SOME RINGS OF DIFFERENTIAL OPERATORS FOR Sl_2-INVARIANTS ARE SIMPLE

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Abstract. It has been conjectured that the ring of differential operators of the algebraic quotient of a connected smooth affine variety under a reductive group action is simple. This is known in the case that the group in question is the extension of a finite group with a torus and in the case of classical representation of classical groups. In these notes we present some tools relevant to this conjecture. In particular we show that it is true for some representations of Sl_2.

1. Introduction

For an affine algebraic variety Z over $\mathbb{C}$ let $\mathcal{D}(Z)$ be the ring of differential operators on $\mathcal{O}_Z$. $\mathcal{D}(Z)$ has a natural filtration by order of differential operators.

Let X be an affine smooth variety over $\mathbb{C}$ on which a reductive group G acts. Work of Kantor, Levasseur, Musson, Schwarz, Stafford and the author suggests the following conjecture

Conjecture 1.1. **(f)** $\operatorname{gr}\mathcal{D}(X/\!\!/G)$ is finitely generated.
(s) If X is connected then $\mathcal{D}(X/\!\!/G)$ is a simple ring.

The simplicity part of this conjecture is important for invariant theory since it implies the Hochster-Roberts theorem [25].

Conjecture 1.1 is known to be true in the case of tori, in the case of finite groups [13][17][18][19][25] and in the case of classical representations of classical groups [14].

In [19] there are powerful results concerning the finite generation part of the conjecture. The basic device is a natural map $\mathcal{D}(X)^G/(\mathcal{D}(X)\mathfrak{g})^G \to \mathcal{D}(X/\!\!/G)$ where $\mathfrak{g} = \mathrm{Lie}(G)$. The (filtered) surjectivity of this map implies 1.1(f). It turns out that the properties of this map are heavily influenced by the properties of the moment map and also by the nature of the slice representations of (G, X).

In these notes we introduce a few tools which might be useful to study the simplicity part of conjecture 1.1. In particular we prove the following results :

Theorem A. Let X, G be as above and assume that X is connected. Then $\mathcal{D}(X/\!\!/G)$ is simple if for all $x \in X$ with closed orbit and with corresponding slice representation (H, N) one has that $\mathcal{D}(N/\!\!/H)$ is simple.

Since X is smooth, it is well-known that $\operatorname{gr}\mathcal{D}_X = \mathcal{O}(T^*X)$ where T^*X is the cotangent bundle on X. Since $\mathcal{O}_X$ is G-equivariant, $\mathfrak{g}$ acts by derivations on $\mathcal{O}_X$ and hence $\mathfrak{g} \subset \mathrm{Der}(\mathcal{O}_X) \subset \mathcal{O}(T^*X)$. Denote by $V(\mathfrak{g})$ the common zeroes of $\mathfrak{g}$.

Theorem B. Assume that X is connected and let $(T^*X)'$ be a saturated open subset of T^*X on which G acts freely. Let $I \subset \mathcal{D}(X)^G/(\mathcal{D}(X)\mathfrak{g})^G$ be a non-zero two-sided ideal with inverse image $J \subset \mathcal{D}(X)^G$ and put $K = \mathcal{D}(X)J$ with the induced filtration. Then

$$V(\operatorname{gr} K) \cap (T^*X)' = \emptyset$$

As an application of these results we consider the case $G = \mathrm{Sl}_2$. For this group 1.1(f) has been proved by Schwarz [19], so we concentrate on simplicity. We can prove the following

1991 *Mathematics Subject Classification.* Primary 16S32,32C38; Secondary 14L30.

Key words and phrases. Rings of differential operators, simple rings, slice representations, Marsden Weinstein reduction.

Theorem C. Assume that $G = \mathrm{Sl}_2$ and $W = \oplus_{i=1}^{e} S^{d_i}\mathbb{C}^2$ where all d_i are odd and $e \geq 2$. Then $\mathcal{D}(W/\!\!/G)$ is simple.

To prove Theorem C we use Theorems A and B to reduce ourselves to the claim that $\mathcal{D}(W/\!\!/G)$ has no finite dimensional representations.

To prove this we have to study holonomic modules on W whose characteristic variety is contained in the unstable part of T^*W. There we have the following :

Theorem D. Let $G = \mathrm{Sl}_2$ and $W = \oplus_{i=1}^{e} S^{d_1}\mathbb{C}^2$ where all d_i are odd and $e \geq 2$. Let $\mathcal{M}$ be in $(G, \mathcal{D}_W)$-qch such that $\mathcal{M}$ is holonomic and $\mathrm{Ch}(\mathcal{M}) \subset (T^*W)^u$. Then $\mathcal{M} \cong \mathcal{L}(W^u, W)^{\oplus r}$ in $(G, \mathcal{D}_W)$-qch.

For the undefined notations in the statement of Theorem D we refer to §2 and §6. We will suffice by saying that W^u is the set of points of W having zero in the closure of their orbit. $(T^*W)^u$ is defined similarly. $\mathcal{L}(W^u, W)$ is the holonomic module on W whose the De Rham complex is the intersection homology complex on W^u, suitably shifted.

Now we give an outline of this paper, section by section.

In section 2 we give some preliminaries concerning differential operators.

In section 3 we show that simplicity of differential operators is, to a certain extent, a local property. This section is more elaborate than what we need afterwards, since we work in the general setting of differential operators on filtered rings.

In section 4 we prove a slightly extended version of Theorem A, which applies also to covariants.

In section 5 we recall some elementary notions from symplectic geometry and we introduce the algebraic Marsden-Weinstein reduction. This is used to prove Theorem B. The material in this section should really be extended to covariants, but time constraints have prevented me from doing so.

In section 6 we give some basic facts concerning G-equivariant quasi-coherent $\mathcal{O}_X$ and $\mathcal{D}_X$-modules. The difference between quasi-coherent and weakly quasi-coherent $(G, \mathcal{D}_X)$-modules is illustrated [4]. All of this is well-known and elementary but it seems to be very hard to find references in the literature. This is aggrevated by the fact that for the sequel we are forced to work with non-affine varieties. In order not to make this section explode, we give very few proofs. Detailed proofs will appear in [24].

In section 7 we prove Theorem D. To do this we make use of the formalism of inverse and direct images for $\mathcal{D}$-modules. This allows us to use an analogous result for the torus case which will be proved in [18].

In section 8 we characterize the finite-dimensional representations for some rings of differential operators for Sl_2-covariants.

Finally in section 9 we prove Theorem C.

2. Preliminaries on differential operators

Let R be a commutative $\mathbb{C}$-algebra and let M, N be R-modules. Then one defines $D_R^n(M, N) \subset \mathrm{Hom}_{\mathbb{C}}(M, N)$ inductively as follows $D_R^n(M, N) = 0$ for $n < 0$ and

$$D_R^n(M, N) = \{f \in \mathrm{Hom}_R(M, N) \mid \forall a \in R : [a, f] \in D_R^{n-1}(M, N)\}$$

In the sequel we use some standard simplified notations :

$$D_R^n(M) = D^n(M, M), \qquad D^n(R) = D_R^n(R) \tag{2.1}$$

Furthermore we put

$$D_R(M, N) = \bigcup D_R^n(M, N) \tag{2.2}$$

and for $D_R(M, N)$ we use similar conventions as in (2.1).

Putting $F_n D_R(M) = D_R^n(M)$ makes $D_R(M)$ into a filtered ring. Furthermore composition makes $D_R(M, N)$ into a $D_R(N) - D_R(M)$-bimodule. A special property in the case $M = R$ is that $\mathrm{gr}\, D(R)$ is commutative.

Let $R^\circ = R \otimes_{\mathbb{C}} R$, $M^\circ = R \otimes_{\mathbb{C}} M$, $I = \ker(R^\circ \to R)$, $P^n_{R,M} = M^\circ / I^{n+1} M^\circ$. $P^n_{R,M}$ is an R-module, but we will consider it as a left R-module via the embedding $R \to R^\circ : r \mapsto r \otimes 1$. Then [19]

$$D^n_R(M,N) = \operatorname{Hom}_R(P^n_{R,M}, N) \tag{2.3}$$

Now we introduce a few notions from filtered ring theory. Let $A := (F_n A)_{n\in\mathbb{Z}}$ be a filtered ring such that $\operatorname{gr} A$ is commutative. Then $\operatorname{gr} A$ is equipped with a Poisson bracket coming from the commutator on A. We recall the following definition

Definition 2.1. Let B be a commutative ring with a Poisson bracket $\{\ ,\}$. Let $J \subset B$ be an ideal. Then

(1) if $\{J,J\} \subset J$ then J is called involutive;
(2) if $\{B,J\} \subset J$ then J is called a Poissson ideal.

