II}\operatorname{Ext}^2$ is effaceable we pick $x \neq y$ in X and we choose n such that ${}^{II}\operatorname{Ext}^2(m_x^n\mathcal{E}, \mathcal{F}) = {}^{II}\operatorname{Ext}^2(m_y^n\mathcal{E}, \mathcal{F}) = 0$. Since the canonical map $m_x^n\mathcal{E} \oplus m_y^n\mathcal{E} \to \mathcal{E}$ is surjective, we are done. □

Corollary 5.2.2. *Let X be as above. Then the canonical functor $F : D^b(\operatorname{coh}(X)) \to D^b_{\operatorname{coh}}(X)$ is an equivalence.*

Proof. By induction over triangles and the above proposition we see that F is fully faithful. That it is essentially surjective also follows by induction over triangles. □

5.3. **The derived category of an exact category.** Assume that $\mathcal{E}$ is an exact category [28]. In [24] Neeman defines the derived category $D(\mathcal{E})$ of $\mathcal{E}$. By definition $D(\mathcal{E}) = K(\mathcal{E})/K(\mathcal{E})^{eac}$ where as usual $K(\mathcal{E})$ is the homotopy category of $\mathcal{E}$ and $K(\mathcal{E})^{eac}$ is the epaisse envelope of the category $K(\mathcal{E})^{ac}$ of acyclic complexes in $K(\mathcal{E})$. By definition a complex

$$\cdots \to X^n \to X^{n+1} \to X^{n+2} \to \cdots$$

is *acyclic* if each map $X^n \to X^{n+1}$ decomposes in $\mathcal{E}$ as a composition of an admissible epimorphism with an admissible monomorphism: $X^n \to D^n \to X^{n+1}$ such that $D^n \to X^{n+1} \to D^{n+1}$ is exact. Since by [24, Lemma 1.1] $K(\mathcal{E})^{ac}$ is triangulated it follows From lemma 2.2.2 that every object in $K(\mathcal{E})^{eac}$ is a direct summand of an object in $K(\mathcal{E})^{ac}$. Furthermore if $\mathcal{E}$ is Karoubian then by [24, Lemma 1.2] $K(\mathcal{E})^{eac} = K(\mathcal{E})^{ac}$.

5.4. **Torsion pairs in abelian categories.** Assume that $\mathcal{C}$ is an abelian category and let $(\mathcal{T}, \mathcal{F})$ be a *torsion pair* in $\mathcal{C}$, i.e. $\mathcal{T}$ and $\mathcal{F}$ are full subcategories in $\mathcal{C}$ such that $\operatorname{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and every object $C \in \mathcal{C}$ fits in a unique exact sequence

$$0 \to T \to C \to F \to 0 \tag{5.3}$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. It follows that $\mathcal{T}$ and $\mathcal{F}$ are respectively closed under quotients and subobjects.

The assignments $C \mapsto T$ and $C \mapsto F$ in the exact sequence (5.3) yield functors $\tau : \mathcal{C} \to \mathcal{T}$ and $\phi : \mathcal{C} \to \mathcal{F}$ which are respectively the right and left adjoint to the inclusions $\mathcal{T} \to \mathcal{C}$, $\mathcal{F} \to \mathcal{C}$.

It is easy to see that $\mathcal{T}$ and $\mathcal{F}$ possess kernels and cokernels. We have formulas

$$\begin{aligned} \ker_{\mathcal{F}} &= \ker_{\mathcal{C}} \\ \operatorname{coker}_{\mathcal{F}} &= \phi \circ \operatorname{coker}_{\mathcal{C}} \end{aligned} \tag{5.4}$$

and dual formulas for $\mathcal{T}$.

Following [13] we say that $(\mathcal{T}, \mathcal{F})$ is *tilting* if every object in $\mathcal{C}$ is a subobject of an object in $\mathcal{T}$. Similarly $(\mathcal{T}, \mathcal{F})$ is *cotilting* if every object in $\mathcal{C}$ is a quotient of an object in $\mathcal{F}$.

