EXISTENCE THEOREMS FOR DUALIZING COMPLEXES OVER NON-COMMUTATIVE GRADED AND FILTERED RINGS

MICHEL VAN DEN BERGH

Abstract. In this note we prove existence theorems for dualizing complexes over graded and filtered rings, thereby generalizing some results by Zhang, Yekutieli and Jørgensen.

1. INTRODUCTION

In this note we aim to complete some of the results in [8, 9, 14, 13]. In these papers the authors are concerned with local cohomology, Serre duality and dualizing complexes over graded rings. Recall that a dualizing complex over a non-commutative ring A is roughly speaking a bounded complex of A-bimodules finitely generated on both sides such that $\mathrm{RHom}_A(-, R)$ defines a duality between suitable subcategories of $D(A)$ and $D(A^\circ)$. In the commutative case, dualizing complexes are fundamental in Grothendieck's duality formalism for coherent sheaves.

In the non-commutative case, it is pointed out in [14] that the existence of a ("balanced") dualizing complex implies Serre duality, but the converse is less clear.

Below we will give a necessary and sufficient criterion for the existence of (balanced) dualizing complexes over connected graded rings (Theorem 6.3). This resembles [9, Thm 3.3] which (roughly) states that the Matlis dual of local cohomology defines a duality between $D^b_{\mathrm{fg}}(A)$ and $D^b_{\mathrm{fg}}(A^\circ)$. However Jørgensen proves his result only for quotients of AS-Gorenstein rings and even in this somewhat more restricted situation he doesn't quite prove that one obtains a balanced dualizing complex in the sense of [13].

Our criterion is based upon a general local duality formula (Theorem 5.1), a slightly restricted version of which is also proved in [9] (Jørgensen's methods would also yield our version).

In section 8 of this paper we develop the rudiments of a theory of dualizing complexes over ungraded rings. This section really poses more problems that it answers, but we are nevertheless able to show the existence of dualizing complexes over some filtered rings, thereby answering a question by Yekutieli at the end of the introduction to [13]. The following corollary (a special case of Corollary 8.7) is a typical example of what can be obtained.

Corollary 1.1. *Let k be a field and assume that A is a k-algebra carrying a filtration $k = F_0A \subset F_1A \subset \cdots$ such that $\mathrm{gr}\, A$ is commutative and finitely generated. Then A has a dualizing complex and in particular $D^b_{\mathrm{fg}}(A)$ is dual to $D^b_{\mathrm{fg}}(A^\circ)$.*

1991 *Mathematics Subject Classification.* Primary 18G10, 16E30.
Key words and phrases. Graded rings, local cohomology, dualizing complexes.
The author is a director of research at the NFWO.

As a byproduct of our methods we obtain a general formula for the dualizing complex over a Gorenstein ring (Proposition 8.4). It is tempting to conjecture that such a formula might be used to prove existence of dualizing complexes over more general classes of non-graded rings, but we have not yet been able to so.

Finally in section 9 we compute the dualizing complex of a Koszul AS-regular algebra [2]. Specializing to dimension three yields a connection between the matrix Q, prominent in [2] and the automorphism defined by the canonical normalizing element in degree three, which was introduced in [3, 4].

This paper was completed before I became aware of [9] so there is some overlap between the papers in their starting sections. For example the results below on Matlis duality are also contained in [9]. Furthermore, as pointed out above, a duality formalism is to some extent developed in [9] for quotients of Gorenstein rings.

Let me close this introduction by thanking Yekutieli for his careful reading of a preliminary version of this paper and for pointing out a serious error in the proof of Proposition 8.2 (and also for telling me how to repair it!).

2. Notations and conventions

Except for section 8, most objects below will be tacitly assumed to be $\mathbb{Z}$-graded. If $M = \oplus M_n$ is a graded object with $M_n = 0$ for $n \gg 0$ then we say that M is right limited. Left limited is defined similarly.

k will always be a field. A *connected* graded k-algebra is an $\mathbb{N}$-graded ring $k + A_1 + A_2 + \cdots$ with $\dim A_i < \infty$ for all i. Unless otherwise specified A, B, C will be connected graded rings.

The category of left graded A-modules is denoted by $\mathrm{Gr}(A)$. Following [13] we view right A-modules as left A°-modules and A-B bimodules as $A \otimes B^\circ$-modules. We denote $A^e = A \otimes A^\circ$. When we write $\mathrm{Hom}_A(-,-)$ we mean graded "Hom". That is, those maps which are finite sums of homogeneous maps. If we really need Hom in $\mathrm{Gr}(A)$ then we write $\mathrm{Hom}_{\mathrm{Gr}(A)}(-,-)$. Similar conventions apply to RHom. Shifting in $\mathrm{Gr}\, A$ will be denote by ( ) and shifting in $D(A)$ will be denoted by [ ].

Forgetting right or left structure defines restriction maps

$$\begin{array}{ccc} \mathrm{Gr}(A \otimes B^\circ) & \longrightarrow & \mathrm{Gr}(A) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(B^\circ) & \longrightarrow & \mathrm{Gr}(k) \end{array}$$

which preserve injectives and projectives [13, lem 2.1]. This means in particular that all derived functors we will use are compatible with it.

If $M \in D(\mathrm{Gr}\, A)$ then $M^\cdot$ stands for an arbitrary complex representing M. Conversely if $N^\cdot$ is a complex then $Q(N^\cdot)$ is the corresponding object in $D(\mathrm{Gr}\, A)$.

We refer to [13] for other notations. In general these should be self-explanatory.