It is easy to see that if $I \subset A$ is a two-sided ideal equipped with the induced filtration then $\operatorname{gr} I$ is a Poisson ideal in $\operatorname{gr} A$. Furthermore, $\operatorname{rad}(\operatorname{gr} I)$ is also a Poisson ideal since $\{f,-\}$ is a derivation and therefore is compatible with taking radicals [9, lemma 2.20].

There is a similar result for left modules which is much deeper. Assume that $\tilde{A} = \oplus_{i\in\mathbb{Z}} F_n A$ is noetherian and let M be a finitely generated A-module. Then it is classical that M is equipped with a "good" filtration (that is a filtration such that $\tilde{M} = \oplus F_n M$ is a finitely generated $\tilde{A}$-module).

Then $\operatorname{ICh}(M) = \operatorname{rad}(\operatorname{Ann}_{\operatorname{gr} A}(\operatorname{gr} M))$ is a graded ideal in $\operatorname{gr} A$ which does not depend upon the choice of the good filtration. One has the following

Theorem 2.2. [8] Let A be as above and assume that A contains $\mathbb{Q}$. Then $\operatorname{ICh}(M)$ is involutive.

The notions connected with differential operators sheaffify well. Let $X/\mathbb{C}$ be a scheme of finite type and let $\mathcal{F}, \mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Then one defines for an affine open $U \subset X$

$$\mathcal{D}^n_X(\mathcal{F},\mathcal{G})(U) = \mathcal{D}^n_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))$$

This defines a coherent $\mathcal{O}_X$-module $\mathcal{D}^n_X(\mathcal{F},\mathcal{G})$ and we use similar abreviations an in (2.1)(2.2). We also put $\mathcal{D}^n(X,\mathcal{F},\mathcal{G}) = \mathcal{D}^n_X(\mathcal{F},\mathcal{G})(X)$, etc ... .

Assume now that X is smooth. Then $\operatorname{gr} \mathcal{D}_X = \mathcal{O}(T^*X)$ where T^*X is the cotangent bundle on X. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module then $\operatorname{Ch}(\mathcal{M}) \subset T^*X$ is the subvariety which is locally defined by $\operatorname{ICh}(\mathcal{M}(U))$. Theorem 2.2 now implies that $\operatorname{Ch}(\mathcal{M})$ is involutive for the canonical symplectic form on T^*X (in local coordinates : $\sum_i dx_i \wedge d\frac{\partial}{\partial x_i}$).

Denote by $\mathcal{D}_X$-qch the category of quasi-coherent $\mathcal{D}_X$ modules and by $D^b(\mathcal{D}_X\text{-qch})$ the associated derived category. We will identify $\mathcal{D}_X$-qch with its essential image in $D^b(\mathcal{D}_X\text{-qch})$.

If $\pi : X \to Y$ is a map between smooth schemes then there is a formalism of direct and inverse images $\pi_!, \pi_+, \pi^!, \pi^+$ between $D^b(\mathcal{D}_X\text{-qch})$ and $D^b(\mathcal{D}_Y\text{-qch})$ for which we refer the reader to [3].

Assume now that Y is a closed subset of X, that X is smooth and that X and Y are irreducible. An important object is $\mathcal{L}(Y,X)$ which is the holonomic $\mathcal{D}_X$-module whose De Rham complex is the intersection homology complex on Y (up to shift) [7]. $\mathcal{L}(Y,X)$ is the unique simple quasi-coherent submodule of $\mathcal{H}^{\operatorname{codim}(Y,X)}_Y(X,\mathcal{O}_X)$ whose support is Y.

An alternative way of obtaining $\mathcal{L}(Y,X)$ is via the intermediate extension functor [3]. Let $U \subset X$ be open such that $Y \cap U$ is non-empty and smooth. Let $s : U \to X$ be the inclusion map. Then

$$\begin{aligned}\mathcal{L}(Y,X) &= s_{!+}\mathcal{H}^{\operatorname{codim}(Y,X)}_{U\cap Y}(U,\mathcal{O}_U)\\ &= \operatorname{im}(\mathcal{H}^0(s_!\mathcal{H}^{\operatorname{codim}(Y,X)}_{U\cap Y}(U,\mathcal{O}_U)) \to s_+\mathcal{H}^0(\mathcal{H}^{\operatorname{codim}(Y,X)}_{U\cap Y}(U,\mathcal{O}_U)))\end{aligned}$$

Below we give two simple lemmas which apply to differential operators, and which we will need.

Lemma 2.3. Let A be a $\mathbb{Z}$-graded $\mathbb{Q}$-algebra and let $\pi \in A_0$ be such that $[\pi,-]$ is multiplication by n on A_n. Then every two-sided ideal in A is graded.

This applies to differential operators in the following way. Let R be a finitely generated commutative $\mathbb{Q}$-algebra and let M be a finitely generated graded R-module. Then $D_R(M)$ is graded. Let $\pi : M \to M$ be the Euler derivation which acts by multiplication by n on M_n. Then π satisfies the hypothesis of the lemma and hence every two-sided ideal in $D_R(M)$ is graded.

Note the Euler derivation is a canonical object which in particular is is compatible with graded group actions on (R, M).

Lemma 2.4. [5] Let A be a ring and let $S \subset A$ be a multiplicative set such that for all $s \in S$ the derivation $[s, -]$ is locally nilpotent. Then S is an Öre set. Furthermore if I is a two-sided ideal in A then I_S is a two-sided ideal in A_S.

This lemma allows one to localize two-sided ideals in $D_R(M)$ at multiplicative sets in R.

3. Differential operators on filtered rings

In this section we prove the following results.

Theorem 3.1. Let R be a connected finitely generated commutative $\mathbb{C}$-algebra and let M be a finitely generated R-module. For a maximal ideal $m \subset R$ denote by $\hat{}$ the m-adic completion. Then $D_R(M)$ is simple if for all maximal ideals m one has that $D_{\hat{R}_m}(\hat{M}_m)$ is simple.

Theorem 3.2. Let R be a finitely generated graded commutative $\mathbb{C}$-algebra of the form $\mathbb{C} \oplus R_1 \oplus R_2 \oplus \cdots$ and let M be a finitely generated graded R-module. Let $\hat{}$ denote completion for the $R_{>0}$-adic filtration on R. Then $D_R(M)$ is simple if and only if $D_{\hat{R}}(\hat{M})$ is simple.

Remark 3.3. It seems quite plausible that the converse to Theorem 3.1 is true, but I am unable to prove it.

To prove Theorems 3.1 and 3.2 we will work in slightly greater generality, in the hope that the results obtained may be useful elsewhere. For related material from a different, but more general, view point, see [26]

We need some notions from filtered ring theory [1].

Let R be a ring equipped with an exhaustive ascending filtration $(F_nR)_{n\in\mathbb{Z}}$. The Rees ring $\tilde{R}$ of R is the graded ring $\oplus_{n\in\mathbb{Z}} F_nR$.

Let $X \in \tilde{R}_1 = F_1R \subset F_0R$ be the element corresponding to $1 \in F_0R$. Then $\tilde{R}/X\tilde{R} = \operatorname{gr} R = \oplus F_nR/F_{n-1}R$ is the associated graded ring of R. Furthermore $\tilde{R}_X = R[X, X^{-1}]$ and hence we recover R as $(\tilde{R}_X)_0$.

The completion $\hat{R}$ is the (filtered) ring $\varprojlim R/F_nR$. One has $\tilde{\hat{R}} = \hat{\tilde{R}}$ where on the right one takes *graded* completion for the (X)-adic topology on $\tilde{R}$.

Similar definitions and properties apply to any filtered R-module $M := (F_nM)_n$. We recall that a filtration on M is said to be good if $\tilde{M}$ is a finitely generated $\tilde{R}$-module.

We need the following general results.

Proposition 3.4. Suppose that R is a filtered ring with $\tilde{R}$ noetherian and let M be a filtered R-module with a good filtration. Then $\hat{R}$ is a flat R-module and $\hat{M} = \hat{R} \otimes_R M$.

Proof. See for example [1] $\square$

Proposition 3.5. Suppose that R is a filtered commutative ring with $\tilde{R}$ noetherian, and let M, N be filtered R-modules with a good filtration. Define a filtration on $\operatorname{Hom}_R(M, N)$ by

$$F_k \operatorname{Hom}_R(M, N) = \{f \in \operatorname{Hom}_R(M, N) \mid \forall l \in \mathbb{Z} : f(F_lM) \subset F_{k+l}N\}$$

Then

(1) this filtration is exhaustive;
(2) this filtration is good;
(3) the canonical map

$$\operatorname{Hom}_R(M, N)\hat{} \to \operatorname{Hom}_{\hat{R}}(\hat{M}, \hat{N})$$

is a filtered isomorphism.