The torsion pair $(\mathcal{T}, \mathcal{F})$ defines a t-structure on $D^b(\mathcal{C})$ by

$$\begin{aligned} {}^pD^b(\mathcal{C})^{\leq 0} &= \{C \in D^b(\mathcal{C})^{\leq 1} \mid H^1(C) \in \mathcal{T}\} \\ {}^pD^b(\mathcal{C})^{\geq 0} &= \{C \in D^b(\mathcal{C})^{\geq 0} \mid H^0(C) \in \mathcal{F}\} \end{aligned}$$

By definition the tilting ${}^p\mathcal{C}$ of $\mathcal{C}$ with respect to $(\mathcal{T}, \mathcal{F})$ is the heart of this t-structure. It is easy to see that $(\mathcal{F}, \mathcal{T}[-1])$ is a torsion pair in ${}^p\mathcal{C}$. Furthermore according to [13, Prop. I.3.2] $(\mathcal{T}, \mathcal{F})$ is tilting if and only if $(\mathcal{F}, \mathcal{T}[-1])$ is cotilting and vice versa.

Let $\mathcal{E}$ be either $\mathcal{T}$ or $\mathcal{F}$. The exact structure on $\mathcal{C}$ induces an exact structure on $\mathcal{E}$. This is intrinsically determined in the following way: a morphism $f : A \to B$ in $\mathcal{E}$ is *strict* if the canonical morphism $\operatorname{coker}\ker f \to \ker\operatorname{coker} f$ is an isomorphism. A diagram

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an admissible exact sequence if f is a strict monomorphism, g is a strict epimorphism and $\operatorname{coker} f = g$, $\ker g = f$.

The following statements are obvious.

Lemma 5.4.1. (1) *A complex over $\mathcal{E}$ is acyclic if and only if it is acyclic in $\mathcal{C}$.*

(2) *$K(\mathcal{E})^{eac} = K(\mathcal{E})^{ac}$.*

(3) *A map between complexes over $\mathcal{E}$ is an isomorphism in $D(\mathcal{E})$ if and only if it is a quasi-isomorphism over $\mathcal{C}$.*

Lemma 5.4.2. [6, Ex. 1.3.23(iii)] *Assume that $(\mathcal{T},\mathcal{F})$ is cotilting. Then the canonical map $D(\mathcal{F}) \to D(\mathcal{C})$ is an equivalence.*

Proof. Since $(\mathcal{T},\mathcal{F})$ is cotilting and $\mathcal{F}$ is closed under subobjects, every object in $\mathcal{C}$ has a resolution of length two by objects in $\mathcal{F}$. Therefore by the (dual version) of [14, Lemma I.4.6] it follows that if X is a complex over $\mathcal{C}$ there exists a quasi-isomorphism $F \to X$ with F a complex over $\mathcal{F}$.

We find for F_1, F_2 complexes over $\mathcal{F}$

$$\begin{aligned}
\operatorname{Hom}_{D(\mathcal{C})}(F_1,F_2) &= \varinjlim_{X \underset{\text{qi}}{\to} F_1} \operatorname{Hom}_{K(\mathcal{C})}(X,F_2)\\
&= \varinjlim_{\substack{F_1' \underset{\text{qi}}{\to} F_1\\ F_1' \in K(\mathcal{F})}} \operatorname{Hom}_{K(\mathcal{C})}(F_1',F_2)\\
&= \operatorname{Hom}_{D(\mathcal{F})}(F_1,F_2)
\end{aligned}$$

The last equality follows From lemma 5.4.1(3). □

This result was also proved by Schneiders in the (equivalent) setting of quasi-abelian categories. This is explained in Appendix B.

The following result is proved in [13] under some additional (unnecessary) conditions.