3. Matlis duality for graded rings

The results in this section are also contained in [9]. For the convenience of the reader I repeat them here.

If M is a graded k-vectorspace then we define its Matlis dual as

$$M' = \mathrm{Hom}_k(M, k) = \bigoplus_n M_n^*$$

Let us call M locally finite if M is finite dimensional in every degree. Then $(-)'$ defines an autoduality on the category of locally finite vectorspaces. If A is a graded ring then $(-)'$ sends $\operatorname{Gr} A$ to $\operatorname{Gr} A^\circ$ and in this way defines a duality between the full subcategories consisting of locally finite objects. Since $(-)'$ is exact it extends in the obvious way to a contravariant functor

$$(-)' : D(\operatorname{Gr} A) \to D(\operatorname{Gr} A^\circ)$$

and we have $H^i(K)' = H^{-i}(K')$, $K[n]' = K'[-n]$.

Let us denote by $D_{\mathrm{lf}}(\operatorname{Gr} A)$ the full subcategory of $D(\operatorname{Gr} A)$ consisting of complexes with locally finite homology.

Proposition 3.1. *(Matlis duality.)*

(1) *$(-)'$ defines a duality between $D_{\mathrm{lf}}(\operatorname{Gr} A)$ and $D_{\mathrm{lf}}(\operatorname{Gr} A^\circ)$.*
(2) *For $M, N \in D_{\mathrm{lf}}(\operatorname{Gr} A)$ we have*

$$\operatorname{Hom}_A(M,N) = \operatorname{Hom}_{A^\circ}(N', M') \tag{3.1}$$

(3) *Assume in addition that either $M \in D^-_{\mathrm{lf}}(\operatorname{Gr} A)$ or $N \in D^+_{\mathrm{lf}}(\operatorname{Gr} A)$. Then*

$$\operatorname{RHom}_A(M,N) = \operatorname{RHom}_{A^\circ}(N', M')$$

Proof. (1) If $K \in D_{\mathrm{lf}}(\operatorname{Gr} A)$ then we have $K' \in D_{\mathrm{lf}}(\operatorname{Gr} A^\circ)$ and by looking at homology we see that the canonical map $K \to K''$ is an isomorphism. This proves 1.

(2) By 1. we find that

$$\operatorname{Hom}_{D(\operatorname{Gr} A)}(M,N) = \operatorname{Hom}_{D(\operatorname{Gr} A^\circ)}(N', M')$$

and by taking the sum over the shifts in $\operatorname{Gr} A$ we obtain (3.1).

(3) Under the current hypotheses we may, and we will, assume that $M^\cdot$ is a right bounded complex of projectives or that $N^\cdot$ is a left bounded complex of injectives.

Taking H^0 of the composition

$$\operatorname{RHom}_A(M,N) = Q\operatorname{Hom}_A(M^\cdot, N^\cdot) \to Q\operatorname{Hom}_{A^\circ}((N^\cdot)', (M^\cdot)') \to \operatorname{RHom}_{A^\circ}(N', M')$$

is precisely the map (3.1) and hence is an isomorphism. If we replace M by shifts in $D(\operatorname{Gr} A)$ we find that this is also the case for H^i. This proves 3. □

4. Some results on local cohomology

In this section we prove some results for the local cohomology of graded rings. Our main interest concerns noetherian rings. However even if A is noetherian then a priori we don't know if this is also the case for A^e. Therefore in this section we are forced to consider non-noetherian rings also.

For a connected k-algebra A we put $m = m_A = A_{\geq 1}$ and we define the functor

$$\Gamma_m : \operatorname{Gr}(A) \to \operatorname{Gr}(A) : M \mapsto \operatorname*{inj\,lim}_n \operatorname{Hom}_A(A/A_{\geq n}, M)$$

On $\operatorname{Gr}(A^\circ)$ and $\operatorname{Gr}(A^e)$ we have corresponding functors, which we denote respectively by Γ_{m° and Γ_{m^e}. We will also use the same notations for some variants of these functors. For example Γ_m defines a functor $\operatorname{Gr}(A \otimes B) \to \operatorname{Gr}(A \otimes B)$. Thanks to [13, lem 2.1] such sloppiness is allowed, even if we work with the corresponding derived functors.

Assume that A is finitely generated. Let us call an object X in $\mathrm{Gr}(A \otimes B)$, m-torsion if for every $x \in X$ there exists an n such that $A_{\geq n}x = 0$. It is easy to see that this defines a torsion theory in $\mathrm{Gr}(A \otimes B)$ and Γ_m is the corresponding torsion functor.

Lemma 4.1. *Assume that E is an injective $A \otimes B$-module.*

1. *If M is an A-module then $\mathrm{Hom}_A(M, E)$ is an injective B-module.*
2. *$\Gamma_m(E)$ is a direct limit of injective B-modules.*

Proof.
1. This follows from the fact that $\mathrm{Hom}_B(-, \mathrm{Hom}_A(M, E)) = \mathrm{Hom}_{A \otimes B}(M \otimes -, E))$ is an exact functor.
2. This follows from 1. together with the definition of Γ_m. □

We will say that A is Ext-finite if $\mathrm{Ext}^i(k, k)$ is finite dimensional for every i. This is equivalent with the minimal free (left or right) resolution of k being of finite rank in every degree. From this we easily deduce:

Lemma 4.2. *Assume that A, B are* Ext*-finite. Then so is $A \otimes B$.*

Out main reason for introducing Ext-finiteness is the following.

