ON THE DOUBLE OF THE HALL ALGEBRA OF A QUIVER

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Abstract. We show that it is possible to define reflection isomorphisms on the double of the (twisted) Hall algebra of a quiver. Combining these reflections with Fourier transform yields an alternative construction of Lusztig's braid group action on a quantum enveloping algebra.

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1. Introduction

Throughout Q will be a quiver (directed graph). For simplicity we assume that Q has no loops, although this is not very essential. Let k be a field with q elements. The twisted Hall algebra $H(Q)$ of Q is a certain $\mathbb{C}$-algebra which has a canonical basis given by the isomorphism classes of finite dimensional representations of Q (see §6 for precise definitions). It was Ringel's insight [13] that the subalgebra of $H(Q)$ generated by the simple representations (the "composition algebra") is isomorphic to the upper triangular part $U_v(\mathfrak{n})$ of a quantum enveloping algebra associated to Q (for $v = q^{-1/2}$ in our setup).

In particular it follows that the composition algebra is independent of the orientation of Q. In [9, §13] Lusztig states that by using a notion of Fourier transform one can prove the same for the full Hall algebra. Since as far as we know nobody

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has taken the trouble to write down an explicit proof of this, we provide one, for the convenience of the reader, in §7.

In the representation theory of quivers there exists another way to change the orientation of edges. This is through the so-called reflection functors [1]. Suppose that i is a source in Q and let Q' be obtained from Q by changing the orientation of all edges adjacent to i. Then for certain *admissible* representations of Q (see below) one can define a corresponding reflected representation of Q'. It is easy to see that in this way one obtains an isomorphism between the subalgebras of $H(Q)$ and $H(Q')$ corresponding to admissible representations. However it is impossible to extend this to an isomorphism between the full algebras. To do this we have to add "negative" elements to $H(Q)$.

It is well-known how this should be done. The starting point is a beautiful result by Green [3] which asserts that $H(Q)$ is a Hopf algebra in a certain twisted sense. To make $H(Q)$ into a real Hopf algebra one introduces appropriate Cartan elements. If one then performs a Drinfeld double construction and divides out certain diagonal elements one obtains a Hopf algebra which we denote by $U(Q)$ and which is an analogue of $U_v(\mathfrak{g})$. In fact it was shown by Xiao Jie in [14] that $U_v(\mathfrak{g})$ is contained in $U(Q)$.

Our main result is (Theorem 9.1)

Theorem A. *Let Q, Q' and i be as above. Then it is possible to define an isomorphism between $U(Q)$ and $U(Q')$ which on the isomorphism classes of admissible representations is given by the usual reflection functor associated to i.*

The next step is to combine the Fourier transform and the reflection isomorphisms to obtain for every vertex i an automorphism of $U(Q)$. This is done as follows. First we make i into a source by a Fourier tranform. Then we reflect and finally we go back to Q by another Fourier transform. We show that the result is in fact independent of the intermediate steps and hence yields a well defined automorphism of $U(Q)$ which we denote by $\hat{t}_i$. We can then prove the following theorem (Theorem 13.1).

Theorem B. *The $\hat{t}_i$ restrict to the automorphisms $T''_{i,-1}$ of $U_v(\mathfrak{g})$ which were defined by Lusztig* [11, 10] *(and independently by Levendorski and Soibelman* [8] *on a certain completion of $U_v(\mathfrak{g})$).*

A slightly restricted version of this result was proved in the Dynkin case by Lusztig [9] (using a different method).

Thus we have obtained an alternative construction of the automorphisms $T''_{i,-1}$ of $U_v(\mathfrak{g})$ (Lusztig considers in fact several other variants of these automorphisms, but these are all fairly obvous modifications of the $T''_{i,-1}$). A distinguishing feature of our approach is that we have defined the automorphisms globally whereas in Lusztig's aproach they are defined on generators.

One very important property of Lusztig's automorphisms is that they yield a braid group action. It would be nice to be able to give proof of this fact using representation theory of quivers, but we have not been able to do so, except in simple cases (such as the Dynkin case). We conjecture that the $\hat{t}_i$ satisfy the braid relations on the whole of $U(Q)$ instead of just on $U_v(\mathfrak{g})$.

2. Conventions

All Hopf-algebras we use are defined over $\mathbb{C}$ and have invertible antipode. We write $\Delta(a) = a_1 \otimes a_2$, ommitting the parentheses and the summation sign in the Sweedler convention.

Throughout q is a fixed prime power. k will usually be the field with q elements, although in §4, k will be arbitrary. Unadorned tensor products are over $\mathbb{C}$ or over k, depending on the context.

In the sequel we will sometimes use v-factorials for $v \in \mathbb{C}^*$, so we state their definition here :

$$[n]_v^! = \frac{(v^n - v^{-n})(v^{n-1} - v^{-n+1}) \cdots (v - v^{-1})}{(v - v^{-1})^n}$$

Note that $[n]_v^! = [n]_{1/v}^!$. This eliminates at least one possibility for sign errors!

If A is an algebra and $a \in A$ then we write $a_v^{(n)}$ for $a^n/[n]_v^!$.

Our definitions regarding Hall algebras are usually taken from Kapranov [7]. Therefore some of our formulas are slightly different from the ones occuring in [3, 13, 14].

3. The Drinfeld double

In this section we define the Drinfeld double by generators and relations. This will be slightly more convenient than the usual definition which puts a multiplication on the tensor product of two Hopf algebras.

Assume that A, B are Hopf algebras and let $(-,-) : A \times B \to \mathbb{C}$ be a Hopf pairing. Then the Drinfeld double [2] $D(A,B)$ of this data is by definition the free poduct $A * B$ divided out by the following relations (with $a \in A$, $b \in B$)

$$(b_1 * a_2)(a_1, b_2) - (a_1 * b_2)(a_2, b_1) \tag{3.1}$$

Note that equivalently we may impose the relations

$$b * a = (a_1, S^{-1}(b_3))(a_2 * b_2)(a_3, b_1) \tag{3.2}$$

(see [5, p72]).

We write $a \otimes b$ for the image of $a * b$ in $D(A,B)$.

Lemma 3.1. *The maps*

$$A \otimes B \to D(A,B) : a \otimes b \to a \otimes b \tag{3.3}$$

$$A \otimes B \to D(A,B) : a \otimes b \to (1 \otimes b)(a \otimes 1) \tag{3.4}$$

are isomorphisms of vector spaces.

Proof. This is classical. Since both maps are similar we only consider the first one. We make $A \otimes B$ into an algebra by defininining the multiplication

$$(a \otimes 1)(1 \otimes b) = a \otimes b$$

$$(1 \otimes b)(a \otimes 1) = (a_1, S^{-1}(b_3))(a_2 \otimes b_2)(a_3, b_1)$$

(this is the usual definition of the Drinfeld double). Then by analogy of (3.1) one has in $A \otimes B$:

$$(1 \otimes b_1)(a_2 \otimes 1)(a_1, b_2) - (a_1 \otimes b_2)(a_2, b_1) = 0$$

In other words the map $D(A,B) \to A \otimes B$ which sends $a \otimes 1$ to $a \otimes 1$ and $1 \otimes b$ to $1 \otimes b$ is well defined. This is obviously an inverse to (3.3) so we are done. □

It is classical that $D(A,B)$ carries a Hopf algebra structure which is uniquely determined by requiring that $A\otimes 1$ and $1\otimes B^{\mathrm{coop}}$ are sub-Hopf algebras of $D(A,B)$.

Below we will need the following lemma.

Lemma 3.2. *Assume that A, B are $\mathbb{N}$-graded. Assume that A is generated over A_0 by a vector space V_A and similarly B is generated by a vector space V_B over B_0. Then we only have to impose the relation* (3.1) *for $a \in A_0 \cup V_A$ and $b \in B_0 \cup V_B$.*

Proof. Let H be the algebra obtained by imposing the relations as in the statement of the lemma. We have to show that the relation (3.1) holds for arbitrary a, b.