Proposition 5.4.3. *Assume that $(\mathcal{T},\mathcal{F})$ is cotilting. Then $D({}^p\mathcal{C}) = D(\mathcal{C})$.*

Proof. According to lemma 5.4.2 we have $D(\mathcal{C}) = D(\mathcal{F})$. Since $(\mathcal{F},\mathcal{T}[-1])$ is tilting, we can invoke the dual result for ${}^p\mathcal{C}$ which is $D({}^p\mathcal{C}) = D(\mathcal{F})$. Since the exact structure on $\mathcal{F}$ is intrinsic, the induced exact structures on $\mathcal{F}$ From the inclusions $\mathcal{F} \to \mathcal{C}$ and $\mathcal{F} \subset {}^p\mathcal{C}$ are the same and this finishes the proof. □

Remark 5.4.4. Lemma 5.4.2 and Propositions 5.4.3 are also valid for D^b (the equivalences preserve boundedness).

5.5. **Tilting in noetherian abelian categories.**

Lemma 5.5.1. *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair in $\mathcal{C}$. Then $\mathcal{C}$ is noetherian if and only if the following conditions hold:*

N1. *Every chain of subobjects of F: $F_0 \subset F_1 \subset F_2 \subset \cdots$ for $F_i \in \mathcal{F}$, $F \in \mathcal{F}$ becomes stationary.*

N2. *Every chain of epimorphisms $T_0 \to T_1 \to T_2 \to \cdots$ for $T_i \in \mathcal{T}$ becomes stationary.*

Proof. Let us show that N1, N2 imply $\mathcal{C}$ noetherian. Let $C_0 \subset C_1 \subset \cdots$ be an ascending chain of subobjects of $C \in \mathcal{C}$. The sequence $F_i = \operatorname{Im}(\phi(C_i) \to \phi(C))$ becomes stationary by N1. Denote by $F \subset \phi(C)$ the limiting subobject of the sequence. We may relace C by the fibred product $C' = C \times_{\phi(C)} F = \ker(C \oplus F \to \phi(C))$. Indeed, the natural morphisms $C_i \to C'$ are monic, because so are the composites $C_i \to C' \to C$.

By construction of C', $\phi(C') = F$ and the maps $\phi(C_i) \to \phi(C')$ are epic for $i \gg 0$. If $R_i = C'/C_i$, then we have a complex $\phi(C_i) \to \phi(C') \to \phi(R_i)$. As ϕ is a left adjoint it takes epi to epi, so both morphisms in this complex are epic. It follows that $\phi(R_i) = 0$, i.e. $R_i \in \mathcal{T}$ for $i \gg 0$. Therefore, the chain of epimorphisms $R_0 \to R_1 \to \dots$ becomes stationary by N2. This proves that the primary chain of C_i's becomes stationary. The converse statement is obvious. □

By (5.4) morphisms in $\mathcal{T}$ are epimorphisms iff they are epimorphisms in $\mathcal{C}$ and morphisms in $\mathcal{F}$ are monomorphisms iff they are monomorphisms in $\mathcal{C}$. So N1 and N2 are intrinsic in $\mathcal{T}, \mathcal{F}$.

We will use the following criterion for ${}^p\mathcal{C}$ to be noetherian.

Lemma 5.5.2. *Assume that $\mathcal{C}$ is noetherian and $(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathcal{C}$. Then ${}^p\mathcal{C}$ is noetherian if and only if the following is true: every ascending chain $F_0 \subset F_1 \subset \cdots$ with $F_i \in \mathcal{F}$ and $\operatorname{coker}(F_0 \to F_i) \in \mathcal{T}$ for all i, is stationary.*

Proof. If there is an ascending chain as in the statement of the lemma which is not stationary then it is easy to see that we have an ascending chain of subobjects of F_0 in ${}^p\mathcal{C}$

$$F_1/F_0[-1] \subset F_2/F_0[-1] \subset F_3/F_0[-1] \subset \cdots .$$

Hence ${}^p\mathcal{C}$ is not noetherian. So we will now concentrate on the converse direction.

By lemma 5.5.1, to check that ${}^p\mathcal{C}$ is noetherian we have to verify N1, N2 with $\mathcal{T}$ and $\mathcal{F}$ exchanged. To this end we have to know the nature of monomorphisms in $\mathcal{T}$ and epimorphisms in $\mathcal{F}$. From (5.4) we obtain:

- Monomorphisms in $\mathcal{T}$ are the maps whose kernel in $\mathcal{C}$ is in $\mathcal{F}$.
- Epimorphisms in $\mathcal{F}$ are the maps whose cokernel in $\mathcal{C}$ is in $\mathcal{T}$.