Lemma 4.3. *Assume that A is* Ext*-finite. Then $R^i\Gamma_m(-)$ commutes with colimits.*

Proof. Since

$$R^i\Gamma_m(-) = \operatorname{inj}\lim_n \mathrm{Ext}^i_A(A/A_{\geq n}, -)$$

it suffices to show that $\mathrm{Ext}^i(A/A_{\geq n}, -)$ commutes with colimits.

Let $F^{\cdot}$ be the minimal free resolution of $A/A_{\geq n}$. Then

$$\mathrm{Ext}^i_A(A/A_{\geq n}, -) = H^i(\mathrm{Hom}_A(F^{\cdot}, -)) \tag{4.1}$$

Since $A/A_{\geq n}$ is finite dimensional it follows from Ext-finiteness that $F^{\cdot}$ has finite rank in every degree. Hence the righthand side of (4.1) commutes with colimits and we are done. □

Lemma 4.4. *Assume that A is* Ext*-finite. Let $M \in D^+(\mathrm{Gr}\,A)$. If M has m-torsion homology then the canonical map $R\Gamma_m(M) \to M$ is an isomorphism. If Γ_m has finite cohomological dimension then the same is true for $M \in D(\mathrm{Gr}\,A)$.*

Proof. We have to show that $R\Gamma_m(M)$ and M have the same homology. By looking at the appropriate spectral sequences for hypercohomology this amounts to showing that

$$R^i\Gamma_m(M) = 0 \tag{4.2}$$

for $i > 0$ and for torsion $M \in \mathrm{Gr}\,A$. Now if M is torsion then M is a direct limit of finite dimensional modules. Hence by lemma 4.3 it suffices to show that (4.2) holds with M finite dimensional. But then it suffices to show it for k. If $F^{\cdot}$ is the minimal (right) free resolution of k then $(F^{\cdot})'$ is an injective resolution of k. $(F^{\cdot})'$ consists of direct sums of copies of A' and thus is torsion in every degree. Hence we find that $R^i\Gamma_m(k) = H^i(\Gamma_m((F^{\cdot})')) = H^i((F^{\cdot})') = 0$, if $i > 0$. □

Lemma 4.5. *Assume that A and B are* Ext*-finite. Then*

$$R\Gamma_{m_{A\otimes B}} = R\Gamma_{m_B} \circ R\Gamma_{m_A}$$

as endofunctors on $D^+(\mathrm{Gr}(A \otimes B))$. If Γ_{m_A} and Γ_{m_B} have finite cohomological dimension then the statement is also true on $D(\mathrm{Gr}(A \otimes B))$.

Proof. Let us first consider the non-derived statement. In the following computation we use the fact that the filters of ideals $(A \otimes B)_{\geq n}$ and $A_{\geq n} \otimes B + A \otimes B_{\geq n}$ are cofinal.

$$\begin{aligned}
\Gamma_{m_{A\otimes B}} &= \operatorname{inj}\lim_n \operatorname{Hom}_{A\otimes B}((A\otimes B)/(A\otimes B)_{\geq n}, -) \\
&= \operatorname{inj}\lim_n \operatorname{Hom}_{A\otimes B}(A/(A_{\geq n}\otimes B + A\otimes B_{\geq n}), -) \\
&= \operatorname{inj}\lim_n \operatorname{Hom}_{A\otimes B}((A/A_{\geq n})\otimes (B/B_{\geq n}), -) \\
&= \operatorname{inj}\lim_n \operatorname{Hom}_{B}(B/B_{\geq n}, \operatorname{Hom}_A(A/A_{\geq n}, -)) \\
&= \operatorname{inj}\lim_n \operatorname{inj}\lim_m \operatorname{Hom}_{B}(B/B_{\geq n}, \operatorname{Hom}_A(A/A_{\geq m}, -)) \\
&= \operatorname{inj}\lim_n \operatorname{Hom}_{B}(B/B_{\geq n}, \operatorname{inj}\lim_m \operatorname{Hom}_A(A/A_{\geq m}, -)) \\
&= \Gamma_{m_B} \circ \Gamma_{m_A}
\end{aligned}$$

If we want to show the corresponding derived statement $R\Gamma_{m_{A\otimes B}}(M) = R\Gamma_{m_B}(R\Gamma_{m_A}(M))$ then as usual we replace M by a complex of injectives in $\operatorname{Gr}(A \otimes B)$. The fact that this is possible if $M \in D(\operatorname{Gr} A \otimes B)$ follows from [6, Apl 2.4]. From lemma 4.1.2. and lemma 4.3 it follows that Γ_{m_A} applied to an injective $A \otimes B$-module yields a module which is acyclic for Γ_{m_B}. This shows what we want. □

Recall that a connected left noetherian graded ring A is said to satisfy "χ" [1] if for every finitely generated graded A-module we have that $\operatorname{Ext}^i(k, M)$ has right limited grading for all i. By [1, Cor 3.6] this is equivalent with $R^i\Gamma_m(M)$ having right limited grading for all finitely generated M and all i.

Lemma 4.6. *Assume that A is left noetherian and satisfies χ. If $M \in D^+(\operatorname{Gr} A)$ has finitely generated homology then $R\Gamma_m(M)$ has right limited homology (not necessarily uniformly). If Γ_m has finite cohomological dimension then the same result is true for $M \in D(\operatorname{Gr} A)$.*

Proof. This follows by considering the appropriate spectral sequences for hypercohomology. □

Theorem 4.7. *Assume that A is left noetherian and satisfies χ and that B is* Ext-*finite. Let M be an object in $D^+(\operatorname{Gr} A \otimes B)$ whose cohomology modules are finitely generated as A-modules. Then*

$$R\Gamma_{m_{A\otimes B}}(M) = R\Gamma_{m_A}(M) \tag{4.3}$$

If Γ_{m_A} and Γ_{m_B} have finite cohomological dimension then the same statement is true if we assume $M \in D(\operatorname{Gr} A \otimes B)$.