Proof. Clearly

$$\begin{aligned}\operatorname{Hom}_R(M,N)^{\sim} &= \oplus_k \operatorname{Hom}_{\tilde{R}\text{-gr}}(\tilde{M},\tilde{N}(k))\\ &= \operatorname{Hom}_{\tilde{R}}(\tilde{M},\tilde{N})\end{aligned}$$

The last equality holds because $\tilde{M}$ is finitely generated and hence $\operatorname{Hom}_{\tilde{R}}(\tilde{M},\tilde{N})$ is graded.

The fact that $\tilde{M}$, $\tilde{N}$ are finitely generated and that $\tilde{R}$ is noetherian implies that $\operatorname{Hom}_{\tilde{R}}(\tilde{M},\tilde{N})$ is finitely generated. This proves (2).

Furthermore, again using that $\tilde{M}$ is finitely generated, one has

$$\begin{aligned}\operatorname{Hom}_{\tilde{R}}(\tilde{M},\tilde{N})_{X,0} &= \operatorname{Hom}_{R[X,X^{-1}]}(M[X,X^{-1}],N[X,X^{-1}])_0\\ &= \operatorname{Hom}_R(M,N)\end{aligned}$$

This proves (1).

For (3) we have to show that

$$\operatorname{Hom}_R(M,N)^{\wedge\sim} \to \operatorname{Hom}_{\hat{R}}(\hat{M},\hat{N})^{\sim}$$

is a graded isomorphism.

On the lefthand side we have

$$\begin{aligned}\operatorname{Hom}_R(M,N)^{\wedge\sim} &= \operatorname{Hom}_R(M,N)^{\sim\wedge}\\ &= \operatorname{Hom}_{\tilde{R}}(\tilde{M},\tilde{N})^{\wedge}\end{aligned}$$

Furthermore $\hat{R},\hat{M},\hat{N}$ satisfy the same hypothese as R,M,N and hence

$$\begin{aligned}\operatorname{Hom}_{\hat{R}}(\hat{M},\hat{N})^{\sim} &= \operatorname{Hom}_{\tilde{\hat{R}}}(\tilde{\hat{M}},\tilde{\hat{N}})\\ &= \operatorname{Hom}_{\hat{\tilde{R}}}(\hat{\tilde{M}},\hat{\tilde{N}})\end{aligned}$$

The result now follows by replacing $\tilde{M}$ with a finite free graded resolution of length 2, and using the fact that (graded) X-adic completion is exact on finitely generated (graded) modules since $\tilde{R}$ is noetherian. □

From now on we assume that R is a commutative filtered $\mathbb{C}$-algebra and that M, N are filtered R-modules. We always assume that $D^n_R(M,N)$ is equipped with the following standard filtration

$$F_k D^n_R(M,N) = \{f \in D^n_R(M,N) \mid \forall l \in \mathbb{Z} : f(F_k M) \subset F_{k+l} N\} \tag{3.1}$$

There is a corresponding filtration on $D_R(M,N)$.

If the filtration on N is separating then this is also the case for the filtration on $D^n_R(M,N)$. Note however that in general there is no reason for the filtration on $D^n_R(M,N)$ to be exhaustive.

By $D^{n,\mathrm{cont}}_R(M,N)$ we denote the differential operators of order $\le n$ that are continuous with respect to the topologies defined by the filtrations. That is

$$D^{n,\mathrm{cont}}_R(M,N) = \{f \in D^n_R(M,N) \mid \forall l \in \mathbb{Z}, \exists k \in \mathbb{Z} : f(F_k M) \subset F_l N\}$$

We use the standard abreviations $D^{\mathrm{cont}}_R(M,N) = \bigcup_n D^{n,\mathrm{cont}}_R(M,N)$, etc ....

Note in particular that

$$\bigcup_k F_k D^n_R(M,N) \subset D^{n,\mathrm{cont}}_R(M,N)$$

Recall that by (2.3) $D^n_R(M,N) = \operatorname{Hom}_R(P^n_{R,M},N)$ Now $P^n_{R,M}$, being a quotient of $R\otimes_{\mathbb{C}} M$, is equipped with a canonical induced filtration. Imitating [19] we obtain

$$D^{n,\mathrm{cont}}_R(M,N) = \operatorname{Hom}^{\mathrm{cont}}_R(P^n_{R,M},N) \tag{3.2}$$

$$F_k D^n_R(M,N) = F_k \operatorname{Hom}_R(P^n_{R,M},N) \tag{3.3}$$

Proposition 3.6. Assume that $\tilde{R}$ is finitely generated and that the filtration on M is good. Then the canonical filtration on $P^n_{R,M}$ is good.

Proof. Clearly $P^n_{\tilde R,\tilde M}$ is finitely generated and one verifies that the surjective mp $\tilde R \otimes_{\mathbb{C}} \tilde M \to (R \otimes_{\mathbb{C}} M)^\sim$ factors through a surjective map $P^n_{\tilde R,\tilde M} \to \tilde P^n_{R,M}$. Therefore $\tilde P^n_{R,M}$ if finitely generated. $\square$

Corollary 3.7. Assume that $\tilde R$ is finitely generated and that the filtrations om M and N are good. Then the standard filtration (3.1) on $D^n_R(M,N)$ is exhaustive and good. In particular $D^{n,\mathrm{cont}}_R(M,N) = D^n_R(M,N)$.

Proof. By (3.3) and by Proposition 3.6 this reduces to Proposition 3.5(1) and 3.5(2). $\square$

For the I-adic filtration for an ideal $I \subset R$ one has a similar result.

Proposition 3.8. Assume that the filtration on R is the I-adic filtration for an ideal $I \subset R$. That is

$$F_k R = \begin{cases} I^{-k} & \text{if } k < 0 \\ R & \text{if } k \geq 0 \end{cases}$$

Assume furthermore that the filtrations on M and N are cofinal for the induced I-adic filtration. Then the standard filtration (3.1) on $D^n_R(M,N)$ is exhaustive.

Proof. We may as well assume that the filtration on M, N is the induced I-adic filtration. Let $D \in D^n_R(M,N)$. Then one proves by induction on k, n that $D(I^k M) \subset I^{k-n} N$. $\square$

Remark 3.9. If the filtration is not the I-adic filtration and if $\tilde R$ is not finitely generated then in in general $D^{n,\mathrm{cont}}(M,N) \neq D^n(M,N)$. This occurs even when $\tilde R$ is noetherian. For example take $R = M = N$, $R = \mathbb{C}((X))$ and let R be filtered by $F_k R = X^{-k}\mathbb{C}[[X]]$. Then $\mathrm{Der}^{\mathrm{cont}}_{\mathbb{C}}(R) = R\frac{\partial}{\partial X}$ but $\mathrm{Der}_{\mathbb{C}}(R)$ is of infinite rank over R since R is a field of infinite trancedence degree.

Lemma 3.10. The canonical map $P^n_{R,M} \to P^n_{\hat R,\hat M}$ becomes a filtered isomorphism after completing.

Proof. Let $I' = \ker(\hat R \otimes_{\mathbb{C}} \hat R \to \hat R)$. We look at the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I'^{n+1}\hat M^\circ & \longrightarrow & \hat R \otimes_{\mathbb{C}} \hat M & \longrightarrow & P_{\hat R,\hat M} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I^{n+1}M^\circ & \longrightarrow & R \otimes_{\mathbb{C}} M & \longrightarrow & P_{R,M} & \longrightarrow & 0 \end{array} \tag{3.4}$$

If we give $R \otimes_{\mathbb{C}} M$ and $\hat R \otimes_{\mathbb{C}} \hat M$ the canonical filtrations and $I^{n+1}M^\circ$, $I'^{n+1}\hat M^\circ$ the induced filtrations then the exact sequences in (3.4) are strict exact and hence they remain exact after completing.

Since $(R \otimes_{\mathbb{C}} M)\hat{} \to (\hat R \otimes_{\mathbb{C}} \hat M)\hat{}$ is a filtered isomorphism, it suffices to show that the image of $I^{n+1}M^\circ$ is dense in $I'^{n+1}\hat M^\circ$.

We first show that the image of I is dense in I'. Hence we look at

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I' & \longrightarrow & \hat R \otimes_{\mathbb{C}} \hat R & \longrightarrow & \hat R & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & R \otimes_{\mathbb{C}} R & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

where we give I, I' the induced filtrations. Completing the two rows in this diagram yields an isomorphism $\hat I = \hat I'$ and hence I is dense in I'.