We do this by induction on the degree of a, b. If for example b is not in $B_0 \cup V_B$ then we can write $b = \sum_i c_i d_i$ where c_i, d_i have smaller degree. Since the relation (3.1) is clearly additive in b be may assume that $b = cd$. Thus we have to show that the following is true in H.

$$(c_1d_1 * a_2)(a_1, c_2d_2) = (a_1 * c_2d_2)(a_2, c_1d_1)$$

We compute the left hand side of this equation.

$$\begin{aligned}(c_1d_1 * a_2)(a_1, c_2d_2) &= c_1(d_1 * a_3)(a_1, c_2)(a_2, d_2)\\ &= c_1 * a_2 * d_2(a_3, d_1)(a_1, c_2) \qquad (d_1 \text{ has lower degree})\\ &= (a_1 * c_2)d_2(a_2, c_1)(a_3, d_1) \qquad (c_1 \text{ has lower degree})\\ &= (a_1 * c_2d_2)(a_2, c_1d_1)\end{aligned}$$

The case where a is not in $A_0 \cup V_A$ is treated similarly. □

We also need the following.

Lemma 3.3. *Assume that $(-,-) : A\times A \to \mathbb{C}$ is a symmetric Hopf algebra pairing. Then*

$$\omega : D(A,A) \to D(A,A) : a \otimes a' \mapsto (1 \otimes a)(a' \otimes 1)$$

is an isomorphism between $D(A,A)$ and $D(A,A)^{\mathrm{coop}}$.

Proof. Left to the reader. □

4. Representations of quivers

In this section k is an arbitrary field. A quiver Q is a quadruple (Q_0, Q_1, t, h) where Q_0, Q_1 are finite sets and $h, t : Q_1 \to Q_0$ are maps of sets. The elements of Q_0 and Q_1 are respectively called the vertices and the edges of Q. If $e \in Q_1$ then $h(e)$ is the head of e and $t(e)$ is the tail of e. If $h(e) = t(e)$ then one refers to e as a loop. *Throughout we will assume that our quivers have no loops.*

A representation A of Q is a pair $((A_i)_{i\in Q_0}, (\phi_e)_{e\in Q_1})$ where the A_i are finite dimensional vector spaces and the ϕ_e are maps $A_{t(e)} \to A_{h(e)}$. It is clear that the representations of Q form a category which we denote by $\mathrm{Rep}(Q)$. Below we use the notation $[A]$ for the isomorphism class of A in $\mathrm{Rep}(Q)$. By S_i we denote the unique representation which satisfies $(S_i)_j = k^{\delta_{ij}}$.

If $A, B \in \mathrm{Rep}(Q)$ then we use the Euler form

$$\langle A, B\rangle = \dim \mathrm{Hom}(A,B) - \dim \mathrm{Ext}^1(A,B)$$

$\langle A, B\rangle$ is additive on short exact sequences. One also uses the symmetrization

$$(A,B) = \langle A, B\rangle + \langle B, A\rangle$$

If we want to emphasize the quiver Q then we will write $\langle A, B\rangle_Q$ and $(A,B)_Q$.

If $A \in \mathrm{Rep}(Q)$ then we define the dimension vector $\bar{A}$ of A by $\bar{A} = (\dim A_i)_i \in \mathbb{N}^{Q_0}$. If $i \in Q_0$ then we write e_i for the dimension vector of S_i. Thus $(e_i)_j = \delta_{ij}$.

One defines a bilinear form $\langle -, - \rangle$ on $\mathbb{C}^{Q_0}$ by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ -a_{ij} & \text{otherwise} \end{cases}$$

where a_{ij} is the number of edges going from i to j. As above the symmetrization of $\langle \alpha, \beta \rangle$ is given by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$. In contrast to $\langle -, - \rangle$ one sees that $(-, -)$ does not depend on the orientation of Q.

It is a basic fact [12] that for $M, N \in \mathrm{Rep}(Q)$ one has

$$\langle M, N \rangle = \langle \bar{M}, \bar{N} \rangle$$

(even if Q has oriented cycles). The same result holds for $(-, -)$ by symmetrization.

To study the representations with a given dimension vector α one introduces the representation space $R(Q, \alpha)$. By definition

$$R(Q, \alpha) = \prod_{e \in Q_i} \mathrm{Hom}(k^{\alpha_{t(e)}}, k^{\alpha_{h(e)}})$$

One also defines

$$\mathrm{GL}(\alpha) = \prod_{i \in Q_0} \mathrm{GL}(\alpha_i)$$

It is clear that $\mathrm{GL}(\alpha)$ acts on $R(Q, \alpha)$ by conjugation of matrices.

Let $x \in R(Q, \alpha)$. By definition $x = (\phi_e)_e$ where the ϕ_e are maps $k^{\alpha_{t(e)}} \to k^{\alpha_{h(e)}}$. We define A_x as the representation $((k^{\alpha_i})_i, (\phi_e)_e)$. It is then easy to see that the map $x \mapsto [A_x]$ defines a bijection between $R(Q, \alpha)/GL(\alpha)$ and the set of isomorphism classes of representations of Q of dimension vector α.

Very important in the representation theory of quivers are the so-called reflection functors [1] which we now define. If $i \in Q_0$ then we call i a source if there are no edges arriving in i. A sink is defined similarly.

Let i be a source and $A \in \mathrm{Rep}(Q)$. We call A (i-)admissible if the induced map $A_i \to \oplus_{t(e)=i} A_{h(e)}$ is injective. We denote by $\mathrm{Rep}_{\mathrm{adm}}(Q)$ the full subcategory of $\mathrm{Rep}(Q)$ consisting of admissible representation. If i is a sink then $\mathrm{Rep}_{\mathrm{adm}}(Q)$ is defined similarly.

Now let i be again a source and let Q' be the quiver obtained from Q by inverting all edges starting in i. Then there is an equivalence of categories

$$s_i : \mathrm{Rep}_{\mathrm{adm}}(Q) \to \mathrm{Rep}_{\mathrm{adm}}(Q')$$

such that for an admissible A one has

$$(s_i A)_j = \begin{cases} A_j & \text{if } j \neq i \\ \mathrm{coker}(A_i \to \oplus_{t(e)=i} A_{h(e)}) & \text{if } j = i \end{cases}$$

As usual one can define a similar functor if i is a sink in Q.

If A is admissible then the dimension vectors of A and $s_i A$ are related by $\overline{s_i A} = \overline{A} - (\overline{A}, e_i) e_i$. This makes it natural to define for all j (not necessarily sources or sinks) linear maps

$$s_j : \mathbb{C}^{Q_0} \to \mathbb{C}^{Q_0} : \alpha \mapsto \alpha - (\alpha, e_j) e_j$$

These $(s_j)_j$ are also called reflections. Together they generate the so-called Weyl group of Q. If i is a source or a sink and A is i-admissible then it is clear that $\overline{s_i A} = s_i \bar{A}$.

The following lemma is easy to check.

Lemma 4.1. *The reflections s_j leave the bilinear form $(-,-)$ invariant. Furthermore for $\alpha, \beta \in \mathbb{C}^{Q_0}$ one has $\langle \alpha, \beta \rangle_Q = \langle s_i\alpha, s_i\beta \rangle_{Q'}$.*

5. The Fourier transform

If X is a finite set then we denote by $F(X)$ the set of functions $X \to \mathbb{C}$. If $f : X \to Y$ is a map of finite set then there is an associate map $f^* : F(Y) \to F(X)$ given by $f^*(g) = g \circ f$.

We make $F(X)$ into an inner product space by defining

$$(f, g) = \frac{1}{|X|} \sum_{x \in X} f(x)g(x) \tag{5.1}$$

In this section we fix a finite set U. Let $k = \mathbb{F}_q$ and let V, W be finite dimensional k-vector spaces. Assume furthermore that there is a non-degenerate pairing $\langle -, - \rangle : V \times W \to k$. Choose a non-trivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^*$. For $f \in F(U \times V)$ one defines for $u \in U$, $w \in W$

$$\hat{f}(u, w) = q^{-\dim V/2} \sum_{v \in V} f(u, v)\psi(\langle v, w \rangle)$$

Thus $\hat{f} \in F(U \oplus W)$. For convenience we also define

$$\bar{f}(u, w) = \hat{f}(u, -w) = q^{-\dim V/2} \sum_{v \in V} f(u, v)\bar{\psi}(\langle v, w \rangle)$$

The following results are well-known and easy to prove (cfr [6])

Lemma 5.1. (1) $\bar{\hat{f}} = \hat{\bar{f}} = f$.