Proof. This is a combination of lemmas 4.5, 4.6 and 4.4. □

Corollary 4.8. *Assume that A, B are left noetherian and satisfy χ. If M is an object in $D^+(\operatorname{Gr} A \otimes B)$ whose cohomology modules are finitely generated as A and as B-modules then*

$$R\Gamma_{m_A}(M) = R\Gamma_{m_B}(M)$$

Furthermore if Γ_{m_A} and Γ_{m_B} have finite cohomological dimension then the same result is true if we take $M \in D(\operatorname{Gr} A \otimes B)$

Proof. This is the obvious combination of (4.3) with the symmetric statement $R\Gamma_{m_A \otimes m_B}(M) = R\Gamma_{m_A}(M)$ □

5. Local duality

The following result was stated in [9] under the additional hypotheses that A is left noetherian and $M \in D^-(A^e)$. For completeness we include our own proof.

Theorem 5.1. *(Local duality) Assume that A is* Ext*-finite and that Γ_m has finite cohomological dimension.*

1. *$R\Gamma_m(A)'$ has finite injective dimension as an object in $D(\mathrm{Gr}(A))$.*
2. *For $M \in D(\mathrm{Gr}(A \otimes C^\circ))$*

$$R\Gamma_m(M)' = \mathrm{RHom}_A(M, R\Gamma_m(A)') \tag{5.1}$$

in $D(\mathrm{Gr}(C \otimes A^\circ))$.

Proof. Below we let $E^\cdot$ be a bounded resolution of A as graded A^e-modules whose restriction consists of Γ_m-acyclic A-modules. We can construct such a resolution by taking a suitable truncation of an A^e-injective resolution of A. Then we have in $D(\mathrm{Gr}(A \otimes B^\circ) : R\Gamma_m(A) = Q\Gamma_m(E^\cdot)$.

To prove 2., we start by assuming that $M \in D^-(\mathrm{Gr}(A \otimes C^\circ))$. Let $K^\cdot$ be a graded projective resolution of M in $\mathrm{Gr}(A \otimes C^\circ)$. This is automatically a projective resolution of M in $\mathrm{Gr}(A)$. Thus

$$\mathrm{RHom}_A(M, R\Gamma_m(A)') = Q\,\mathrm{Hom}_A(K^\cdot, \Gamma_m(E^\cdot)') \tag{5.2}$$

On the other hand by lemma 4.3 $R^i\Gamma_m$ commutes with direct sums. So $E^\cdot \otimes_A K^\cdot$ is a Γ_m-acyclic right bounded complex, quasi-isomorphic to M. Hence it can be used to compute $R\Gamma_m(M)$. We find

$$R\Gamma_m(M)' = Q\Gamma_m(E^\cdot \otimes_A K^\cdot)' = Q(\Gamma_m(E^\cdot) \otimes_A K^\cdot)' \tag{5.3}$$

It is now easy to see that the righthand sides of (5.2) and (5.3) are equal. This proves 2. for $M \in D^-(\mathrm{Gr}\, A \otimes C^\circ)$.

Applying (5.1) to $M[n]$ for all n and all $M \in \mathrm{Gr}(A)$ we see that the fact that Γ_m has finite cohomological dimension implies that $R\Gamma_m(A)'$ has finite injective dimension, which proves 1.

We now prove 2. in general. If $M \in D(\mathrm{Gr}(A \otimes C^\circ))$ then by [6], M is quasi-isomorphic to a complex $K^\cdot$ of projectives. Since $R\Gamma_m(A)'$ has finite injective dimension and Γ_m has finite cohomological dimension the formulas (5.2)(5.3) remain valid. So we can simply copy the proof for $M \in D^-(\mathrm{Gr}\, A \otimes C^\circ)$. □

6. An existence theorem for dualizing complexes.

Let us recall the following definition by Yekutieli [13].

Definition 6.1. Assume that A is a left and right (graded) noetherian ring. Then an object of $D^b(\mathrm{Mod}\, A^e)$ $(D^b(\mathrm{Gr}\, A^e))$ is called a dualizing complex (in the graded sense) if it satisfies the following conditions

1. R has finite injective dimension over A and A°.
2. The cohomology of R is given by bimodules which are finitely generated on both sides.
3. The natural morphisms $\Phi : A \to \mathrm{RHom}_A(R, R)$ and $\Phi^\circ : A \to \mathrm{RHom}_{A^\circ}(R, R)$ are isomorphisms in $D(\mathrm{Mod}\, A^e)$ $(D(\mathrm{Gr}\, A^e))$.

Since our rings are noetherian it is clear from this definition that a dualizing complex in the graded sense, is automatically one in the ungraded sense (one uses the fact that a graded injective has injective dimension ≤ 1 as ungraded modules, which we leave to the reader as a pleasant exercise in homological algebra).

It is shown in [13] (following [7]) that if $R \in D(\operatorname{Mod} A)$ is a dualizing complex then $\operatorname{RHom}_A(-, R)$ defines a duality between the full subcategories of $D(\operatorname{Mod} A)$ and $D(\operatorname{Mod} A^\circ)$ consisting of complexes with finitely generated homology, justifying to some extent the terminology.