A simple verification shows that for the above filtrations on $I^{n+1}M^\circ$ and $I'^{n+1}\hat M^\circ$ one has

$$(F_{i_1}I)\cdots(F_{i_{n+1}}I)F_j\hat M^\circ \subset F_{i_1+\cdots+i_{n+1}+j}(I^{n+1}\hat M^\circ)$$

A standard continuity argument now implies that the image of $I^{n+1}M^\circ$ is dense in $I'^{n+1}\hat M^\circ$. $\square$

Corollary 3.11. Assume that $\tilde R$ is finitely generated and that the filtrations on M and N are good. Let $D^n_R(M,N)$ and $D^{n,\mathrm{cont}}_R(M,N)$ be equipped with the standard filtration (3.1). Then there is a filtered isomorphism between $D^n_R(M,N)\hat{}$ and $D^{n,\mathrm{cont}}_{\hat R}(\hat M,\hat N)$. In particular the standard filtration on $D^{n,\mathrm{cont}}_R(M,N)\hat{}$ is exhaustive.

Proof. We have

$$\begin{aligned} D^{n,\mathrm{cont}}(M,N) &= \mathrm{Hom}^{\mathrm{cont}}_{\hat{R}}(P^n_{\hat{R},\hat{M}},\hat{N}) && \text{by (3.2)} \\ &= \mathrm{Hom}^{\mathrm{cont}}_{\hat{R}}(\hat{P}^n_{\hat{R},\hat{M}},\hat{N}) && \text{because } \hat{N} \text{ is complete} \\ &= \mathrm{Hom}^{\mathrm{cont}}_{\hat{R}}(\hat{P}^n_{R,M},\hat{N}) && \text{by lemma 3.10} \end{aligned}$$

Now by Proposition 3.6, $P_{R,M}$ carries a good filtration, and hence the same is true for its completion $\hat{P}_{R,M}$. Therefore

$$\begin{aligned} \mathrm{Hom}^{\mathrm{cont}}_{\hat{R}}(\hat{P}^n_{R,M},\hat{N}) &= \mathrm{Hom}_{\hat{R}}(\hat{P}^n_{R,M},\hat{N}) && \text{by Prop. 3.5(1)} \\ &= \mathrm{Hom}_{R}(P^n_{R,M},N)^\wedge && \text{by Prop. 3.5(3)} \\ &= D^n_R(M,N)^\wedge \quad \square \end{aligned}$$

Assume now that $\tilde{R}$ is finitely generated and that M,N have good filtrations. Then corollary 3.11 together with Proposition 3.4 gives us an isomorphism between $D^{n,\mathrm{cont}}_{\hat{R}}(\hat{M},\hat{N})$ and $\hat{R}\otimes D^n_R(M,N)$. Taking direct limits this extends to an isomorphism between $D^{\mathrm{cont}}_{\hat{R}}(\hat{M},\hat{N})$ and $\hat{R}\otimes_R D_R(M,N)$.

This leads us to the following lemma :

Lemma 3.12. Assume that $\tilde{R}$ is finitely generated and that the filtration on M is good. If I is a two-sided ideal in $D_R(M)$ then under the above isomorphism, $\hat{R}\otimes_R I$ defines a two-sided ideal in $D^{\mathrm{cont}}_{\hat{R}}(\hat{M})$.

Proof. Let $I_n = I\cap D^n_R(M)$. Then by the fact that $\tilde{R}$ is noetherian and by corollary 3.7, the induced filtration on I_n is good. Hence we have by Proposition 3.4

$$\hat{R}\otimes_R I = \varinjlim \hat{R}\otimes_R I_n = \lim \hat{I}_n \tag{3.5}$$

and it is easily seen that the right hand side of (3.5) is a two-sided ideal in $\varinjlim D^n_R(M)^\wedge = D^{\mathrm{cont}}_{\hat{R}}(\hat{M})$. $\square$

Proof of Theorem 3.1. Clearly if $m\subset R$ is a maximal ideal then for the m-adic filtration one has that $\tilde{R}$ is finitely generated and that the induced filtration on M is good. So the above theory applies.

Let $I\subset D_R(M)$ be a non-trivial two-sided ideal. Then there exists some n such that $I_n = I\cap D^n_R(M)$ is neither zero, nor equal to $D^n_R(M)$.

$D^n_R(M)$ contains R and hence we have

$$0 = \mathrm{Ann}_R\, D^n_R(M) \supset \mathrm{Ann}_R\, I_n \cdot \mathrm{Ann}_R(D^n_R(M)/I_n)$$

By hypotheses R is connected. This implies that

$$\mathrm{Ann}_R\, I_n + \mathrm{Ann}_R(D^n_R(M)/I_n) \neq R$$

Hence choose a maximal ideal $m \supset \mathrm{Ann}_R\, I_n + \mathrm{Ann}_R(D^n_R(M)/I_n)$. Then for this m one has $\hat{R}_m\otimes_R I\neq 0$ and $\hat{R}_m\otimes_R (D_R(M)/I)\neq 0$. This yields a contradiction with lemma 3.12 which asserts that $\hat{R}_m\otimes_R I$ defines a two-sided ideal in $D^{\mathrm{cont}}_{\hat{R}_m}(\hat{M}_m) = D_{\hat{R}_m}(\hat{M}_m)$ (by Prop. 3.8). $\square$

Now we discuss the following situation. Assume that R is a graded ring, finitely generated over $\mathbb{C}$ and let M,N be finitely generated R-modules. We consider R as a filtered ring via

$$F_k R = \oplus_{l\geq k} R_l \tag{3.6}$$

M,N are considered as filtered R-modules in a similar way. Note that $\tilde{R} = R[X]$, etc .... In particular $\tilde{R}$ is finitely generated and the filtrations on $\tilde{M}$ and $\tilde{N}$ are good, so the above theory applies.

It follows easily from the hypotheses that $D^n(M,N)$ is graded and hence

$$\mathrm{gr}\, D^n_R(M,N) = D^n_R(M,N) \tag{3.7}$$

From this one deduces the following

Proposition 3.13. Let R, M, N be as above. Then

$$\operatorname{gr} D^{n,\mathrm{cont}}_{\hat{R}}(\hat{M}, \hat{N}) = D^n_R(M, N)$$

Proof. We have

$$\begin{aligned} \operatorname{gr} D^{n,\mathrm{cont}}_{\hat{R}}(\hat{M}, \hat{N}) &= \operatorname{gr} D^n_R(M, N)\hat{} && \text{by cor. 3.11} \\ &= \operatorname{gr} D^n_R(M, N) \\ &= D^n_R(M, N) && \text{by (3.7)} \quad \square \end{aligned}$$

Proof of Theorem 3.2. The $R_{>0}$-adic filtration is cofinal with the filtration introduced by (3.6), hence we may use the latter. In particular by Proposition 3.8 one has $D^{\mathrm{cont}}_{\hat{R}}(\hat{M}) = D_{\hat{R}}(\hat{M})$.

By Proposition 3.13 we have that $\operatorname{gr} D_{\hat{R}}(\hat{M}) = D_R(M)$ and hence if $D_R(M)$ is simple then $D_{\hat{R}}(\hat{M})$ is simple. Note that this conclusion uses the fact that the filtration (3.1) on $D_{\hat{R}}(\hat{M})$ is separating, which follows easily from the definition.

Conversely assume that $D_{\hat{R}}(\hat{M})$ is simple and let I be a two-sided ideal in $D_R(M)$. By lemma 2.3 one knows that I is graded.

Furthermore $\hat{R} \otimes_R I$ defines a two-sided ideal in $\hat{R} \otimes_R D_R(M) = D_{\hat{R}}(\hat{M})$ and hence, since $D_{\hat{R}}(\hat{M})$ is simple by hypothesis one either has $\hat{R} \otimes_R I = 0$ or $\hat{R} \otimes_R I = D_{\hat{R}}(\hat{M}) = \hat{R} \otimes_R D_R(M)$.

Now $\hat{R}$ is faithfully flat for graded modules and hence $I = 0$ or $I = D_R(M)$. $\square$

4. Simplicity of rings of differential operators and slice representations

In this section we follow some of the conventions and notations of [19, §1,2,4]. In particular X will be a smooth connected affine variety over $\mathbb{C}$ and G will be a reductive group acting on X. $\mathcal{E}$ will be a G-vector bundle on X (the naive definition of this notion suffices, for a more complete discussion see §6).

Let $x \in X$ be such that Gx is closed. Let H be the stabilizer of x. Then the tangent space $T_x(Gx)$ is isomorphic to $\mathfrak{g}/\mathfrak{h}$ where $\mathfrak{g}, \mathfrak{h}$ are the Lie algebras of G, H. Since H is reductive there is a H-decomposition $T_x(X) = N \oplus \mathfrak{g}/\mathfrak{h}$. The representation (H, N) is called the slice representation at x.

Of course H also acts on the fiber $\mathcal{E}_x$, which gives another H-representation associated to the point x. We denote by $\mathcal{E}_{x,N}$ the corresponding trivial H-vectorbundle on N.

Let $\mathcal{E}^G$ be the coherent sheaf on $\mathcal{O}_{X/\!\!/G}$ with global sections $H^0(X, \mathcal{E})^G$.