Let us now check that N2 holds if we replace $\mathcal{T}$ by $\mathcal{F}$. Thus we have a chain of maps in $\mathcal{F}$

$$F_0 \to F_1 \to F_2 \to \cdots \tag{5.5}$$

whose cokernel is in $\mathcal{T}$. Using the fact that $\mathcal{C}$ is noetherian we see that the kernel $K_{ij} = \ker(F_i \to F_j)$ will become stationary for $j \gg 0$. Let $K_i = K_{ij}$ for $j \gg 0$. Then the maps $F_i/K_i \to F_{i+1}/K_{i+1}$ are injective. Using the fact that F_i/K_i injects in F_j for $j \gg 0$ we see that $F_i/K_i \in \mathcal{F}$. Furthermore $(F_{i+1}/K_{i+1})/(F_i/K_i)$ is a quotient of $\operatorname{coker}(F_i \to F_{i+1})$ so it lies in $\mathcal{T}$.

It follows that the condition given in the statement of the lemma holds for the sequence $(F_i/K_i)_i$, i.e. this sequence will become stationary. Hence by left shifting if necessary we may assume that $F_i/K_i \to F_{i+1}/K_{i+1}$ is an isomorphism for all $i \geq 0$. From the snake lemma we then deduce that $\operatorname{coker}(K_i \to K_{i+1})$ is isomorphic to $\operatorname{coker}(F_i \to F_{i+1})$ and hence is in $\mathcal{T}$. This implies that for $j \gg 0$, $K_j = \operatorname{coker}(K_0 \to K_j) \in \mathcal{T}$. Since also $K_j \in \mathcal{F}$ this implies $K_j = 0$ for $j \gg 0$. Truncating the beginning of the sequence by sufficiently big j we obtain a sequence which satisfies the conditions in the statement of the lemma. This implies that $F_j \to F_{j+1}$ is an isomorphism for $j \gg 0$.

Let us now assume N2 and check that N1 holds if we replace $\mathcal{F}$ by $\mathcal{T}$. Thus we have a chain of maps in $\mathcal{T}$

$$T_0 \to T_1 \to T_2 \to \cdots \to T$$

whose kernel is in $\mathcal{F}$. Since $\mathcal{C}$ is noetherian, the images of the maps $T_i \to T$ will become stationary. Since these images are in $\mathcal{T}$ we may without loss of generality assume that the maps $T_i \to T$ are surjective. Put $F_i = \ker(T_i \to T)$. Then $\operatorname{coker}(F_i \to F_{i+1})$, being isomorphic to $\operatorname{coker}(T_i \to T_{i+1})$, is in $\mathcal{T}$. Hence the chain $(F_i)_i$ is like that in (5.5), hence it becomes stationary. This implies that the chain $(T_i)_i$ also becomes stationary. $\square$

Remark 5.5.3. If $\mathcal{T} \subset \mathcal{C}$ is the subcategory of torsion sheaves in the category of coherent sheaves on an analitic or algebraic variety (the case of our interest in the next subsection), then $\mathcal{T}$ has a property to be closed under subobjects in $\mathcal{C}$. Under this additional condition the proof of the lemma can be simplified in two places: $\operatorname{coker}(K_i \to K_{i+1})$ are torsion being subobjects of $\operatorname{coker}(F_i \to F_{i+1})$ and N1 with $\mathcal{F}$ replaced by $\mathcal{T}$ is automatically satisfied as the kernels of $T_i \to T_{i+1}$ and $T_i \to T$ are trivial.

5.6. **Non-saturation for analytic surfaces.** We can now prove the following result:

Theorem 5.6.1. *Let X be a smooth compact analytic surface with no curves. Then $D^b(\operatorname{coh}(X))$ is not saturated.*

By Corollary 5.2.2 the result also holds for $D^b_{\operatorname{coh}}(X)$.

Proof.