Unfortunately, even over a connected graded ring, a dualizing complex is only determined up to shifting in the derived category, and up to left or right multiplication with an invertible bimodule. In order to rigidify the definition of a dualizing complex, Yekutieli introduced the notion of a balanced dualizing complex which we give below. From now on we assume again that we work in the graded category (except in section 8). As usual A, B, C are connected graded rings.

Definition 6.2. Assume that A is left and right noetherian and has a dualizing complex R. Then we say that R is a balanced dualizing complex if

$$R\Gamma_m(R^\cdot) \cong A' \quad \text{and} \quad R\Gamma_{m^\circ}(R^\cdot) \cong A' \tag{6.1}$$

in $D(\operatorname{Gr} A^e)$.

The following theorem generalizes [8, Thm 3.3].

Theorem 6.3. *Assume that A is left and right noetherian. If A has a balanced dualizing complex R then R is given by*

$$R = R\Gamma_m(A)' \tag{6.2}$$

and furthermore the following holds.

1. *Γ_m, Γ_{m° have finite cohomological dimension.*
2. *A and A° satisfy χ.*

Conversely if these conditions are satisfied then A has a balanced dualizing complex, given by (6.2).

Proof. By [13, Cor 4.21], a balanced dualizing complex, if it exists, is given by (6.2). The fact that 1.,2. hold is proved in [14, Th 4.2].

Now assume that A satisfies 1. and 2., and define R by (6.2). By Corollary 4.8, we also have $R = R\Gamma_{m^\circ}(A)'$ and it follows from Theorem 5.1.2 that R has finite injective dimension on the left and on the right. This proves 6.1.1.

We have $H^i(R) = R^i\Gamma_m(A)'$ and hence by [1, Prop 7.9], $H^i(R)$ is finitely generated on the right. Similarly $H^i(R)$ is also finitely generated on the left. This proves 6.1.2. In particular $R \in D_{\mathrm{lf}}(A)$.

We now prove 6.1.3. By Proposition 3.1.3

$$\operatorname{RHom}_A(R, R) = \operatorname{RHom}_{A^\circ}(R\Gamma_m(A), R\Gamma_m(A)) = \operatorname{RHom}_{A^\circ}(R\Gamma_m(A), R\Gamma_{m^\circ}(A))$$

Now Γ_{m° preserves injectives (for example [13, Prop 4.6]) and $R\Gamma_m(A)$ has right torsion homology by Lemma 4.6. From this we deduce.

$$\begin{aligned}
\operatorname{RHom}_{A^\circ}(R\Gamma_m(A), R\Gamma_{m^\circ}(A)) &= \operatorname{RHom}_{A^\circ}(R\Gamma_m(A), A) && \\
&= \operatorname{RHom}_A(A', R\Gamma_m(A)') && \text{(Proposition 3.1.3)}\\
&= R\Gamma_m(A')' && \text{(Theorem 5.1)}\\
&= A'' && \text{(Lemma 4.4)}\\
&= A &&
\end{aligned}$$

The second part of 6.1.3 is similar.

Now we verify what we have to do in order to prove the first part of (6.1). Since A is locally finite, we may as well prove

$$R\Gamma_m(R)' \cong A$$

in $D(\operatorname{Gr} A^e)$. By local duality this is equivalent to

$$\operatorname{RHom}_A(R, R\Gamma_m(A)') = A$$

Hence the first part of (6.1) follows from 6.1.3! The second part is similar. □

7. An application

Theorem 7.1. *Assume that A and B have balanced dualizing complexes and that $A \otimes B$ is left and right noetherian. Then $A \otimes B$ has a balanced dualizing complex, which is the tensor product of the dualizing complexes of A and B.*

Proof. We have to verify the hypotheses for Theorem 6.3. By lemma 4.5

$$R\Gamma_{m_{A\otimes B}} = R\Gamma_{m_A} \circ R\Gamma_{m_B} \tag{7.1}$$

as endofunctors on $D(\operatorname{Gr} A \otimes B)$. Thus in particular $R\Gamma_{m_{A\otimes B}}$ has finite cohomological dimension, and by symmetry the same holds for $R\Gamma_{m^\circ_{A\otimes B}}$. This proves 6.3.1.

By checking the proof of lemma 4.5 one sees that (7.1) also holds as endofunctors on $D(\operatorname{Gr} A^e \circ B^e)$. Thus we find

$$R\Gamma_{m_{A\otimes B}}(A \otimes B) = R\Gamma_{m_A}(A) \otimes R\Gamma_{m_B}(B) \tag{7.2}$$

To verify χ we have to show that if M is a finitely generated $A \otimes B$-module then $R\Gamma_{m_{A\otimes B}}(M)$ has right limited homology. Since $R\Gamma_{m_{A\otimes B}}$ has finite cohomological dimension it suffices to do this for $M = A \otimes B$. However this is clear by (7.2). The fact that $(A \otimes B)^\circ$ satisfies χ is similar.

Hence we can now invoke Theorem 6.3 to obtain that $A \otimes B$ has a dualizing complex given by $R\Gamma_{m_{A\otimes B}}(A \otimes B)'$. By (7.2) this is equal to the tensor product of the dualizing complexes of A and B. □

8. Dualizing complexes over filtered rings

In this section we drop the convention that A, B, C are connected graded rings. Our aim is to give an existence criterion for dualizing complexes over filtered rings.

In [13] Yekutieli introduced the notion of a balanced dualizing complex, in order to rigidify the definition of a dualizing complex. Unfortunately "balanced" makes a priori no sense for non-graded rings, so it is convenient to have a substitute.