Theorem A'. Let $X, G, \mathcal{E}$ be as above. Then $\mathcal{D}(X/\!\!/G, \mathcal{E}^G)$ is simple if and only if for all $x \in X$ with closed orbit and with corresponding slice representation (H, N) one has that $\mathcal{D}(N/\!\!/H, \mathcal{E}^H_{x,N})$ is simple.

Proof. Let $x \in X$ be such that Gx is closed. By the Luna slice theorem [15] there exists a H-invariant "slice" S to Gx containing x such that there are etale maps

$$N/\!\!/H \xleftarrow{\beta} S/\!\!/H \xrightarrow{\alpha} X/\!\!/G$$

where α is induced from the inclusion $S \hookrightarrow X$ and β sends $\bar{x}$ to $\bar{0}$.

Furthermore by [19, lemma 2.3] we may shrink S in such a way that $\alpha^* \mathcal{E}^G = \beta^* \mathcal{E}^H_{x,N}$.

Hence $\hat{\mathcal{O}}_{X/\!\!/G, \bar{x}} = \hat{\mathcal{O}}_{N/\!\!/H, \bar{0}}$ and likewise $(\widehat{\mathcal{E}^G})_{\bar{x}} = (\widehat{\mathcal{E}^H_{x,N}})_{\bar{0}}$.

Therefore Theorem 3.1 implies that $\mathcal{D}(X/\!\!/G, \mathcal{E}^G)$ is simple if all $D_{\hat{\mathcal{O}}_{N/\!\!/H,\bar{0}}}((\widehat{\mathcal{E}^H_{x,N}})_{\bar{0}})$ are simple.

Now $\mathcal{O}(N/\!\!/H) = \mathcal{O}(N)^H$ and $\mathcal{E}^H_{x,N}(N/\!\!/H) = (\mathcal{E}_x \otimes_{\mathbb{C}} \mathcal{O}(N))^H$ are graded and therefore we may invoke Theorem 3.2 to deduce that $D_{\hat{\mathcal{O}}_{N/\!\!/H,\bar{0}}}((\widehat{\mathcal{E}^H_{x,N}})_{\bar{0}})$ is simple if and only if $D_{\mathcal{O}(N/\!\!/H)}(\mathcal{E}^H_{x,N}(N/\!\!/H)) = \mathcal{D}_{N/\!\!/H}(\mathcal{E}^H_{x,N})$ is simple. $\square$

Theorem A' allows one to give a seemingly weaker version of conjecture 1.1(s)

Conjecture 4.2. Let G be a reductive group and let W be a finite dimensional G-representation. Let $J \subset \mathbb{C}[W/\!\!/G]$ be a graded ideal of finite codimension. Then

$$\mathcal{D}(W/\!\!/G)J\mathcal{D}(W/\!\!/G) = \mathcal{D}(W/\!\!/G)$$

Proposition 4.3. Conjecture 1.1(s) and conjecture 4.2 are equivalent.

Proof. It is clear that conjecture 1.1(s) implies conjecture 4.2. So assume that conjecture 1.1(s) is false. Then by Theorem A' there exists a counterexample of the form (G, W) with W a representation. Choose this counterexample in such a way that $(\dim G, \dim W)$ is miminal in the lexicographic ordering.

If $W = W' \oplus F$ where F is a trivial representation then $\mathcal{D}(W/\!\!/G)$ is a Weyl algebra over $\mathcal{D}(W'/\!\!/G)$ and it is easily seen that this implies that $\mathcal{D}(W'/\!\!/G)$ is also a counterexample. Hence $W^G = 0$ or equivalently every point in W has a stabilizer strictly smaller than G. Theorem 3.1 and the minimality of (G, W) imply that if $f \in \mathbb{C}[W/\!\!/G]$ such that $f(\bar{0}) = 0$ then $\mathcal{D}((W/\!\!/G)_f)$ is simple.

Let $I \subset \mathcal{D}(W/\!\!/G)$ be a non-trivial two-sided ideal. By lemma 2.3 we know that I is graded. Put $J = \mathbb{C}[W/\!\!/G] \cap I$. If J is not of finite codimension then there exist $f \in \mathbb{C}[W/\!\!/G] \setminus \operatorname{rad} J$ such that $f(\bar{0}) = 0$.

By lemma 2.4 I_f is then a non-trivial two-sided ideal in $\mathcal{D}((W/\!\!/G)_f)$ but it has been shown above that this is impossible. Hence J is of finite codimension.

Now suppose that conjecture 4.2 is true. Then $I \supset \mathcal{D}(W/\!\!/G)J\mathcal{D}(W/\!\!/G) = \mathcal{D}(W/\!\!/G)$ contradicting the choice of I. $\square$

5. The Marsden-Weinstein reduction

In this section we recall some elementary notions from symplectic geometry. We assume that Z is a smooth connected affine variety over $\mathbb{C}$ with a symplectic form $\omega \in H^0(Z, \Omega^2_{Z/\mathbb{C}})$ and with corresponding Poisson bracket $\{\ ,\ \}$.

We assume that G is a reductive group acting symplectically on Z (that is ω is G-invariant) with Lie algebra $\mathfrak{g}$ and with moment map $\mu : Z \longrightarrow \mathfrak{g}^*$. That is μ is a G-equivariant map of affine varieties over $\mathbb{C}$ such that for all $x \in Z$, $\zeta \in T_x X$ and $v \in \mathfrak{g}$ one has

$$d\mu_x(\zeta) \cdot v = \omega_x(\zeta, v_x) \tag{5.1}$$

Here $x \mapsto v_x$ denotes the vector field obtained from the left action of G on Z and we have identified $T_{\mu(x)}\mathfrak{g}^*$ with $\mathfrak{g}^*$. For $f \in \mathcal{O}(Z)$ we let H_f stand for the corresponding Hamiltonian vector field on Z. That is if $g \in \mathcal{O}(Z)$ then $H_f(g) = \{f, g\}$. The relation between ω and $\{\ ,\ \}$ is given by $\omega(H_f, H_g) = \{f, g\}$.

If $v \in \mathfrak{g}$ then we define f_v as the function $x \mapsto \mu(x) \cdot v$. One has the following relation :

Lemma 5.1. Let $v \in \mathfrak{g}$ then for $x \in Z : (H_{f_v})_x = v_x$.

Proof. We have to show for $g \in \mathcal{O}(Z) : \{f_v, g\}(x) = v_x(g)$.

Let $\zeta \in T_x Z$. Then by the definition of f_v and the moment map $\zeta(f_v) = d\mu_x(\zeta) \cdot v = \omega_x(\zeta, v_x)$. Hence $\{f_v, g\}(x) = -(H_g)_x(f_v) = \omega_x(v_x, (H_g)_x)$.

On the other hand for $h \in \mathcal{O}(Z)$ one has $H_h(g) = \{h, g\} = \omega(H_h, H_g)$ and since the H_h span TZ we obtain $\omega_x(v_x, (H_g)_x) = v_x(g)$. $\square$

Lemma 5.2. Let $v \in \mathfrak{g}$ and $g \in \mathcal{O}(Z)^G$. Then

(1) $\{f_v, g\} = 0$
(2) The defining ideal of the (scheme theoretic) fiber $\mu^{-1}(0)$ is closed under $\{-, g\}$.

Proof. (1) One has $\{f_v, g\}(x) = v_x(g) = 0$ since g is G-invariant.
(2) The defining ideal of $\mu^{-1}(0)$ is spanned by $(f_v)_{v \in \mathfrak{g}}$ so (2) follows from (1). $\square$

The Marsden-Weinstein reduction Y of Z is defined as

$$Y = \mu^{-1}(0)/\!\!/G$$

by analogy with the corresponding C^∞-notion [16][20]. Perhaps one should call this the scheme-theoretic Marsden-Weinstein reduction, and the set theoretic version would then be given by Y_{red}. However we have no use for this concept.

On Y there is a natural Poisson bracket defined as follows. Let $f,g\in\mathcal{O}(Y)$. Then we may choose $f',g'\in\mathcal{O}(Z)^G$ such that $f'\,|\,Y=f$, $g'\,|\,Y=g$. We put $\{f,g\}_Y=\{f',g'\}$. It follows from lemma 5.2 that this is well defined.

Proposition 5.3. Assume that G acts freely on Z. Then Y is a smooth symplectic variety.

Proof. First note that now (5.1) implies that $d\mu_x$ is surjective and hence μ is smooth. This implies that $\mu^{-1}(0)$ is also smooth. By hypothesis G acts freely on $\mu^{-1}(0)$ which implies that Y is smooth.