Step 1. Let $\mathcal{T} \subset \operatorname{coh}(X)$ be the full subcategory of objects in $\operatorname{coh}(X)$ whose support is strictly smaller than X. Since X contains no curves and is compact, this support must be a finite set of points. Let $\mathcal{F}$ be the full subcategory of objects F in $\operatorname{coh}(X)$ such that $\operatorname{Hom}(\mathcal{T}, F) = 0$. It is clear that $(\mathcal{T}, \mathcal{F})$ is a torsion pair.

Step 2. $\mathcal{T}$ is closed under essential extensions. To prove this let $T \in \mathcal{T}$ and let $T \subset T'$ be an essential extension. Let $\{x_1, \dots, x_n\} \in X$ be the support of T. By the Artin-Rees property of the stalks of $\mathcal{O}_{X,x_i}$ there exists $t \geq 0$ such that $m^t_{x_i} T'_{x_i} \cap T_{x_i} = 0$ for all i. Thus $m^t_{x_1} \cdots m^t_{x_n} T' \cap T = 0$ and since we are in an essential extension it follows $m^t_{x_1} \cdots m^t_{x_n} T' = 0$. Hence $T' \in \mathcal{T}$.

Step 3. Let $E \in \operatorname{coh}(X)$. Then E is a quotient of an object in $\mathcal{F}$. In [34] Schuster proved the more general result that every coherent sheaf on a complex surface is a quotient of a vector bundle. We give a simple proof of the weaker statement that we need.

We write E as an extension

$$0 \to T \to E \to F \to 0$$

where $T \in \mathcal{T}$ is torsion and $F \in \mathcal{F}$. Take the maximal $E' \subset E$, such that $E' \cap T = 0$. As $T \subset E/E'$ is an essential extension, then by the previous step $E/E' \in \mathcal{T}$. $E' \in \mathcal{F}$ by the choice of T. We now obtain and exact sequence

$$0 \to E' \to E \to T' \to 0$$

with $T' \in \mathcal{T}$. It is easy to see that every object in $\mathcal{T}$ is a quotient of a free $\mathcal{O}_X$-module. So write T' as a quotient of $F' \in \mathcal{F}$ and let E'' be the corresponding pullback of E. Then E'' is an extension of E' and F' and hence $E'' \in \mathcal{F}$. Thus we have written E as a quotient of $E'' \in \mathcal{F}$.

Step 4. By the previous step $(\mathcal{T}, \mathcal{F})$ is cotilting. Hence by lemma 5.4.3 $D^b(\operatorname{coh}(X)) = D^b({}^p\operatorname{coh}(X))$.

Step 5. Now we claim that ${}^p\operatorname{coh}(X)$ is noetherian. By lemma 5.5.2 we need to show that every ascending chain

$$F_0 \subset F_1 \subset F_2 \subset \cdots$$

with $F_i \in \mathcal{F}$, $F_i/F_0 \in \mathcal{T}$ becomes stationary.

This is satisfied in our case because we must have $F_n \subset F_0^{**}$ and F_0^{**}/F_0 has finite length.

Step 6. Note that ${}^p\mathrm{coh}(X)$ is self-dual under $R\mathcal{H}om(-,\mathcal{O}_X)$. Hence it is both noetherian and Artinian. Thus ${}^p\mathrm{coh}(X)$ has finite length.

Step 7. Assume that $\mathrm{coh}(X)$ is saturated. By Step 4 ${}^p\mathrm{coh}(X)$ will also be saturated. Since this is a finite length category it follows From lemma 2.5.1 that it has to be of the form $\mathrm{mod}(\Lambda)$ for a finite dimensional algebra Λ.

By proposition 5.1, $S[-2]$ is a functor which preserves $\mathrm{coh}(X)$, $\mathcal{T}$ and $\mathcal{F}$. Hence it preserves ${}^p\mathrm{coh}(X)$ (regarded as a subcategory in $D^b(\mathrm{coh}(X))$. For a finite dimensional algebra the Serre functor takes projectives into injectives. Therefore, its shift by -2 cannot preserve the category $\mathrm{mod}(\Lambda)$. We have obtained a contradiction.