(2) *If U, V, W are respectively a H-set and H-representations for a finite group H, and if $\langle -, - \rangle$ is H-invariant then one has $\widehat{hf} = h\hat{f}$ for $h \in H$.*

(3) $(\hat{f}, \bar{g}) = (f, g)$.

The following lemma is also clear.

Lemma 5.2. *Assume that V_1, V_2, W_1, W_2 are finite dimensional k-vector spaces, equipped with non-degenerate pairings $\langle -, - \rangle_1 : V_1 \times W_1 \to \mathbb{C}$ and $\langle -, - \rangle_2 : V_2 \times W_2 \to \mathbb{C}$. Then the following diagram is commutative*

$$< -1\text{`}1\text{`}1; > [F(U \oplus V_1 \oplus W_2)\text{`}F(U \oplus V_1 \oplus V_2)\text{`}F(U \oplus W_1 \oplus W_2); (\hat{-})\text{`}(\hat{-})\text{`}(\hat{-})]$$

The horizontal Fourier transform is defined with respect to the pairing $\langle -, - \rangle_1 + \langle -, - \rangle_2$.

Finally we have the following.

Lemma 5.3. *Assume that $f : U \to U'$ is a map between finite sets. Then the following diagram is commutative.*

$$\begin{array}{ccc} F(U' \times V) & \xrightarrow{(\hat{-})} & F(U' \times W) \\ {\scriptstyle (f,\mathrm{id})^*}\downarrow & & {\scriptstyle (f,\mathrm{id})^*}\downarrow \\ F(U \times V) & \xrightarrow{(\hat{-})} & F(U \times W) \end{array}$$

6. Twisted Hall algebras

As usual $k = \mathbb{F}_q$. Let Q be a quiver without loops. The twisted Hall algebra (sometimes called Ringel algebra) [13] $H(Q)$ is the free $\mathbb{C}$-vector space with basis the isomorphism classes of objects in $\mathrm{Rep}(Q)$. The product is given by

$$[A][B] = q^{(1/2)\langle B,A\rangle} \sum_{[C]} g_{AB}^{C}[C] \tag{6.1}$$

where by definition g_{AB}^{C} is the cardinality of the set

$$\{X \subset C \mid X \cong A, \quad C/X \cong B\}$$

Note that we have taken Kapranov's definition of the product [7]. This yields an algebra which is the opposite of the one defined by Ringel.

It is clear that $H(Q)$ is a $\mathbb{Z}^{Q_0}$ graded algebra, where $H(Q)_\alpha$ is defined as the vector space with basis the isomorphism classes of Q-representations with dimension vector α.

For simplicity of notations put $F(Q,\alpha) = F(R(Q,\alpha))$. It is then also clear that

$$H(Q)_\alpha = F(Q,\alpha)^{\mathrm{GL}(\alpha)} \tag{6.2}$$

where a representation is identified with the characteristic function of its orbit.

The formula (6.2) suggests the posibility of defining a product on $F(Q) = \oplus_\alpha F(Q,\alpha)$ which restricts to (6.1) on $H(Q)$. It is well known how this should be done. One defines for $f \in F(Q,\alpha)$, $g \in F(Q,\beta)$

$$(fg)(z) = \frac{q^{(1/2)\langle \beta,\alpha\rangle}}{|\,\mathrm{GL}(\alpha)||GL(\beta)|} \sum_{x,y,\phi,\theta} f(x)g(y) \tag{6.3}$$

where the sum is over all exact sequences

$$0 \to A_x \xrightarrow{\phi} A_z \xrightarrow{\theta} A_y \to 0$$

This product is assocative, but does not have a unit element.

Denote by $(-,-)$ the bilinear form on $F(Q,\alpha)$ given by (5.1). Then for the restriction of $(-,-)$ to $H(Q)_\alpha = F(Q,\alpha)^{\mathrm{GL}(\alpha)}$ one finds

$$([A],[B]) = \frac{1}{|\,\mathrm{Aut}(A)|}\delta_{[A],[B]} \tag{6.4}$$

One extends $(-,-)$ by linearity to a bilinear form on $H(Q)$.

7. The Fourier transform and the twisted Hall algebra

Notations are as in the previous section. Assume that Q' is obtained from Q by inverting a set of edges $E \subset Q_1$. Then

$$R(Q,\alpha) = U_\alpha \times V_\alpha$$
$$R(Q',\alpha) = U_\alpha \times W_\alpha$$

where

$$U_\alpha = \prod_{e \notin E} \mathrm{Hom}(k^{\alpha_{t(e)}}, k^{\alpha_{h(e)}})$$
$$V_\alpha = \prod_{e \in E} \mathrm{Hom}(k^{\alpha_{t(e)}}, k^{\alpha_{h(e)}})$$
$$W_\alpha = \prod_{e \in E} \mathrm{Hom}(k^{\alpha_{h(e)}}, k^{\alpha_{t(e)}})$$

Fix $\epsilon_e \in \{-1, +1\}$ for $e \in Q_1$. We use the $(\epsilon_e)_e$ to define a non-degenerate pairing

$$\langle -, - \rangle : V_\alpha \times W_\alpha : (A_e, B_e)_{e \in E} \mapsto \sum_{e \in E} \epsilon_e \,\mathrm{Tr}(A_e B_e)$$

With this pairing we may define a Fourier transform $F(Q, \alpha) \to F(Q', \alpha)$. By linearity we extend this to a Fourier transform $F(Q) \to F(Q')$, which restricts to a map $H(Q) \to H(Q')$. The following result was stated without proof in [9].

Theorem 7.1. [9] *The Fourier transform defines an isomorphism $H(Q) \to H(Q')$ as algebras. In particular the twisted Hall algebra is independent of the orientation of Q.*

Proof. We have to show that if $f \in F(Q, \alpha)$ and $g \in F(Q, \beta)$ then $\widehat{fg} = \hat{f}\hat{g}$. By transitivity (lemma 5.2) it is clear that we only have to show this in the case that E consists of one edge. Hence we will assume $E = \{e\}$.

By definition we have

$$(\hat{f}\hat{g})(u, w) = \frac{q^{(1/2)\langle \beta, \alpha \rangle_{Q'}}}{|GL(\alpha)||GL(\beta)|} \sum_{(u_1, w_1, u_2, w_2, \phi, \theta)} \hat{f}(u_1, w_1)\hat{g}(u_2, w_2) \tag{7.1}$$

Here $u \in U_{\alpha+\beta}$, $w \in W_{\alpha+\beta}$, etc... and the sum runs over all exact sequences

$$0 \to A_{u_1, w_1} \xrightarrow{\phi} A_{u,w} \xrightarrow{\theta} A_{u_2, w_2} \to 0$$

We find that (7.1) is equal to

$$\frac{q^{(1/2)(\langle \beta, \alpha \rangle_{Q'} - (\alpha_{h(e)}\alpha_{t(e)} + \beta_{h(e)}\beta_{t(e)}))}}{|GL(\alpha)||GL(\beta)|} \sum_{(u_1, w_1, u_2, w_2, \phi, \theta, v_1, v_2)} f(u_1, v_1)g(u_2, v_2)\psi(\langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle) \tag{7.2}$$

The various maps occuring in this formula are as in the following diagram.