Let $\phi: Z\longrightarrow Z/\!\!/G$ be the quotient map. We define a 2-form on Y by the following device. Let $\zeta,\eta\in T\mu^{-1}(0)$. Then

$$\omega_Y(d\phi(\zeta),d\phi(\eta))\overset{\mathrm{def}}{=}\omega(\zeta,\eta)$$

It follows easily from (5.1) that this is well defined, and defines a symplectic form on Y. Furthermore the associated Poisson bracket is the one we have defined earlier. □

We recall the following result :

Proposition 5.4. Assume that Y is an affine connected smooth symplectic variety over $\mathbb{C}$. Then there are no non-trivial Poisson ideals in $\mathcal{O}(Y)$.

Proof. This is well-known. We give a sketch of a proof. Let $I\subset\mathcal{O}(Y)$ be a non-trivial Poisson ideal. Since Y is connected, there exists $y\in Y$ such that $I_y\neq 0$, $I_y\neq\mathcal{O}(Y)_y$. Now completing defines a non-trivial Poission ideal $\hat{I}_y$ in $\hat{\mathcal{O}}_{Y,y}$. However as a Poisson ring $\hat{\mathcal{O}}_{Y,y}$ is isomorphic to

$$\hat{\mathcal{O}}_{Y,y}=\mathbb{C}[[x_1,\dots,x_n,y_1,\dots,y_n]]$$

with Poisson bracket $\{x_i,x_j\}=0$, $\{y_i,y_j\}=0$ for all i,j and

$$\{x_i,y_j\}=\begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j\end{cases}$$

Hence $\{x_i,-\}=\frac{\partial}{\partial y_i}$, $\{y_i,-\}=-\frac{\partial}{\partial x_i}$. So a Poisson ideal in $\hat{\mathcal{O}}_{Y,y}$ must be stable under $\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_i}$ which implies that $\hat{I}_{Y,y}=0$ or $\hat{I}_{Y,y}=\hat{\mathcal{O}}_{Y,y}$. This yields a contradiction. □

Proof of Theorem B. A moment map for the action of G on T^*X is given by $\mu: T^*\longrightarrow\mathfrak{g}^*:(x,\omega_x)\mapsto(v\mapsto\langle v_x,\omega_x\rangle)$.

There is a surjective map $\mathcal{O}(\mu^{-1}(0)/\!\!/G)\longrightarrow\mathrm{gr}(\mathcal{D}(X)^G/(\mathcal{D}(X)\mathfrak{g})^G$ compatible with Poisson brackets. Then Theorem B follows easily from Proposition 5.3 and Proposition 5.4. □

6. Preliminaries on G-equivariant quasi-coherent $\mathcal{O}_X$ and $\mathcal{D}_X$-modules

In this section we state, without proofs, some properties of G-equivariant sheaves. Detailed proofs occur in [24]. Including them here would blow up this section out of proportion. Furthermore much of what we say is clear if the underlying scheme is affine. However for use in the sequel we are forced to adopt a more general framework.

6.1. G-equivariant quasi-coherent $\mathcal{O}_X$-modules. Below X will be an arbitrary scheme over $\mathbb{C}$ and G will be a linear algebraic group over $\mathbb{C}$, acting on X. $e\in G$ will be the unit element and $m_e\subset\mathcal{O}(G)$ will be the corresponding maximal ideal. The category of quasi-coherent $\mathcal{O}_X$-modules will be denoted by $\mathcal{O}_X$-qch.

We start with the following diagram of objects and maps

$$G\times G\times X \begin{array}{c}\xrightarrow{d_0}\\ \xrightarrow{d_1}\\ \xrightarrow{d_2}\end{array} G\times X \begin{array}{c}\xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xrightarrow{d_1}\end{array} X \tag{6.1}$$

$$d_0(g_1,x) = g_1^{-1}x \qquad d_0(g_1,g_2,x) = (g_2, g_1^{-1}x)$$
$$d_1(g_1,x) = x \qquad d_1(g_1,g_2,x) = (g_1g_2, x)$$
$$s_0(x) = (e,x) \qquad d_2(g_1,g_2,x) = (g_1,x)$$

Definition 6.1.1. [2] A G-equivariant quasi-coherent $\mathcal{O}_X$-module is a pair $(\mathcal{F},\theta)$ where $\mathcal{F} \in \mathcal{O}_X$-qch and θ is an isomorphism $d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ in $\mathcal{O}_{G\times X}$-qch such that

$$\begin{aligned} d_0^*\theta \circ d_2^*\theta &= d_1^*\theta \\ s_0^*\theta &= \mathrm{id}_{\mathcal{F}} \end{aligned} \tag{6.2}$$

The corresponding category is denoted by $(G,\mathcal{O}_X)$-qch.

If there is no possibility for confusion we will simply write $\mathcal{F}$ for $(\mathcal{F},\theta)$.

Example 6.1.2. If $\mathcal{F} = \mathcal{O}_X$ then we may take $\theta = \mathrm{id}$. Then (6.2) is obviously satisfied.

Let $g \in G$ and let $i_g : X \to G \times X : x \mapsto (g,x)$ be the corresponding inclusion. Then $i_g^*(\theta)$ defines for *all* open $U \subset X$ a map $\mathcal{F}(U) \to \mathcal{F}(g^{-1}U)$. We denote by $g : \mathcal{F}(U) \to \mathcal{F}(gU)$ the inverse of this map. This defines an action on $\mathcal{F}$ of the discrete group G^{dis} associated to G (see [11]). So, with obvious notations, one has a functor $(G,\mathcal{O}_X)$-qch $\to (G^{\mathrm{dis}},\mathcal{O}_X)$-qch which is fully faithful, but not an equivalence (see below for the affine case).

Functors compatible with (flat or smooth) base-change preserve G-equivariance, since they preserve (6.2).

For example if $\pi : Y \to X$ is G-equivariant and satisfies some benign conditions (see [12]) then $\pi_*(\mathcal{F},\theta) = (\pi_*\mathcal{F},\theta')$ where θ' makes the following diagram commutative

$$\begin{tikzcd} (\mathrm{id}\times\pi)_* d_1^*\mathcal{F} \arrow[r, "(\mathrm{id}\times\pi)_*(\theta)"] \arrow[d] & (\mathrm{id}\times\pi)_* d_0^*\mathcal{F} \arrow[d] \\ d_1^*\pi_*\mathcal{F} \arrow[r, "\theta'"] & d_0^*\pi_*\mathcal{F} \end{tikzcd}$$

Here the vertical maps are the canonical identifications given by base-change. In the sequel we will use such "G-equivariant functors" implicitly.

Now let $(\mathcal{F},\theta)$ be in $(G,\mathcal{O}_X)$ and let $U \subset X$ be an affine open. Take $f \in \mathcal{F}(U)$ and put $l(f) = p^*(\theta(d_1^*f))$ where $p(g,x) = (g,gx)$. $l(f)$ is a section of $(d_1^*\mathcal{F})(p^{-1}(G\times U))$. Now $p^{-1}(G\times U)$ is a neighborhood of $e \times U$ and hence we may consider $l(f)$ as a section of $(s_0^{-1}\mathcal{F})(U)$. Hence we have defined a map of sheaves on X

$$l : \mathcal{F} \to s_0^{-1}\mathcal{F}$$

Let $\mathcal{O}(\hat{G}) = \varprojlim \mathcal{O}(G)/m_e^n$ be the formal group associated to G. Then $s_0^{-1}\mathcal{F}$ is embedded in $\mathcal{O}(\hat{G})\hat{\otimes}\mathcal{F}$ ($\hat{\otimes}$ = completed tensor product) and hence we may view l as a "coaction"

$$l : \mathcal{F} \to \mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{F}$$

Denote by $\langle\ ,\ \rangle$ the canonical pairing between $\mathcal{O}(G)$ and $\mathcal{O}(\hat{G})^* \overset{\mathrm{def}}{=} \varinjlim(\mathcal{O}(G)/m_e^n)^*$ and let $\mathfrak{g} \subset (m_e/m_e^2)^* = \mathcal{O}(\hat{G})^*$ be the Lie algebra associated to G. Then we may define a left action $l : \mathfrak{g}\otimes\mathcal{F} \to \mathcal{F}$ by the composition

$$\mathfrak{g}\otimes\mathcal{F} \xrightarrow{1\otimes l} \mathfrak{g}\otimes\mathcal{O}(\hat{G}) \mathbin{\hat{\otimes}} \mathcal{F} \xrightarrow{-\langle\ ,\ \rangle\otimes 1} \mathcal{F}$$

If $v \in \mathfrak{g}$, $f \in \mathcal{F}(U)$ then, if no confusion is possible, we write vf for $l(v\otimes f)$.