Remark 5.6.2. It seems likely that this counter example is only the tip of the iceberg and that in fact a compact analytic manifold is saturated if and only if it is an algebraic space. This would mean that saturatedness would be a criterion for a triangulated category to be of algebraic nature.

Among the surfaces to which the theorem is applicable are K3, 2-dimensional tori and surfaces of type VII in the Kodaira classification [23].

Appendix A. An alternative proof in the commutative case

Theorem 1.1, as stated follows From the non-commutative result Theorem 4.2.12. However in the commutative case it is possible to give a straightforward proof of a more general result.

Theorem A.1. *Assume that X is a projective variety over a field k. Let $\mathcal{D}$ be the triangulated category of perfect complexes on X. Then every contravariant cohomological functor of finite type on $\mathcal{D}$ is representable by a bounded complex with coherent homology.*

Proof. According to [11, lemma 2.13] H is represented by an object E in $D(\mathrm{Qch}(X))$. We have to show that this object is in $D^b(\mathrm{coh}(X))$. To prove this we repeat the argument of [8].

Choose an embedding $\pi : X \to \mathbb{P}^n$ and consider the functor $H' = H\circ R\pi^* : D^b(\mathrm{coh}(\mathbb{P}^n)) \to \mathrm{Vect}(k)$. According to Beilinson's result [5] as it was reformulated in [3, 9] there is an equivalence $\theta : D^b(\mathrm{mod}(\Lambda)) \to D^b(\mathrm{coh}(\mathbb{P}^n))$ where Λ a finite dimensional algebra of finite global dimension. Put $H'' = H' \circ \theta$. Invoking [11, lemma 2.13] again we see that H'' is representable by an object G in $D(\Lambda)$. Since H'' is still of finite type it follows that $\sum_n \dim H''^n(\Lambda) = \sum_n \dim \mathrm{Hom}(\Lambda[n], G) < \infty$. Thus $G \in D^b(\mathrm{mod}(\Lambda))$. This implies that H' is represented by $F = \theta^{-1}(G) \in D^b(\mathrm{coh}(X))$.

Thus if $A \in D^b(\mathrm{coh}(\mathbb{P}^n))$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{P}^n}(A, R\pi_*E) &= \mathrm{Hom}_X(R\pi^*A, E)\\ &= H(R\pi^*A)\\ &= H'(A)\\ &= \mathrm{Hom}_{\mathbb{P}^n}(A, F)\end{aligned}$$

Putting $A = F$ we obtain a map $\mu : F \to \pi_*E$ which becomes an isomorphism if we apply $\mathrm{Hom}_{\mathbb{P}^n}(A, -)$ for $A \in D^b(\mathrm{coh}(\mathbb{P}^n))$. In other words the cone of μ is right orthogonal to $D^b(\mathrm{coh}(\mathbb{P}^n)))$. By taking $A = \mathcal{O}(n)_n$ it follows easily that the cone of μ is zero and hence μ is an isomorphism. Thus $\pi_*E \in D^b(\mathrm{coh}(\mathbb{P}^n))$. This implies $E \in D^b(\mathrm{coh}(X))$ $\square$

Appendix B. Quasi-abelian categories

In this appendix we discuss quasi-abelian categories. Let $\mathcal{E}$ be an addititive category with kernels and cokernels. A morphism $f : A \to B$ is said to be *strict* if the canonical map $\operatorname{coker}\ker f \to \ker\operatorname{coker} f$ is an isomorphism.

We say $\mathcal{E}$ is quasi-abelian if $\mathcal{E}$ satisfies the property that the pullback of any strict epi is strict epi and the pushout of any strict mono is strict mono. Quasi-abelian category appear frequently in the literature, often under different names. They are called "preabelian" in [16], "semiabelian" in [29] and quasi-abelian in [33, 12]. It can also be seen that quasi-abelian categories are additive categories which are regular and coregular [4]. In this appendix we show that the notion of a quasi-abelian category is the same as that of a (co)tilting torsion theory.