Definition 8.1. Let A be a left and right noetherian (graded) ring. A dualizing complex R over A is *rigid* (in the graded sense) if

$$\mathrm{RHom}_{A^e}(A, {}_AR \otimes R_A) \cong R \tag{8.1}$$

in $D(\mathrm{Mod}\, A^e)$ ($D(\mathrm{Gr}\, A^e)$) (the notations R_A, R_A mean respectively the left and right A-structures of R).

Note again that the corresponding graded version of this definition implies the ungraded version if A^e is (left or equivalently right) noetherian.

The justification of Definition 8.1 is in the following proposition.

Proposition 8.2. *Let A be left and right noetherian.*

(1) *If R_1, R_2 are rigid dualizing complexes for A then $R_1 \cong R_2$ in $D(\mathrm{Mod}\, A^e)$.*
(2) *If A is connected graded and R is a balanced dualizing complex for A then R is rigid (in the graded sense).*

Proof. (1) We thank Amnon Yekutiely for suggesting a correction to the original faulty proof of this statement.

Put $L = \mathrm{RHom}_A(R_1, R_2)$, $L' = \mathrm{RHom}_A(R_2, R_1)$, $\tilde{L} = \mathrm{RHom}_{A^\circ}(R_1, R_2)$, $\tilde{L}' = \mathrm{RHom}_{A^\circ}(R_2, R_1)$. According to [13, Lem 3.10], L, L' have finite Tor-dimension on the left and $R_2 = R_1 \overset{L}{\otimes}_A L$, $R_1 = R_2 \overset{L}{\otimes}_A L'$. Furthermore by the proof of [13, Thm 3.9] we also have

$$L \overset{L}{\otimes}_A L' = A = L' \overset{L}{\otimes}_A L$$

in $D(\mathrm{Mod}\, A^e)$. Symmetric statements hold of course for $\tilde{L}$, $\tilde{L}'$.

We may now compute

$$\begin{aligned} R_1 \overset{L}{\otimes}_A L = R_2 &= \mathrm{RHom}_{A^e}(A, R_2 \otimes R_2) \\ &= \mathrm{RHom}_{A^e}(A, R_1 \overset{L}{\otimes}_A L \otimes \tilde{L} \overset{L}{\otimes}_A R_1) \\ &= \tilde{L} \overset{L}{\otimes}_A \mathrm{RHom}_{A^e}(A, R_1 \otimes R_1) \overset{L}{\otimes}_A L \\ &= \tilde{L} \overset{L}{\otimes}_A R_1 \overset{L}{\otimes}_A L \\ &= R_2 \overset{L}{\otimes}_A L \end{aligned}$$

Tensoring on the right with L' yields $R_1 = R_2$.

(2) By local duality combined with Theorem 4.7 and lemma 4.5 we find

$$\begin{aligned} R &= R\Gamma_m(A)' \\ &= R\Gamma_{m^e}(A)' \\ &= \mathrm{RHom}_{A^e}(A, R\Gamma_{m^e}(A \otimes A^\circ)') \\ &= \mathrm{RHom}_{A^e}(A, R\Gamma_m(A)' \otimes R\Gamma_{m^\circ}(A)') \\ &= \mathrm{RHom}_{A^e}(A, R \otimes R) \quad \square \end{aligned}$$

Remark 8.3. The previous proposition leaves two obvious questions, which I have been unable to answer so far.

(1) If a connected graded ring has a rigid dualizing complex, is it balanced?
(2) If a graded (not necessarily connected) ring has a rigid dualizing complex, is this dualizing complex automatically graded?

Assume that A is a left and right noetherian ring possessing a rigid dualizing complex R. Then we say that A is AS-Cohen-Macaulay (see Remark 8.5 below for the terminology) if R has homology in only one degree. In that case $R = \omega_A[\delta]$ for some ω_A in $\operatorname{Gr} A^e$ and $\delta \in \mathbb{Z}$. We call ω_A the dualizing module of A.

If A is AS-Cohen-Macaulay then we say that A is AS-Gorenstein if ω_A is a left progenerator. It then follows immediately from Definition 6.1.3 that ω_A is an invertible bimodule. Thus

$$R^{-1} = \operatorname{RHom}_A(R, A)$$

has homology in only one degree and

$$R^{-1} \overset{L}{\otimes}_A R = A$$
$$R \overset{L}{\otimes}_A R^{-1} = A$$

in $D(\operatorname{Mod} A^e)$. We can now prove the following.

Proposition 8.4. *Assume that A is AS-Gorenstein. Then*

$$R^{-1} = \operatorname{RHom}_{A^e}(A, A^e)$$

in $D(\operatorname{Mod} A^e)$.

Proof. We have

$$\begin{aligned}\operatorname{RHom}_{A^e}(A, A^e) &= \operatorname{RHom}_{A^e}(A, R \otimes R) \otimes_{A^e} (R^{-1} \otimes R^{-1})\\ &= R \otimes_{A^e} (R^{-1} \otimes R^{-1})\\ &= R^{-1} \quad \square\end{aligned}$$

Remark 8.5. The "AS" in the definitions of AS-Cohen-Macaulay and AS-Gorenstein stands for "Artin-Schelter" since these definitions are generalization of the notion of regular rings as introduced by Artin and Schelter in [2].

There are stronger versions in the literature of the Cohen-Macaulay and Gorenstein property. For these we refer to [10].

Assume that A is a k-algebra, equipped with an ascending filtration $(F_n A)_{n \in \mathbb{N}}$ such that $\operatorname{gr} A$ is connected. In that case we say that the filtration on A is connected. Our aim is to prove the following theorem.