The $\mathfrak{g}$-action on $\mathcal{F}$ is compatible with the $\mathfrak{g}$-action on $\mathcal{O}_X$ ($\mathcal{O}_X$ is in $(G,\mathcal{O}_X)$-qch by example 6.1.2)in the following sense : if $a \in \mathcal{O}_X(U)$, $f \in \mathcal{F}(U)$, $v \in \mathfrak{g}$ then $v(af) = (va)f + a(vf)$.

If we denote by $(\mathfrak{g},\mathcal{O}_X)$-qch the category of quasi-coherent $\mathcal{O}_X$-modules, equipped with a compatible $\mathfrak{g}$-action then we have defined a functor $(G,\mathcal{O})$-qch $\to (\mathfrak{g},\mathcal{O}_X)$-qch. This functor has some good properties [24]. In particular we mention the fact that it is fully faithful if G is connected.

If X happens to be affine then things are easier. In that case $\mathcal{O}(X)$ is a rational G-representation via $g(f) = f \circ g^{-1}$. Furthermore the coaction l defined above is in this case obtained from a coaction of $\mathcal{O}(G)$ on $\mathcal{F}(X)$ and hence $\mathcal{F}(X)$ also becomes a rational G-representation. If we denote by $(G, \mathcal{O}(X))$-mod the category of $\mathcal{O}(X)$-module equipped with a rational G-action, compatible with the G-action $\mathcal{O}(X)$ then $\mathcal{F} \mapsto \mathcal{F}(X)$ defines an equivalence between $(G, \mathcal{O}_X)$-qch and $(G, \mathcal{O}(X))$-mod. The induced $\mathfrak{g}$-action on $\mathcal{F}(X)$ is now obtained by differentiating the G-action. This makes sense since $\mathcal{F}(X)$ is rational (= locally finite).

6.2. G-equivariant differential operators. Our conventions are as before but now we assume that X is of finite type over $\mathbb{C}$. $\mathcal{F}, \mathcal{G}$ will be coherent G-equivariant $\mathcal{O}_X$-modules. Then $\mathcal{D}_X(\mathcal{F}, \mathcal{G})$ is also G-equivariant in a canonical way. It is clear how to define the action of the discrete group G^{dis} associated to G. Let $g \in G$ and let $D \in \mathcal{D}_X(\mathcal{F}, \mathcal{G})(U)$. Then we put

$$g(D) = g \circ D \circ g^{-1} \tag{6.3}$$

which defines an element of $\mathcal{D}_X(\mathcal{F}, \mathcal{G})(gU)$.

It is slightly less obvious why $\mathcal{D}_X(\mathcal{F}, \mathcal{G})$ is in $(G, \mathcal{O}_X)$-qch. In fact this depends on X being of finite type. One possibility is to introduce relative differential operators $\mathcal{D}_{X/S}(\mathcal{F}, \mathcal{G})$ where X is a scheme over some base-scheme S. Under our hypotheses such relative differential operators are compatible with flat base-change.

Example 6.2.1. Let T be a one dimensional torus and let X be an infinite dimensional rational T-representation with an infinite number of different weights. Then $\mathcal{D}(X)$ contains $\mathrm{Der}(\mathcal{O}_X)$ which is an infinite product of copies of $\mathbb{C}$. It is easy to see that the induced torus action on $\mathrm{Der}(\mathcal{O}_X)$ is not rational.

By the discussion in §6.1 there is an induced $\mathfrak{g}$-action on $\mathcal{D}_X(\mathcal{F}, \mathcal{G})$. One may show [24] that this $\mathfrak{g}$-action is as follows. Let $U \subset X$ be an open affine, $D \in \mathcal{D}_X(\mathcal{F}, \mathcal{G})(U) = \mathcal{D}_X(\mathcal{F}(U), \mathcal{G}(U))$, $f \in \mathcal{F}(U)$, $v \in \mathfrak{g}$. Then

$$(vD) * (f) = v(D * f) - D * (vf) \tag{6.4}$$

Here "$*$" denotes the action of a differential operator. For $v \in \mathfrak{g}$ denote by $D_v \in \mathcal{D}(X, \mathcal{F})$ the element which corresponds to the action of v on $\mathcal{F}$. Then (6.4) may be written as $v(D) = [D_v, D]$. Note that in the affine case this follows easily by differentiating (6.3).

We denote the category of G-equivariant quasi-coherent-$\mathcal{O}_X$ modules $\mathcal{M}$, equipped with a G-equivariant $\mathcal{D}_X(\mathcal{F})$-action

$$\mathcal{D}_X(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$$

by $(G, \mathcal{D}_X(\mathcal{F}))$-wqch. Object in $(G, \mathcal{D}_X(\mathcal{F}))$-wqch are called "weakly G-equivariant quasi-coherent $\mathcal{D}_X(\mathcal{F})$-modules."

A G-equivariant quasi-coherent $\mathcal{D}_X(\mathcal{F})$ module is an object $\mathcal{M}$ of $(G, \mathcal{D}_X(\mathcal{F}))$-wqch for which the action of all $v \in \mathfrak{g}$ (obtained from the fact that $\mathcal{M}$ is in particular in $(G, \mathcal{O}_X)$-qch) coincides with the action of D_v (obtained from that fact that $\mathcal{M}$ is in $\mathcal{D}_X(\mathcal{F})$-qch). The corresponding category is denoted by $(G, \mathcal{D}_X(\mathcal{F}))$-qch.

The following example illustrates the difference between $(G, \mathcal{D}_X(\mathcal{F}))$-wqch and $(G, \mathcal{D}_X(\mathcal{F}))$-qch.

Example 6.2.2. Assume that X is a point. Then $(G, \mathcal{D}_X)$-wqch is the category of rational G-representations, whereas $(G, \mathcal{D}_X)$-qch is the category of $\mathbb{C}$ vectorspaces, with trivial G-action.

There is a functor

$$\mathcal{M} \mapsto \mathcal{D}_X(\mathcal{F}, \mathcal{G}) \otimes_{\mathcal{D}_X(\mathcal{F})} \mathcal{M}$$

from $\mathcal{D}_X(\mathcal{F})$-qch to $\mathcal{D}_Y(\mathcal{G})$-qch extending to a functor from $(G, \mathcal{D}_X(\mathcal{F}))$-wqch to $(G, \mathcal{D}_X(\mathcal{G}))$-wqch. Using (6.4) one may show :

Lemma 6.2.3. The above functor sends $(G, \mathcal{D}_X(\mathcal{F}))$-qch to $(G, \mathcal{D}_X(\mathcal{G}))$-qch

The following result will be extremely useful for us [23][24]

Theorem 6.2.4. Assume that G is connected. Then the forgetful functor $(G, \mathcal{D}_X(\mathcal{F}))$-qch $\to$ $\mathcal{D}_X(\mathcal{F})$-qch is fully faithful. Furthermore if $\mathcal{M} \in (G, \mathcal{D}_X(\mathcal{F}))$-qch and $\mathcal{N} \subset \mathcal{M}$ in $\mathcal{D}_X(\mathcal{F})$-wqch then $\mathcal{N} \in (G, \mathcal{D}_X(\mathcal{F}))$-qch.

We now look at the case where X is smooth and $\mathcal{F} = \mathcal{O}_X$. Then it is well-known [4] (see also [24]) that there is the following alternative description of $(G, \mathcal{D}_X)$-qch. A G-equivariant quasi-coherent $\mathcal{D}_X$-module can be represented by a pair $(\mathcal{F}, \theta)$ where $\mathcal{F}$ is in $\mathcal{D}_X$-qch and $\theta : d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ is in $\mathcal{D}_{G\times X}$-qch such that (6.1) holds. Note that this makes sense since both $d_1^*\mathcal{F}$ and $d_0^*\mathcal{F}$ are in $\mathcal{D}_{G\times X}$-qch [3].

In a similar way as in §6.1 this implies that object in $(G, \mathcal{D}_X)$-qch are compatible with standard functors, since these commute with smooth base-change. We think in particular of $H^i\pi_+$, $H^i\pi^!$ for a G-equivariant map π and $\mathcal{H}^i_Y(X, -)$ for a G-equivariant closed subset Y of X. It also follows from the description in §2 that $\mathcal{L}(Y, X)$ is in $(G, \mathcal{D}_X)$-qch.

Theorem 6.2.4 and the discussion thereafter dispenses us to a certain extent of having to work with G-equivariant derived categories [2] (if G is connected).

That is, instead of directly computing in $(G, \mathcal{D}_X)$-qch we work in $\mathcal{D}_X$-qch (or in $D^b(\mathcal{D}_X\text{-qch})$) and in the end we know that we obtain G-equivariant sheaves, having a unique G-structure.

We need the following generalization of a well-known result on quasi-coherent sheaves.