Let $\mathcal{E}$ be quasi-abelian. $\mathcal{E}$ carries an intrinsic exact structure with the admissible mono- and epimorphism being respectively the strict mono- and epimorphisms [33, §1.1.4].

In [33, §1.2.3] it is shown that $\mathcal{E}$ has two canonical embeddings into abelian categories $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$ preserving and reflecting exactness. Furthermore $\mathcal{E}$ is stable under extensions in these embeddings.

Proposition B.1. [33, Prop. 1.2.35] *The embedding $\mathcal{E} \subset \mathcal{LH}(\mathcal{E})$ is characterized by the following properties: $\mathcal{E} \subset \mathcal{LH}(\mathcal{E})$ is a fully faithful embedding of $\mathcal{E}$ into an abelian category, $\mathcal{E}$ is closed under subobjects in $\mathcal{LH}(\mathcal{E})$ and every object in $\mathcal{LH}(\mathcal{E})$ is a quotient of an object in $\mathcal{E}$.*

The following result is [33, Prop. 1.2.31].

Proposition B.2. *The inclusion $\mathcal{E} \subset \mathcal{LH}(\mathcal{E})$ extends to an equivalence of derived categories $D(\mathcal{E}) \cong D(\mathcal{LH}(\mathcal{E}))$.*

The following result shows that the notion of a quasi-abelian category is the same as that of (co)tilting torsion theory.

Proposition B.3. *Let $\mathcal{E}$ be an additive category. The following are equivalent.*

(1) *$\mathcal{E}$ is quasi-abelian.*
(2) *There exists a cotilting torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{C}$ with $\mathcal{E} \cong \mathcal{F}$.*
(3) *There exists a tilting torsion pair $(\mathcal{T}', \mathcal{F}')$ in an abelian category $\mathcal{C}'$ with $\mathcal{E} \cong \mathcal{T}'$.*

In the situation of (2) we have $\mathcal{C} \cong \mathcal{LH}(\mathcal{F})$ and in the situation of (3) we have $\mathcal{C}' \cong \mathcal{RH}(\mathcal{F})$.

Proof. That $\mathcal{C} \cong \mathcal{LH}(\mathcal{E})$ and $\mathcal{C}' \cong \mathcal{RH}(\mathcal{E})$ follows directly From Proposition B.1 (and its dual version).

To prove the stated equivalence we note that by symmetry we only need to prove the equivalence of (1) and (2).

(2)⇒(1) Since $\mathcal{F}$ is exact, pullbacks of admissible epimorphisms are admissible epimorphisms. Since the admissible epimorphisms are precisely the strict epimorphisms this shows that pullbacks of strict epimorphisms are strict epimorphisms. The corresponding result for strict monomorphisms is proved in the same way.

(1)⇒(2) Put $\mathcal{F} = \mathcal{E}$ and $\mathcal{C} = \mathcal{LH}(\mathcal{E})$. Let $\mathcal{T}$ be the full subcategory of $\mathcal{C}$ consisting of objects $\operatorname{coker}_{\mathcal{C}} f$ where the morphism f is an epimorphism in $\mathcal{F}$.

We claim that $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion pair in $\mathcal{C}$. If $T = \operatorname{coker}_{\mathcal{C}} f \in \mathcal{T}$ and $F \in \mathcal{F}$ then From the fact that f is an epimorphism in $\mathcal{F}$ we immediately obtain $\operatorname{Hom}(T, F) = 0$.

Now let C be an arbitrary object in $\mathcal{C}$. According to Proposition B.1 there exist a short exact sequence in $\mathcal{C}$

$$F \xrightarrow{f} F' \to C \to 0 \tag{B.1}$$

with $F, F' \in \mathcal{C}$. In particular if $(\mathcal{T}, \mathcal{F})$ is a torsion theory then it will certainly be cotilting.