(7.3)

There are $q^{\beta_{t(e)}\alpha_{h(e)}}$ possibilities of inserting a map $v : k^{\alpha_{t(e)} + \beta_{t(e)}} \to k^{\alpha_{h(e)} + \beta_{h(e)}}$ in this diagram compatible with v_1 and v_2. The definition of $\langle -, - \rangle$ by traces yields $\langle v, w \rangle = \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle$, or averaging over all possible choices of v :

$$\langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle = q^{-\beta_{t(e)}\alpha_{h(e)}} \sum_v \langle v, w \rangle$$

so that we can rewrite (7.2) as

$$\frac{q^{(1/2)(\langle \beta, \alpha \rangle_{Q'} - (\alpha_{h(e)}\alpha_{t(e)} + \beta_{h(e)}\beta_{t(e)} + 2\beta_{t(e)}\alpha_{h(e)}))}}{|\,\mathrm{GL}(\alpha)||\,\mathrm{GL}(\beta)|} \sum_{(u_1, w_1, u_2, w_2, \phi, \theta, v_1, v_2, v)} f(u_1, v_1)g(u_2, v_2)\psi(\langle v, w \rangle) \tag{7.4}$$

We compare this with

(7.5)

$$\widehat{fg}(u,w)=\frac{q^{(1/2)(\langle\beta,\alpha\rangle_Q-(\alpha_{h(e)}+\beta_{h(e)})(\alpha_{t(e)}+\beta_{t(e)}))}}{|\operatorname{GL}(\alpha)||\operatorname{GL}(\beta)|}\sum_{(v,u_1,v_1,u_2,v_2,\phi,\theta)}f(u_1,v_1)g(u_2,v_2)\psi(\langle v,w\rangle)$$

where now the sum is over all exact sequences

$$0\to A_{u_1,v_1}\xrightarrow{\phi}A_{u,v}\xrightarrow{\theta}A_{u_2,v_2}\to 0$$

It is clear that (7.4) will be equal to (7.5) provided the following two statements hold. First one should have

(7.6)

$$\langle\beta,\alpha\rangle_{Q'}-(\alpha_{h(e)}\alpha_{t(e)}+\beta_{h(e)}\beta_{t(e)}+2\beta_{t(e)}\alpha_{h(e)})=\langle\beta,\alpha\rangle_Q-(\alpha_{h(e)}+\beta_{h(e)})(\alpha_{t(e)}+\beta_{t(e)})$$

and secondly if w is such that there do not exist w_1,w_2 making diagram (7.3) commutative then if v runs trough the maps $v:k^{\alpha_{t(e)}+\beta_{t(e)}}\to k^{\alpha_{h(e)}+\beta_{h(e)}}$ which are compatible with v_1, v_2 in (7.3), one should have.

$$\sum_v\psi(\langle v,w\rangle)=0 \tag{7.7}$$

(7.6) is trival and (7.7) is a pleasant exercise in linear algebra. □

Denote the Fourier transform we have defined above by $F_{Q,Q',\epsilon}$. For another quiver Q'' with the same vertices and edges as Q but with possibly different orientation we would like to define an ϵ' such that

$$F_{Q,Q'',\epsilon}=F_{Q',Q'',\epsilon'}F_{Q,Q',\epsilon}$$

From lemmas 5.1 and 5.2 it follows that the correct choice is

$$\epsilon'_e=\begin{cases}-\epsilon_e & \text{if } e\in E\\ \epsilon_e & \text{otherwise}\end{cases}$$

Having made this choice for all Q' we simply drop the ϵ's from the notation and write $F_{Q,Q'}$ for the Fourier transform.

8. The algebra $U(Q)$

We use the same notations as above. Thus Q is a quiver without loops. Following Green [3] we introduce the algebra $B(Q)$ with basis $K_\alpha[A]$ where $\alpha\in\mathbb{Z}^{Q_0}$ and $A\in\operatorname{Rep}(Q)$ and multiplication

$$K_\alpha[A]\cdot K_\beta[B]=q^{(1/2)(\langle B,A\rangle+(\beta,\bar{A}))}\sum_{[C]}g_{AB}^CK_{\alpha+\beta}[C]$$

Thus $B(Q)$ is a crossed product of $H(Q)$ with the groups algebra of $\mathbb{Z}^{Q_0}$.

We will write K_α for $K_\alpha[0]$ and $[A]$ for $K_\alpha[A]$. We also write K_i for K_{e_i}.

It was shown by Green [3] that $B(Q)$ has a comultiplication given by

$$\Delta[A]=\sum_{[B],[C]}q^{(1/2)\langle C,B\rangle}g_{BC}^A\frac{|\operatorname{Aut}(B)|\cdot|\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|}[B]\otimes K_{\bar{B}}[C]$$

$$\Delta(K_\alpha)=K_\alpha\otimes K_\alpha$$

It turns out that $B(Q)$ is actually a Hopf algebra [7, 14] with counit $\epsilon(K_\alpha[A]) = \delta_{[A],[0]}$ and antipode

$$S[A] = \sum_{n=0}^{\infty}(-1)^n \sum_{0=A_0\subsetneq\cdots\subsetneq A_n=A} \prod_{i=1}^{n} q^{(1/2)\langle A_i/A_{i-1},A_{i-1}\rangle} \frac{\prod_{j=1}^{n} |\operatorname{Aut}(A_j/A_{j-1})|}{|\operatorname{Aut}(A)|} \cdot [A_1/A_0]\cdots[A_n/A_{n-1}]K_{\bar{A}}^{-1}$$

$$S(K_\alpha) = K_{-\alpha}$$

Furthermore $B(Q)$ carries a Hopf pairing with itself, given by

$$(K_\alpha[A], K_\beta[B]) = q^{1/2(\alpha,\beta)}([A],[B]) = q^{1/2(\alpha,\beta)}\frac{\delta_{[A],[B]}}{|\operatorname{Aut}(A)|}$$

Using this Hopf pairing we can define the Drinfeld double of $B(Q)$. We denote the resulting Hopf algebra by $D(Q)$. Dividing out $D(Q)$ by the ideal generated by $K_\alpha \otimes 1 - 1 \otimes K_{-\alpha}$ yields a new Hopf algebra, which we denote by $U(Q)$.

For use below we write down the defining relations of $U(Q)$. We write respectively $[A]$, $[A]^-$ and K_α for the images of $[A]\otimes 1$, $1\otimes[A]$ and K_α in $U(Q)$. Then $U(Q)$ is the algebra generated by $[A], K_\alpha, [A]^-$ satisfying the following relations

$$K_0 = [0] = [0]^- = 1 \tag{8.1}$$

$$K_\alpha K_\beta = K_{\alpha+\beta} \tag{8.2}$$

$$[A][B] = q^{1/2\langle B,A\rangle}\sum_{[C]} g_{AB}^{C}[C] \tag{8.3}$$

$$[A]^-[B]^- = q^{1/2\langle B,A\rangle}\sum_{[C]} g_{AB}^{C}[C]^- \tag{8.4}$$

$$K_\alpha^{-1}[A]K_\alpha = q^{1/2(\alpha,\bar{A})}[A] \tag{8.5}$$

$$K_\alpha^{-1}[A]^-K_\alpha = q^{-1/2(\alpha,\bar{A})}[A]^- \tag{8.6}$$

$$\sum_{[M],[N]} g_{AB}^{MN} q^{(1/2)\langle \bar{B}-\bar{N},\bar{M}-\bar{N}\rangle}[M][N]^-K_{\bar{B}-\bar{N}}^{-1} = \sum_{[M],[N]} g_{BA}^{NM} q^{(1/2)\langle \bar{A}-\bar{M},\bar{N}-\bar{M}\rangle}[N]^-[M]K_{\bar{A}-\bar{M}} \tag{8.7}$$

where for $M, N, A, B \in \operatorname{Rep}(A)$ we write (as in [7]) g_{AB}^{MN} for the number of exact sequences

$$0 \to M \xrightarrow{u} A \xrightarrow{\phi} B \xrightarrow{v} N \to 0$$

divided by $|\operatorname{Aut}(A)|\cdot|\operatorname{Aut}(B)|$. One has the following easy identity.

$$g_{AB}^{MN} = \frac{|\operatorname{Aut}(M)|\cdot|\operatorname{Aut}(N)|}{|\operatorname{Aut}(A)|\cdot|\operatorname{Aut}(B)|}\sum_{[P]} g_{MP}^{A} g_{PN}^{B} |\operatorname{Aut}(P)| \tag{8.8}$$

It is perhaps useful say something on how (8.1-8.7) are obtained. Relations (8.1-8.6) are simply obtained from the defining relations of $B(Q)$, so we only have to comment on (8.7). This relation is obtained from (3.1), but by lemma 3.2 we only need to impose this relation for a selected set of a, b. It turns out that if we take $a = K_\alpha$ then the relation we obtain from (3.1) already follows from the other relations and similarly if $b = K_\beta$. Hence we only have to consider the case $a = [A]$, $b = [B]$ and for these a, b we obtain (8.7).