Theorem 8.6. *Assume that A has a connected filtration such that $\operatorname{gr} A$ has a balanced dualizing complex. Then A has a rigid dualizing complex.*

Proof. To work conveniently with filtered rings we recall the formalism of Rees rings [5]. The Rees ring of A is the graded ring

$$\tilde{A} = \bigoplus_i (F_n A) t^n \subset A[t, t^{-1}]$$

We identify t with the element $1 \cdot t$ of $\tilde{A}_1 = (F_1 A)t$. Note that t is a regular central element. We have

$$\tilde{A}/t\tilde{A} = \operatorname{gr} A \tag{8.2}$$

$$\tilde{A}_t = A[t, t^{-1}] \tag{8.3}$$

The fact that the filtration on A is indexed by $\mathbb{N}$ implies that $\tilde{A}$ is positively graded.

We have to verify the hypotheses for Theorem 6.3 for $\tilde{A}$, knowing that they hold for $\operatorname{gr} A$. Luckily for us this follows directly from [1, Thm 8.8].

Hence $\tilde{A}$ has a balanced, and thus rigid dualizing complex given by (6.2) which we will denote by $\tilde{R}$. By construction t acts centrally on R and hence we may define $R = (\tilde{R}_t)_0$ which by (8.3) is a complex of bimodules over A.

We claim that R is in fact a rigid dualizing complex for A. By (8.3) there is a category equivalence between (left or right) A-modules and graded (left or right) $\tilde{A}_t$. A similar statement holds for A-bimodules and graded $\tilde{A}_t$-bimodules with central t-action. Thus we only have to show that $\tilde{R}_t$ is a rigid dualizing complex in the graded sense over $\tilde{A}_t$. Since it is easily seen that all defining properties of a rigid dualizing complex are compatible with localization at t, we are done. □

We immediately deduce the following corollary.

Corollary 8.7. *Assume that A is connected filtered in such a way that $\operatorname{gr} A$ is commutative and finitely generated. Then A has a rigid dualizing complex.*

9. Dualizing complexes for Koszul and Artin-Schelter regular algebras

Let V be a finite dimensional k-vector space and let TV be the tensor algebra of V over k. Suppose that $TV/(R)$ where $R \subset V \otimes V$. Thus A is a quadratic k-algebra. The dual algebra of A, denoted by $A^!$ is the quadratic algebra $TV^*/(R^\perp)$. Let $(x_i)_{1=1,\dots,n}$ be a basis of V and $(\zeta_i)_{i=1,\dots,n}$ be the dual basis of V^*. Then

$$e = \sum_{i=1}^{n} x_i \otimes \zeta_i \in A \otimes A^!$$

has the property $e^2 = 0$.

Right multiplication by e defines a complex

$$A \otimes (A^!_1)^* \to A \otimes (A^!_0)^* \tag{9.1}$$

A is said to be Koszul if (9.1) is a resolution of ${}_A k$. We have the following result.

Theorem 9.1. *Assume that A is a Koszul algebra. Let $C(A) = A \otimes A^! \otimes A$ be graded by $C(A)_m = A \otimes (A^!)_m \otimes A$. Consider $C(A)$ as an A^e bimodule via $(a \otimes b)(c \otimes d \otimes e) = ac \otimes d \otimes eb$ and define $d : C_m(A) \to C(A)_{m+1}$ by*

$$d(c \otimes d \otimes e) = \sum_i c \otimes \zeta_i d \otimes x_i e \pm c x_i \otimes d\zeta_i \otimes e$$

where the sign is $+$ if m is odd, and $-$ otherwise. Then

$$\operatorname{RHom}_{A^e}(A, A^e) = (C(A), d)$$

Proof. The proof is based upon the explicit resolution of A as an A-bimodule, given in [12]. Define $K(A) = A \otimes (A^!)^* \otimes A$, with the usual A-bimodule structure. Put $K(A)_m = A \otimes (A^!)^*_m \otimes A$ and define $d : K(A)_m \to K(A)_{m-1}$ by

$$d(c \otimes d \otimes e) = \sum_i \pm c \otimes \zeta_i d \otimes x_i e + c x_i \otimes d\zeta_i \otimes e$$

where this time the sign is "+" is if m is *even.* Then $(K(A), d)$ is the desired resolution of A as A^e-module.

It is easy to see that the dual of $K(A)$ is $C(A)$. □

Recall that a connected ring of finite global dimension and polynomial growth is said to be Artin-Schelter regular if $\mathrm{Ext}^*(k, A) = k$ [2]. It has been shown by Yekutieli that a twosided noetherian AS-regular algebra has a balanced dualizing complex [13, Cor. 4.14] (note that this also follows very easily from Theorem 6.3). Below we will explicitly compute the dualizing complex for a Koszul AS-regular algebra.

Let A be a twosided noetherian Koszul AS-regular algebra of global dimension n. By [11, Prop 5.10] $A^!$ is Frobenius. This means that as $A^!$-bimodule, $(A^!)^* = A^!_{\phi^!}$ for some automorphism $\phi^!$ of $A^!$. By functoriality $\phi^!$ is obtained by dualizing an automorphism ϕ of A. More precisely: $\phi^!$ restricted to $A^!_1 = V^*$ and ϕ restricted to $A_1 = V$ are dual to each other as automorphisms of vectorspaces.