We will now show that C is an extension of the form (5.3). We have the following commutative diagram

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\alpha"'] & F' \arrow[r, "g'"] \arrow[d, equal] & \operatorname{coker}_{\mathcal{F}} f \arrow[d, equal] \\ \ker_{\mathcal{F}} g' \arrow[r, "f'"'] & F' \arrow[r, "g'"'] & \operatorname{coker}_{\mathcal{F}} f \end{tikzcd} \tag{B.2}$$

It is easily checked that $\operatorname{coker}_{\mathcal{F}} f$ satifies the universal property for being a cokernel of f'. Thus $\operatorname{coker}_{\mathcal{F}} f' = \operatorname{coker}_{\mathcal{F}} f$ and hence g' a strict epimorphism.

Hence we obtain in particular the following: *a cokernel of an arbitrary morphism in $\mathcal{F}$ is a strict epimorphism.* Dually we also obtain: *a kernel of an arbitrary morphism in $\mathcal{F}$ is a strict monomorphism.* Thus in particular f' is a strict monomorphism. It also follows that the lower sequence in (B.2) is an admissible exact sequence.

We claim that α is an epimorphism in $\mathcal{F}$. To show this assume that there is a morphism $\beta : \ker_{\mathcal{F}} g' \to Z$ in $\mathcal{F}$ whose composition with α is zero. We have to prove $\beta = 0$.

We extend the commutative diagram (B.2) as follows:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\alpha"'] & F' \arrow[r, "g'"] \arrow[d, equal] & \operatorname{coker}_{\mathcal{F}} f \arrow[d, equal] \\ \ker_{\mathcal{F}} g' \arrow[r, "f'"'] \arrow[d, "\beta"'] & F' \arrow[r, "g'"'] \arrow[d, "\gamma"] & \operatorname{coker}_{\mathcal{F}} f \\ Z \arrow[r, "f''"] & Z' & \end{tikzcd} \tag{B.3}$$

where the lower square is a pushout in $\mathcal{F}$. We now have $f'' \circ \beta \circ \alpha = 0$ and hence $\gamma \circ f = 0$. Thus $\gamma = \phi \circ g'$ for some morphism $\phi : \operatorname{coker}_{\mathcal{F}} f \to Z'$.

We deduce $f'' \circ \beta = \gamma \circ f' = \phi \circ g' \circ f' = 0$. Since we had assumed that $\mathcal{F} = \mathcal{E}$ is quasi-abelian we know that f'' is a strict monomorphims and in particular a monomorphism. Thus it follows that $\beta = 0$ and hence α is an epimorphism.

Furthermore by looking at the decomposition

$$F \xrightarrow{\alpha} \ker_{\mathcal{F}} g' \xrightarrow{f'} F'$$

of f in $\mathcal{C}$ we find that $C = \operatorname{coker}_{\mathcal{C}} f$ is an extension of $\operatorname{coker}_{\mathcal{C}} f'$ by $\operatorname{coker}_{\mathcal{C}} \alpha$. From the fact that α is an epimorphism in $\mathcal{F}$ we obtain that $\operatorname{coker}_{\mathcal{C}} \alpha$ is in $\mathcal{T}$. Now since the lower sequence in (B.2) is an admissible exact sequence and the embedding of $\mathcal{F} \subset \mathcal{C}$ preserves exactness, we have $\operatorname{coker}_{\mathcal{C}} f' = \operatorname{coker}_{\mathcal{F}} f' \in \mathcal{F}$. This finishes the proof of (1)$\Rightarrow$(2). □

Corollary B.4. *If $(\mathcal{T}, \mathcal{F})$ is a cotilting torsion theory in an abelian category $\mathcal{C}$ then $\mathcal{C} \cong \mathcal{LH}(\mathcal{F})$ and ${}^{p}\mathcal{C} \cong \mathcal{LR}(\mathcal{F})$.*

Proof. By Proposition B.3 we have $\mathcal{C} = \mathcal{LH}(\mathcal{F})$. Now $(\mathcal{F}, \mathcal{T}[-1])$ is a tilting torsion pair in ${}^p\mathcal{C}$ and hence, again by Proposition B.3, we have $\mathcal{C} = \mathcal{RH}(\mathcal{F})$. □

Hence we find that Lemma 5.4.2 follows From Proposition B.2.

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