We will often need the special case $B = S_i$ of (8.7) so we state the resulting formula below (slightly rewritten).

$$\begin{aligned}[S_i]^-[A] - [A][S_i]^- = &-\sum_{[C]} q^{(1/2)\langle C,S_i\rangle}\frac{|\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|}g^A_{S_iC}K_i[C] \\ &+\sum_{[C]} q^{(1/2)\langle S_i,C\rangle}\frac{|\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|}g^A_{CS_i}[C]K_i^{-1}\end{aligned} \tag{8.9}$$

It has been shown in [14] that $U(Q)$ has a basis of the form $[A]^-K_\alpha[B]$. Thus there is a triangular decomposition $U(Q) = U^-(Q)\otimes U^0(Q)\otimes U^+(Q)$ where $U^0(Q)$ is simply the group algebra over $\mathbb{Z}^{Q_0}$ and $U^\pm(Q)$ are copies of $H(Q)$.

Now note that the automorphism ω of $D(Q)$ (cfr. lemma 3.3) factors through $U(Q)$. Thus $U(Q)$ has a canonical automorphism of order two, given by

$$\begin{aligned}\omega(K_\alpha) &= K_{-\alpha}\\ \omega[A] &= [A]^-\end{aligned}$$

Let Q' be obtained for Q by inverting some edges. From the functoriality of the Drinfeld double construction, the compatibility of the bilinear forms on $H(Q)$ and $F(Q)$ (cfr. (6.4)) and lemma 5.1 it follows easily that one can extend the Fourier transform to $D(Q)$ and $U(Q)$. On $U(Q)$ we obtain

$$\begin{aligned}F^{U(Q)}_{Q,Q'}(K_\alpha) &= K_\alpha\\ F^{U(Q)}_{Q,Q'}([A]) &= F_{Q,Q'}([A])\\ F^{U(Q)}_{Q,Q'}([A]^-) &= \bar F_{Q,Q'}([A])^-\end{aligned} \tag{8.10}$$

where $\bar F_{Q.Q'}$ is defined as $F_{Q,Q'}$, but with $\bar\psi$ replacing ψ. Having made this definition we write $F_{Q,Q'}$ instead of $F^{U(Q)}_{Q,Q'}$ for the extension of the Fourier transform to $U(Q)$.

9. Reflections act on $U(Q)$

In this section Q will be a quiver without loops possessing a source i and Q' will be obtained from Q by inverting all edges starting in i. For simplicity we call a representation admissible if it is i-admissible. We will prove the following result.

Theorem 9.1. *There is a unique algebra isomorphism $t_i : U(Q) \to U(Q')$, satisfying*

$$\omega t_i = t_i\omega \tag{9.1}$$

$$t_i[S_i] = q^{-1/2}[S_i]^-K_i^{-1} \tag{9.2}$$

$$t_i[A] = [s_iA] \qquad \text{if } A \text{ is admissible} \tag{9.3}$$

$$t_iK_\alpha = K_{s_i\alpha} \tag{9.4}$$

Proof. Every representation A of Q can be written as a direct sum of the form $S_i^b \oplus B$ where B is admissible. Furthermore in the twisted Hall algebra we have

$$[A] = q^{-\langle \bar B,e_i\rangle/2}[S_i^a][B] \tag{9.5}$$

$$[S_i^{a+1}] = (1+q+\cdots+q^a)^{-1}q^{-a/2}[S_i][S_i^a] \tag{9.6}$$

Thus t_i is defined on generators of $U(Q)$. This obviously means that t_i is unique if it exists.

Let us first show that our definition of t_i defines an algebra morphism $H(Q) \to U(Q')$. From (9.6) we deduce by induction that if t_i exists then

$$t_i[S_i^a] = q^{-a^2/2}[S_i^a]^- K_i^{-a}$$

and if A and B are as in (9.5) then

$$t_i[A] = q^{-\langle \bar{B}, e_i\rangle/2} t_i[S_i^a][s_i B]$$

In this way we have defined t_i on the standard basis of $H(Q)$. We now have to show that t_i is compatible with the relation (8.3). Since our choice of t_i is obviously compatible with the relations (9.5)(9.6) it is easy to see that in addition we only have to check the following cases in (8.3).

(1) The case A, B admissible.
(2) The case A admissible and $B = S_i$.

The first case is easy, so let us consider the second case. Rewriting (8.3) a bit using (9.5) we obtain the following relation

$$[A][S_i] - q^{(1/2)(e_i,\bar{A})}[S_i][A] = q^{(1/2)\langle e_i,\bar{A}\rangle} \sum_{[A'] \text{ adm}} g_{AS}^{A'}[A'] \tag{9.7}$$

Thus we have to check that the following holds in $U(Q')$.

$$q^{-1/2}([s_iA][S_i]^- K_i^{-1} - q^{(1/2)(e_i,\bar{A})}[S_i]^- K_i^{-1}[s_iA]) = q^{(1/2)\langle e_i,\bar{A}\rangle_Q} \sum_{[A'] \text{ adm}} g_{AS_i}^{A'}[s_iA'] \tag{9.8}$$

One has

$$q^{(1/2)(e_i,\bar{A})}[S_i]^- K_i^{-1}[s_iA] = q^{(1/2)((e_i,\bar{A})+(e_i,s_i\bar{A}))}[S_i]^-[s_iA]K_i^{-1} = [S_i]^-[s_iA]K_i^{-1}$$

where we have used $(e_i, s_i\bar{A}) = (s_ie_i, \bar{A}) = -(e_i, \bar{A})$.

Hence (9.8) can be written as

$$[s_iA][S_i]^- - [S_i]^-[s_iA] = q^{1/2} \cdot q^{(1/2)\langle e_i,\bar{A}\rangle_Q} \sum_{[A'] \text{ adm}} g_{AS_i}^{A'}[s_iA']K_i$$

We make some further changes to write this equation intrinsically in terms of Q'. First of all we have $\langle e_i, \bar{A}\rangle_Q = \langle s_ie_i, s_i\bar{A}\rangle_{Q'} = -\langle e_i, s_i\bar{A}\rangle_{Q'}$. Secondly by lemma 10.1 below

$$g_{AS_i}^{A'} = \frac{|\operatorname{Aut}(A')|}{|\operatorname{Aut}(A)|} g_{S_i\, s_iA'}^{s_iA}$$

Putting $C = s_iA$, $C' = s_iA'$, (9.8) finally becomes

$$[C][S_i]^- - [S_i]^-[C] = q^{1/2} \cdot q^{-(1/2)\langle e_i,\bar{C}\rangle_{Q'}} \sum_{[C']} g_{S_iC'}^{C} \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|}[C']K_i \tag{9.9}$$

Thus (9.9) should hold in $U(Q')$ for all admissible $C \in \operatorname{Rep}(Q')$.

From (8.9) we obtain the following identity in $U(Q')$.

$$\begin{aligned}[S_i]^-[C] - [C][S_i]^- = &-\sum_{[C']} q^{(1/2)\langle \bar{C}', e_i\rangle_{Q'}} \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|} g_{S_iC'}^{C} K_i[C'] \\ &+ \sum_{[C']} q^{(1/2)\langle e_i, \bar{C}'\rangle_{Q'}} \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|} g_{C'S_i}^{C}[C']K_i^{-1}\end{aligned} \tag{9.10}$$

Since i is a sink in Q' and C is admissible we have $g^C_{C'S_i} = 0$. Interchanging the order of K_i and $[C']$ we may rewrite (9.10) as

$$[S_i]^-[C] - [C][S_i]^- = -\sum_{[C']} q^{-(1/2)\langle e_i, \bar{C}'\rangle_{Q'}} \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|} g^C_{S_iC'}[C']K_i$$

This equation becomes equivalent to (9.9), if we observe that

$$\langle e_i, \bar{C}'\rangle_{Q'} = \langle e_i, \bar{C}\rangle_{Q'} - 1$$

This finishes the proof that t_i defines an algebra homomorphism $H(Q) = U^+(Q) \to U(Q')$.