Theorem 9.2. *Let A be as above and let ϵ be the automorphism of A which is multiplication by $(-1)^m$ on A_m. Then the balanced dualizing complex of A is given by $A_{\phi\epsilon^{n+1}}[n](-n)$.*

Proof. Choose a non-zero element w in $(A^!_n)^*$. Then for $\beta \in A^!$ we have $\beta w = w(-\beta)$, $w\beta = w(\beta-)$. Since $w\beta = \phi^!(\beta)w$, this yields $w(\beta-) = w(-\phi^!(\beta))$. Hence we obtain

$$\beta\alpha = \alpha\phi^!(\beta) \tag{9.2}$$

for $\beta \in A^!_k$, $\alpha \in A^!_{n-k}$, $0 \le k \le n$. Choose a non-zero $u \in A^!_n$ and define the bilinear form $\langle\ ,\ \rangle : A^!_k \otimes A^!_{n-k} \to k$ by

$$\alpha\beta = \langle \alpha, \beta \rangle u$$

Then (9.2) reads as

$$\langle \beta, \alpha \rangle = \langle \alpha, \phi^!(\beta) \rangle$$

By [13, p61, bottom] it follows that A is AS-Gorenstein and that the balanced dualizing complex lives purely in degree n. Thus by Theorem 9.1 it suffices to compute

$$\mathrm{coker}(A \otimes A^!_{n-1} \otimes A \xrightarrow{d} A \otimes A^!_n \otimes A)$$

Since we also know that this cokernel is left (and hence right) free of rank one it suffices to analyze how the generators $(x_i)_i$ of A act on $1 \otimes u \otimes 1$. As above, let $\zeta_i \in A^!_1$ be the basis dual to $(x_i)_i$. Furthermore let $(\zeta_i^*)_i \in A^!_{n-1}$ be the right dual basis to $(\zeta_i)_i$ for $\langle\ ,\ \rangle$. Then we have

$$\begin{aligned} d(1 \otimes \zeta_j^* \otimes 1) &= \sum_i 1 \otimes \zeta_i\zeta_j^* \otimes x_i + (-)^n x_i \otimes \zeta_j^*\zeta_i \otimes 1 \\ &= 1 \otimes u \otimes x_j + (-)^n \sum_i \langle (\phi^!)^{-1}(\zeta_i), \zeta_j^* \rangle x_i \otimes u \otimes 1 \\ &= 1 \otimes u \otimes x_j + (-)^n \phi^{-1}(x_j) \otimes u \otimes 1 \end{aligned}$$

for which we deduce that modulo $A \otimes A^!_{n-1} \otimes A$

$$(1 \otimes u \otimes 1)x_j = (-)^{n+1}\phi^{-1}(x_j)(1 \otimes u \otimes 1)$$

and thus

$$\mathrm{RHom}_{A^e}(A, A^e) = A_{\epsilon^{n+1}\phi^{-1}}[-n](n)$$

Now Proposition 8.4 yields what we want. □

Let us briefly recall how to apply this result to three dimensional AS-regular algebras. Those were the main study objects of [2].

We restrict ourselves to the Koszul case. In that case the algebras are generated by three generators x_1, x_2, x_3 and have three relations f_1, f_2, f_3. Let x, f stand for the corresponding column vectors. x and f may be chosen is such a way that there exists a linear 3×3-matrix M such that $f = Mx$, $x^t M = (Qf)^t$ for some invertible 3×3 matrix Q with scalar entries. Then we have:

Corollary 9.3. *Let A be as above and let ϕ be the automorphism of A given by $x \mapsto Q^{-t}x$. Then the balanced dualizing complex of A is given by $A_\phi[3](-3)$.*

Proof. As usual (ζ_i) is the dual basis of (x_i). By the previous theorem we have to compute $A^!$. It is well known how this should be done. Define $C = k \oplus V \oplus R \oplus W$ with $W = V \otimes R \cap R \otimes V$. Then C is a graded subcoalgebra of TV and $C^* = A^!$. We have to describe the bimodule structure on $(A^!)^* = C$. Put $w = x^t f = f^t Q^t x$. This a generator of W. Then a simple verification shows that $\zeta_i w = (1 \otimes \zeta_i)(w) = ((f^t Q^t)_i = (Qf)_i$, $w\zeta_i = (\zeta_i \otimes 1)(w) = f_i$. Thus $(A^!)^* = A^!_\psi$ where ψ is given in matrix form by $y \mapsto Q^{-1}y$. Then the automorphism ϕ of A, dual to ψ has matrix form $x \mapsto Q^{-t}x$. Since $n + 1 = 3 + 1 = 4$ is even we are done. □

The previous corollary has an interesting consequence if we combine it with [13, Theorem 7.18]. If $\lambda \in k$ then we denote by ϕ_λ the automorphism of A which is multiplication by λ^n on A_n.

Corollary 9.4. *Assume that A is a three generator three dimensional regular algebra associated to the triple $(E, \sigma, \mathcal{L})$ with E smooth (see [3, 4]). Let $g \in A_3$ be the canonical normalizing element. Let ϕ_g be the automorphism of A given by $g \cdot g^{-1}$. Let λ be the eigenvalue of σ^*, acting on $\Gamma(E, \omega_E)$ and let ϕ be the automorphism of A given by $x \mapsto Q^{-t}x$. Then*

$$\phi = \phi_g \phi_\lambda$$

Briefly, this says that Q^{-t} and the matrix determined by conjugation by g differ only by the scalar λ.

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Limburgs Universitair Centrum, Departement WNI, Universitaire Campus, 3590 Diepenbeek, Belgium

E-mail address: `vdbergh@luc.ac.be, http://www.luc.ac.be/Research/Algebra`