Since t_i is obviously compatible with the relations (8.1)(8.2) and (8.5) we see that t_i defines in fact an algebra homomorphism $B(Q) = U^0(Q) \otimes U^+(Q) \to U(Q')$. Since t_i is by definition compatible with ω we also obtain an algebra homomorphism $U^0(Q) \otimes U^-(Q) \to U(Q')$. To obtain from this an algebra homomorphism $U(Q) \to U(Q')$ we have to check the cross relations (8.7). As before may invoke lemma 3.2, which tells us that we have to check compatibility with t_i in the following cases.

(1) $A = S_i$ and $B = S_i$.
(2) $A = S_i$ and B is admissible.
(3) A is admissible and $B = S_i$.
(4) A and B are admissible.

However applying ω amounts to exchanging A and B. So we only have to treat cases 1,3,4.

Below we will need how t_i acts on $[S_i^a]^-$. Using compatibility with ω we obtain the following formula.

$$t_i[S_i^a]^- = q^{-a^2/2}[S_i^a]K_i^a \tag{9.11}$$

Case 1 In this case we have by (8.9)

$$[S_i]^-[S_i] - [S_i][S_i]^- = \frac{K_i^{-1} - K_i}{q-1}$$

Applying t_i yields.

$$q^{-1}([S_i]K_i[S_i]^-K_i^{-1} - [S_i]^-K_i^{-1}[S_i]K_i) = \frac{K_i - K_i^{-1}}{q-1}$$

It is clear that this formula holds in $U(Q')$.

Case 3 Now we have in $U(Q)$ (by (8.9))

$$[S_i]^-[A] - [A][S_i]^- = \sum_{[C]} q^{(1/2)\langle e_i, \bar{C}\rangle} \frac{|\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|} g^A_{CS_i}[C]K_i^{-1}$$

where we have taken into account that $g^A_{S_iC} = 0$. Applying t_i yields

$$q^{-1/2}([S_i]K_i[s_iA] - [s_iA][S_i]K_i) = \sum_{[C]} q^{(1/2)\langle e_i, \bar{C}\rangle} \frac{|\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|} g^A_{CS_i}[s_iC]K_i \tag{9.12}$$

Multiplying on the right with $q^{1/2}K_i^{-1}$ and using the fact that

$$\langle e_i, \bar{C}\rangle_Q = \langle e_i, \bar{A}\rangle_Q - 1 = -\langle e_i, s_i\bar{A}\rangle_{Q'} - 1$$

together with

$$\frac{|\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|} g^A_{CS_i} = g^{s_iC}_{S_is_iA}$$

(see lemma 10.1 below) we find that (9.12) is equivalent to

$$q^{-(1/2)(s_i\bar{A},e_i)}[S_i][s_iA]-[s_iA][S_i]=q^{-(1/2)\langle e_i,s_i\bar{A}\rangle}\sum_{[C]}g^{s_iC}_{S_is_iA}[s_iC]$$

Multiplying by $q^{(1/2)(s_iA,e_i)}$ and putting $D=s_iA$ and $D'=s_iC$ we obtain

$$[S_i][D]-q^{(1/2)(\bar{D},e_i)}[D][S_i]=q^{(1/2)\langle\bar{D},e_i\rangle}\sum_{[D']\text{ adm.}}g^{D'}_{S_iD}[D']$$

This formula is the analogue of (9.7) for $U(Q')$ and is proved in the same way.
Case 4 This is the most interesting case. We will check invariance under t_i of the lefthand and righthand side of (8.7).

The lefthand side of (8.7) is given by

$$\sum_{[M],[N]}g^{MN}_{AB}q^{(1/2)\langle\bar{B}-\bar{N},\bar{M}-\bar{N}\rangle}[M][N]^-K^{-1}_{\bar{B}-\bar{N}} \tag{9.13}$$

Since the M's occuring in this sum are subrepresentations of A, they are automatically admissible. However this is not the case for the N's. Therefore we write $N=N'\oplus S_i^a$ with N' admissible. Thus (9.13) becomes

$$\sum_{[M],[N'],a}g^{MN}_{AB}q^{(1/2)(\langle\bar{B}-\bar{N},\bar{M}-\bar{N}\rangle-\langle\bar{N}',ae_i\rangle)}[M][S_i^a]^-[N']^-K^{-1}_{\bar{B}-\bar{N}} \tag{9.14}$$

Applying t_i yields (using (9.11))

$$\sum_{[M],[N'],a}g^{AB}_{MN}q^{(1/2)(\langle\bar{B}-\bar{N},\bar{M}-\bar{N}\rangle-\langle\bar{N}',ae_i\rangle-a^2)}[s_iM][S_i^a]K_i^a[s_iN']^-K^{-1}_{s_i\bar{B}-s_i\bar{N}} \tag{9.15}$$

We rewrite this expression using the following identities.

$$\begin{aligned}[s_iM][S_i^a]&=q^{(1/2)\langle S_i^a,s_iM\rangle_{Q'}}[s_iM\oplus S_i^a]\\&=q^{-(1/2)\langle ae_i,\bar{M}\rangle}[U]\end{aligned}$$

where for simplicity we have written $U=s_iM\oplus S_i^a$.

$$\begin{aligned}K_i^a[s_iN']^-&=q^{(1/2)(ae_i,s_i\bar{N}')}[s_iN']^-K_i^a\\&=q^{-(1/2)(ae_i,N')}[s_i\bar{N}']^-K_i^a\end{aligned}$$

$$K_i^aK^{-1}_{s_i\bar{B}-s_iN}=K^{-1}_{s_i\bar{B}-s_i\bar{N}-a\bar{S}_i}=K^{-1}_{s_i\bar{B}-s_i\bar{N}'}$$

Furthermore by lemma 10.3 below, we have

$$g^{MN}_{AB}=q^{\langle ae_i,\bar{M}\rangle+\langle N',ae_i\rangle}g^{U\,s_iN'}_{s_iA\,s_iB}$$

Combining everything we find that (9.15) is equal to

$$\sum_{[U],[N']}g^{U\,s_iN'}_{s_iA\,s_iB}q^{u/2}[U][s_iN']K^{-1}_{s_i\bar{B}-s_i\bar{N}'} \tag{9.16}$$

where

$$u=\langle\bar{B}-\bar{N},\bar{M}-\bar{N}\rangle-\langle\bar{N}',ae_i\rangle-a^2-\langle ae_i,\bar{M}\rangle-(ae_i,\bar{N}')+2\langle ae_i,\bar{M}\rangle+2\langle\bar{N}',ae_i\rangle$$

A tedious but straightforward verification shows that

$$u=\langle s_i\bar{B}-s_i\bar{N}',\bar{U}-s_i\bar{N}'\rangle_{Q'}$$

Thus, putting $V = s_i N'$ we find that applying t_i to the lefthand side of (8.7) yields

$$\sum_{[U],[V]} g^{U\,V}_{s_i A\; s_i B} q^{(1/2)\langle s_i \bar{B}-\bar{V}, \bar{U}-\bar{V}\rangle_{Q'}} [U][V]^- K^{-1}_{s_i \bar{B}-\bar{V}}$$

Applying t_i to the right hand side of (8.7) yields in an entirely similar way

$$\sum_{[U],[V]} g^{VU}_{s_i B\; s_i A} q^{(1/2)\langle s_i \bar{A}-\bar{U}, \bar{V}-\bar{U}\rangle_{Q'}} [V]^- [U] K_{s_i \bar{A}-\bar{U}}$$

Hence applying t_i to (8.7) when A, B are admissible yields the analog of (8.7) for $U(Q')$. This finishes our proof that t_i is well defined.

To finish the proof we have to show that t_i is invertible. The most straightforward way of doing this, is by defining an explicit inverse $t_i' : U(Q') \to U(Q)$ by a formula similar to (9.1-9.4). Of course one has to show that this t_i' is also an algebra morphism, but this is entirely similar to the above verification for t_i. □

10. Identities

In this section we verify some identities that were used in the last section.

Throughout we assume that i is a source in Q.

Lemma 10.1. *Assume that $A, B \in \mathrm{Rep}(Q)$ are admissible and $a \in \mathbb{N}$. Then*

$$|\operatorname{Aut}(B)| g^{A}_{BS_i^a} = |\operatorname{Aut}(A)| g^{s_i B}_{S_i^a\; s_i A}$$

Proof. We only have to prove this in the case $\bar{A} = \bar{B} + se_i$. Let

$$I = |\{\text{injective maps } B \to A\}|$$
$$S = |\{\text{surjective maps } s_i B \to s_i A\}|$$

Using functoriality one sees that $I = S$.

One also has

$$g^{A}_{BS_i^a} = \frac{I}{|\operatorname{Aut}(B)|}$$
$$g^{s_i B}_{S_i^a\; s_i A} = \frac{S}{|\operatorname{Aut}(s_i A)|}$$

The fact that $|\operatorname{Aut}(s_i A)| = |\operatorname{Aut}(A)|$ now yields the lemma. □

Lemma 10.2. *Assume that $A, B, C \in \mathrm{Rep}(Q)$ are admissible and $a \in \mathbb{N}$. Then*

$$g^{A}_{B\, C \oplus S_i^a} = \sum_{[P]} g^{A}_{PC} g^{P}_{BS_i^a}$$

Proof. Put $D = C \oplus S_i^a$. If $X \subset A$ is such that $A/X \cong D$ then there exists a unique P such that $X \subset P \subset D$ and such that $P/X \cong S_i^a$. In that case one has $A/P \cong C$. Thus we obtain

$$\begin{aligned} g^{A}_{BD} &= \sum_{P \subset A, A/P \cong C} g^{P}_{BS_i^a} \\ &= \sum_{[P]} g^{A}_{PC} g^{P}_{BS_i^a} \end{aligned}$$

□

Lemma 10.3. *Assume that A, B, M, N are admissible and $a \in \mathbb{N}$. Then*

$$g_{AB}^{M\,N\oplus S_i^a} = q^{\langle S_i^a, M\rangle + \langle N, S_i^a\rangle} g_{s_iA\,s_iB}^{s_iM\oplus S_i^a\;s_iN}$$

Proof. From the formula (8.8) together with the lemma 10.2 we deduce

$$\begin{aligned} g_{AB}^{MN\oplus S_i^a} &= \frac{|\operatorname{Aut}(M)|\cdot|\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)|\cdot|\operatorname{Aut}(B)|}\sum_{[P]} g_{MP}^A g_{PN\oplus S_i^a}^B |\operatorname{Aut}(P)| \\ &= \frac{|\operatorname{Aut}(M)|\cdot|\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)|\cdot|\operatorname{Aut}(B)|}\sum_{[P],[Q]} g_{MP}^A g_{QN}^B g_{PS_i^a}^Q |\operatorname{Aut}(P)| \end{aligned}$$

The terms in the last sum can only be non-zero if P, Q are admissible. Hence we may rewrite this sum using lemma 10.1

$$\frac{|\operatorname{Aut}(M)|\cdot|\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)|\cdot|\operatorname{Aut}(B)|}\sum_{[P][Q]\text{ adm}} g_{s_iMs_iP}^{s_iA} g_{s_iQs_iN}^{s_iB} g_{S_i^a s_iQ}^{s_iP} |\operatorname{Aut}(Q)|$$

Now we invoke the version of lemma 10.2 for a sink. Then the previous equation becomes

$$\frac{|\operatorname{Aut}(M)|\cdot|\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(A)|\cdot|\operatorname{Aut}(B)|}\sum_{[Q]\text{ adm}} g_{s_iQs_iN}^{s_iB} g_{s_iM\oplus S_i^a s_iQ}^{s_iA} |\operatorname{Aut}(Q)|$$

Now, invoking (8.8) once again, we obtain

$$\frac{|\operatorname{Aut}(M)|\cdot|\operatorname{Aut}(N\oplus S_i^a)|}{|\operatorname{Aut}(s_iM\oplus S_i^a)|\cdot|\operatorname{Aut}(N)|} g_{s_iAs_iB}^{s_iM\oplus S_i^a s_iN} \tag{10.1}$$

It is clear that

$$\operatorname{Aut}(N\oplus S_a^i) = \begin{pmatrix} \operatorname{Aut}(N) & \operatorname{Hom}(N, S_i^a) \\ 0 & \operatorname{GL}(a) \end{pmatrix}$$

from which we obtain

$$|\operatorname{Aut}(N\oplus S_a^i)| = |\operatorname{Aut}(N)||\operatorname{GL}(a)||\operatorname{Hom}(N, S_i^a)| = q^{\langle N, S_i^a\rangle}|\operatorname{Aut}(N)||\operatorname{GL}(a)|$$

In a similar way we obtain

$$|\operatorname{Aut}(s_iM\oplus S_a^i)| = q^{\langle S_i^a, s_iM\rangle}|\operatorname{Aut}(M)||\operatorname{GL}(a)| = q^{-\langle S_i^a, M\rangle}|\operatorname{Aut}(M)||\operatorname{GL}(a)|$$

Thus finally we obtain that (10.1) is equal to

$$g_{s_iAs_iB}^{s_iM\oplus S_i^a s_iN} q^{\langle S_i^a, M\rangle + \langle N, S_i^a\rangle}$$

which is what we had to show. □

11. Automorphisms of $U(Q)$

In this section Q is a quiver. Let i be an arbitrary vertex of Q. We write

$$\hat{t_i} = F_{Q'',Q} t_i F_{Q,Q'}$$

where Q' is obtained from Q by changing the orientation in such a way that Q' becomes a source. Then Q'' is obtained from Q' by inverting all edges starting in i.

Of course we have to show that this is well defined. Suppose we have chosen Q_1', Q_1'' with the same properties as Q', Q''. Then we must have

$$F_{Q'',Q} t_i F_{Q,Q'} = F_{Q_1'',Q} t_i F_{Q,Q_1'}$$

which is equivalent to

$$F_{Q'',Q_1''} t_i = t_i F_{Q',Q_1'}$$

It is now clear that we need the following lemma.

Lemma 11.1. *Let Q be a quiver and let i be a source in Q. Define quivers Q', Q'', Q''' in the following way.*

1. *Q' is obtained from Q by inverting all edges adjacent to i.*
2. *Q'' is obtained from Q by inverting some edges, not adjacent to i.*
3. *Q''' is obtained by simultaneously inverting the edges that were inverted in Q' and Q''.*

Then the following diagram is commutative.

$$\begin{array}{ccc} U(Q) & \xrightarrow{t_i} & U(Q') \\ {\scriptstyle F_{Q,Q''}}\downarrow & & {\scriptstyle F_{Q',Q'''}}\downarrow \\ U(Q'') & \xrightarrow{t_i} & U(Q''') \end{array}$$

Proof. We have to check this on $U^0(Q)$ and on $U^{\pm}(U)$. The case $U^0(Q)$ is clear and by applying ω and replacing ψ by $\bar{\psi}$ (cfr. (8.10)) we may reduce the case $U^-(Q)$ to $U^+(Q)$.

Thus we have to show that $F_{Q',Q'''}t_i(a) = t_i F_{Q,Q''}(a)$ in the following cases.

1. $a = [A]$ with $[A]$ admissible.
2. $a = [S_i]$.

The second case follows directly from the formulas (9.2)(9.4). The first case can be deduced from lemma 5.3. □

12. A formula for the automorphisms in a special case

In this section we prove the following theorem.

Theorem 12.1. *Let Q be a quiver with a vertex i and let $A \in \mathrm{Rep}(Q)$ be such that $A_i = 0$. Then*

$$\hat{t}_i[A] = \sum_{s+t=n} (-1)^s q^{s/2} [S_i]^{(s)}_{q^{1/2}} [A] [S_i]^{(t)}_{q^{1/2}} \tag{12.1}$$

where $n = -(\bar{A}, e_i)$.

Proof. Without loss of generality we may assume that i is a source. Let Q' be the quiver obtained from Q by inverting all edges adjacent to i. Put $B = t_i A$ and let C be a quotient of B which differs only from B in i.

For $a \in U(Q')$ of degree α we define

$$r_i(a) = [S_i]a - q^{(1/2)(\alpha, e_i)} a [S_i]$$

For $t \in \{0, \dots n\}$ let P_t be the set of Q'-representations D which satisfy $B \twoheadrightarrow D \twoheadrightarrow C$ and $\dim D_i = t$, up to isomorphisms which leave B, C fixed.

By analogy to (9.7) we have the following formula

$$r_i[C] = q^{(1/2)\langle \bar{C}, e_i \rangle} \sum_{[C'] \,\mathrm{adm}} g^{C'}_{S_i C} [C']$$

By lemma 10.1 we have

$$g^{C'}_{S_i C} = \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|} g^{s_i C}_{s_i C' \, S_i}$$

Hence we can now compute

$$\begin{aligned} r_i[C] &= q^{(1/2)\langle \bar{C}, e_i\rangle} \sum_{[C'']} g^{s_iC}_{C''S_i} \frac{|\operatorname{Aut}(C'')|}{|\operatorname{Aut}(C)|}[s_iC''] \\ &= q^{(1/2)\langle \bar{C}, e_i\rangle} \sum_{s_iC/C'' \cong S_i} \frac{|\operatorname{Aut}(C'')|}{|\operatorname{Aut}(C)|}[s_iC''] \\ &= q^{(1/2)\langle \bar{C}, e_i\rangle} \sum_{C' \in P_{\dim C_i+1}} \frac{|\operatorname{Aut}(C')|}{|\operatorname{Aut}(C)|}[C'] \end{aligned}$$

This formula simplifies if we introduce $\widetilde{[C]} = \operatorname{Aut}(C)[C]$. Then we obtain.

$$r_i\widetilde{[C]} = q^{(1/2)\langle \bar{C}, e_i\rangle} \sum_{C' \in P_{\dim C_i+1}} \widetilde{[C']} \tag{12.2}$$

Since $A_i = 0$ we may consider A as a Q' representation in the obvious way. We will use (12.2) to compute $r_i^n[A]$. We find

$$r_i^n\widetilde{[A]} = q^{-n/2} \cdot q^{(-n+1)/2} \cdots q^{1/2} f_n \widetilde{[B]}$$

where f_n is the number of complete flags in an n-dimensional vectorspace over k. This is easily seen to be equal to

$$\frac{(q^n-1)\cdots(q-1)}{(q-1)^n} = q^{n(n-1)/4}[n]^!_{q^{1/2}}$$

Thus, using the fact that $|\operatorname{Aut}(A)| = |\operatorname{Aut}(B)|$ we obtain

$$\begin{aligned} r_i^n[A] &= q^{-n(n+1)/4} \cdot q^{n(n-1)/4}[n]^!_{q^{1/2}}[B] \\ &= q^{-n/2}[n]^!_{q^{1/2}}[B] \end{aligned}$$

Now we compute $\hat{t}_i([A])$. Let r_i be defined in the same way on $U(Q)$ as it was on $U(Q')$. Clearly

$$F_{Q',Q} \circ r_i = r_i \circ F_{Q',Q}$$

Thus we find

$$\hat{t}_i([A]) = F_{Q',Q}[B] = \frac{q^{n/2}}{[n]^!_{q^{1/2}}} F_{Q',Q} r_i^n[A] = \frac{q^{n/2}}{[n]^!_{q^{1/2}}} r_i^n[A]$$

A straightforward verification shows that this is equal to (12.1) □

13. Comparison with Lusztig's automorphisms

As before Q is a quiver without loops. The bilinear form $(-,-)$ defines a (simply laced) Cartan datum with simple roots $(e_i)_i$. Associated to this Cartan datum is a quantum enveloping algebra [4] $U_v(\mathfrak{g})$ which is the $\mathbb{C}$-algebra with generaters E_i,

F_i (for $i \in Q_0$) and L_α (for $\alpha \in \mathbb{Z}^{Q_0}$) and relations

$$L_0 = 1 \tag{13.1}$$

$$L_\alpha L_\beta = L_{\alpha+\beta} \tag{13.2}$$

$$L_\alpha E_i L_\alpha^{-1} = v^{(\alpha,e_i)} E_i \tag{13.3}$$

$$L_\alpha F_i L_\alpha^{-1} = v^{-(\alpha,e_i)} F_i \tag{13.4}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{L_i - L_i^{-1}}{v - v^{-1}} \tag{13.5}$$

$$\sum_{s+t=1-b_{ij}} (-1)^s (E_i)_v^{(s)} E_j (E_i)_v^{(t)} = 0 \qquad (\text{for } i \neq j) \tag{13.6}$$

$$\sum_{s+t=1-b_{ij}} (-1)^s (F_i)_v^{(s)} F_j (F_i)_v^{(t)} = 0 \qquad (\text{for } i \neq j) \tag{13.7}$$

where $b_{ij} = (e_i, e_j)$. It is well known that $U_v(\mathfrak{g})$ is in fact a Hopf algebra. For the comultiplication, counit and antipode we refer to [4].

Let $u(Q)$ be the subalgebra of $U(Q)$ generated by $[S_i]_i$, $[S_i]_i^-$ and K_α. It is clear that $u(Q)$ is a sub-Hopf algebra of $U(Q)$. From (12.1) together with the definition of $\hat{t}_i$ and the compatibility with ω we find the following formulas.

$$\begin{aligned}
\hat{t}_i([S_i]) &= q^{-1/2}[S_i]^- K_i^{-1} \\
\hat{t}_i([S_i]^-) &= q^{-1/2}[S_i] K_i \\
\hat{t}_i([S_j]) &= \sum_{s+t=n} (-1)^s q^{s/2} [S_i]_{q^{1/2}}^{(s)} [S_j] [S_i]_{q^{1/2}}^{(t)} \qquad (\text{for } i \neq j) \\
\hat{t}_i([S_j^-]) &= \sum_{s+t=n} (-1)^s q^{s/2} ([S_i]^-)_{q^{1/2}}^{(s)} [S_j] ([S_i]^-)_{q^{1/2}}^{(t)} \qquad (\text{for } i \neq j) \\
\hat{t}_i(K_\alpha) &= K_{s_i\alpha}
\end{aligned} \tag{13.8}$$

In particular $u(Q)$ is preserved under $\hat{t}_i$.

As in [14] one constructs an isomorphism as Hopf algebras $\theta : U_{q^{-1/2}}(\mathfrak{g}) \to u(Q)$ by $\theta(E_i) = -q^{1/2}[S_i]^-$, $\theta(F_i) = [S_i]$, $\theta(L_i) = K_i^{-1}$.

In [11, 10] Lusztig defines several actions of the braid group on $U_v(\mathfrak{g})$. We will consider one of those. For every $i \in Q_0$ one defines the automorphism $T''_{i,-1}$ of $U_v(\mathfrak{g})$ by

$$\begin{aligned}
T''_{i,-1}(E_i) &= -F_i L_i^{-1} \\
T''_{i,-1}(F_i) &= -L_i E_i \\
T''_{i,-1}(E_j) &= \sum_{s+t=-b_{ij}} (-1)^s v^s (E_i)_v^{(t)} E_j (E_i)_v^{(s)} \qquad (\text{for } i \neq j) \\
T''_{i,-1}(F_j) &= \sum_{s+t=-b_{ij}} (-1)^s v^{-s} (F_i)_v^{(s)} F_j (F_i)_v^{(t)} \qquad (\text{for } i \neq j) \\
T''_{i,-1}(L_\alpha) &= L_{s_i\alpha}
\end{aligned} \tag{13.9}$$

Theorem 13.1. *One has $\theta T''_{i,-1} = \hat{t}_i \theta$.*

Proof. The proof simply amounts to comparing (13.8) with (13.9). We leave this to the reader. □

Remark 13.2. We have now obtained a new proof that the $T''_{i,-1}$ are automorphisms of $U_v(\mathfrak{g})$. One distinguishing feature of our approach is that we have defined the $T''_{i,-1}$ globally (through the $\hat{t}_i$) whereas in Lusztig's approach they are defined on generators.

It would be nice if one could also find a direct proof (or at least a conceptual explanation) that the $T''_{i,-1}$ satisfy the braid relations, but we have not been able to do so, except when $|b_{ij}| \leq 1$ for all i, j, but this is a trivial case anyway.

In fact a natural question is whether the $\hat{t}_i$ satisfy the braid relations on the whole Hall algebra.

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