Assume that $H \subset G$ acts on a smooth scheme $Y/\mathbb{C}$ and assume that $X = G \times^H Y = G \times Y /\!\!/ H$ exists. Let $[e]$ stand for the class of H in G/H. Then

Lemma 6.2.5. $\mathcal{M} \mapsto \mathcal{M}_{[e]}$ induces an equivalence between $(G, \mathcal{D}_X)$-qch and $(H, \mathcal{D}_Y)$-qch.

The inverse to $\mathcal{M} \mapsto \mathcal{M}_{[e]}$ will be denoted by $\tilde{}$.

Now we let X be again as in the beginning of §6.2, but, for simplicity, we assume in addition that X is affine and that G is reductive. $X/\!\!/G$ is the usual algebraic quotient and $\phi : X \to X/\!\!/G$ is the quotient map. If $\mathcal{F} \in (G, \mathcal{O}_X)$-qch then we denote by $\mathcal{F}^G$ the sheaf in $\mathcal{O}_{X/\!\!/G}$-qch whose global sections are $\mathcal{F}(X)^G$.

We look at the folllowing objects :

$$\mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G$$

which is in $\mathcal{O}_{X/\!\!/G}$-qch and

$$\mathcal{D}_X(\mathcal{F})/\mathcal{D}_X(\mathcal{F})\mathfrak{g}$$

which is in $\mathcal{O}_X$-qch. Note that we have identified $\mathfrak{g}$ with its image in $\mathcal{D}_X(\mathcal{F})$.

Lemma 6.2.6. (1) $\mathcal{D}_X(\mathcal{F})^G$ and $\mathfrak{g}$ commute inside $\mathcal{D}_X(\mathcal{F})$

(2) $\mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G$ is a sheaf of rings on $X/\!\!/G$.

(3) $\mathcal{D}_X(\mathcal{F})/\mathcal{D}_X(\mathcal{F})\mathfrak{g}$ is a $\mathcal{D}_X(\mathcal{F}) - \phi^{-1}(\mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G)$-bimodule.

(4) There is a functor $\mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G$-qch $\to (G, \mathcal{D}_X(\mathcal{F}))$-qch given by

$$\mathcal{N} \mapsto \mathcal{D}_X(\mathcal{F})/\mathcal{D}_X(\mathcal{F})\mathfrak{g} \otimes_{\phi^{-1}(\mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G)} \phi^{-1}\mathcal{N}$$

with quasi-inverse

$$(\ )^G : (G, \mathcal{D}_X(\mathcal{F}))\text{-qch} \to \mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G\text{-qch} : \mathcal{M} \mapsto \mathcal{M}^G$$

(5) Taking G-invariants defines a homomorphism of seaves of rings $\mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G \to \mathcal{D}_{X/\!\!/G}(\mathcal{F}^G)$

Proof. (1) If D is an invariant differential operator then $g(D) = D$ for all $g \in G$. Differentiating yields that $v(D) = 0$ for all $v \in \mathfrak{g}$. Now we have seen above that $v(D) = [D_v, D]$ and hence D and D_v commute.

(2),(3),(4) follow from (1). For (5) we use that if $f \in \mathcal{F}(X)^G$ then $\mathfrak{g}f = 0$ which follows from differentiating the G-action. □

Remark 6.2.7. Sometimes $\mathcal{D}_X(\mathcal{F})^G/(\mathcal{D}_X(\mathcal{F})\mathfrak{g})^G$ is a more interesting object than $\mathcal{D}_{X/\!\!/G}(\mathcal{F}^G)$. For example if $X = \mathbb{C}^r$ and $G = \mathbb{C}^*$, acting diagonally on X then $X/\!\!/G$ is a point and hence $\mathcal{D}_{X/\!\!/G}$ is trivial whereas $\mathcal{D}_X^G/(\mathcal{D}_X\mathfrak{g})^G$ corresponds to the global differential operators on $\mathbb{P}^{r-1}$.

7. Unstable holonomic modules for Sl_2

Below we will assume $G = \mathrm{Sl}_2(\mathbb{C})$ and $W = \oplus_{i=1}^{e} W^{(i)}$ is a G-representation with $W^{(i)} = S^{d_i}\mathbb{C}^2$.

Let T^*W be the cotangent bundle of W and $p : T^*W \longrightarrow W$ the natural projection map. G acts on T^*W by $g(x, \omega_x) = (gx, (g^{-1})^*_x \omega_x)$ and p is obviously G-equivariant.

We denote by W^u the unstable part of W. That is

$$W^u = \{x \in W \mid 0 \in \overline{Gx}\}$$

$(T^*W)^u$ is defined similarly.

In this section we aim to prove Theorem D by reduction to the torus case where we have the following [18]

Theorem 7.1. Assume that T is a torus and W a finite-dimensional T-representation which is unstable (that is 0 is in the closure of every orbit). Let $\mathcal{M} \in (T, \mathcal{D}_W)$-qch be coherent as $\mathcal{O}_W$-module. Then $\mathcal{M} = \mathcal{O}_W^{\oplus r}$ in $(T, \mathcal{D}_W)$-qch, for some $r \in \mathbb{N}$.

We will prove Theorem D at the end of this section. First we need to introduce some more notations.

Let $T = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \subset \mathrm{Sl}_2(\mathbb{C})$ be resp. a maximal torus and a Borel subgroup. Let $\lambda : z \mapsto \mathrm{diag}(z, z^{-1})$ be a one-parameter subgroup of T. Put

$$W_\lambda = \{x \in W \mid \lim_{t \longrightarrow 0} \lambda(t)x = 0\}$$

It follows from the Hilbert-Mumford criterion that $W^u = GW_\lambda$.

W_λ is B-stable, and hence there is a canonical quotient map $G \times^B W_\lambda \longrightarrow GW_\lambda$ which is birational, and which is a bijection outside $\{0\}$.

There is a similar description for $(T^*W)^u$. If we identify T^*W with $W \oplus W^*$ then $(T^*W)^u = G(W_\lambda \oplus (W^*)_\lambda)$ and $G \times^B (W_\lambda \oplus (W^*)_\lambda) \longrightarrow G(W_\lambda \oplus (W^*)_\lambda)$ is a bijection outside $(0,0)$. From this one deduces the following lemma

Lemma 7.2. Let $x \in W_\lambda \setminus \{0\}$. Then the settheoretic fiber $p^{-1}(x) \cap (T^*W)^u$ of x is equal to $(W^*)_\lambda$.

Let $X(T)$ be the character group of T. We identify $X(T) = \mathbb{Z}$ where 1 corresponds to $\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right) \mapsto z$. Then W_λ is a linear subspace of W whose T-weights are all > 0. Let $W_1 \subset W$ be spanned by all weight-vectors of weight > 2.

We choose a basis H, E, F for $\mathfrak{g} = \mathrm{Lie}(G)$ such that $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. Differentiation the G-action yields a $\mathfrak{g}$-action on W.

Each $W^{(i)}$ has a basis $w_0^{(i)}, \dots w_{d_i}^{(i)}$ such that the action of $\mathfrak{g}$ on $W^{(i)}$ is given by

$$\begin{aligned} Hw_j^{(i)} &= (d_i - 2j)w_j^{(i)} \\ Ew_j^{(i)} &= jw_{j-1}^{(i)} \\ Fw_j^{(i)} &= (d_i - j)w_{j+1}^{(i)} \end{aligned} \tag{7.1}$$

There are corresponding decompositions

$$W_\lambda = \oplus_{i=1}^{e} W_\lambda^{(i)}, \qquad W_1 = \oplus_{i=1}^{e} W_1^{(i)}$$

where $W_\lambda^{(i)}$ ($W_1^{(i)}$) is spanned by those $w_j^{(i)}$ such that $d_i - 2j > 0(2)$.

For $x \in W$ and $v \in \mathfrak{g}$ we denote by v_x the tangent vector v of G transported to x through the action of G on W. To be more precise let γ be a curve on G with tangent vector v at e. Then $v_x = \frac{d}{dt}\gamma(t)x\Big|_{t=0}$.

If we identify T_xW with W then the pairing $\mathfrak{g} \times W \longrightarrow W : (v, x) \mapsto v_x$ is precisely the action of $\mathfrak{g}$ on W obtained by differentiating the action of G on W.

Lemma 7.3. The map $\pi : G \times^B W_\lambda \longrightarrow GW_\lambda \subset W$ is a closed immersion precisely on $G \times^B (W_\lambda - W_1)$.

Proof. Let $x \in W_1$. We have to determine when

$$T_{(e,x)}(G \times^B W_\lambda) \xrightarrow{(d\pi)_x} T_x(W)$$

is an injection. That is : when does the image of $(d\pi)_x$ have dimension $1 + \dim W_\lambda$?

The quotient map $\psi : G \times W_\lambda \to G \times^B W_\lambda$ is smooth and hence

$$T_e G \times T_x W_\lambda = T_{(e,x)}(G \times W_\lambda) \xrightarrow{(d\psi)_x} T_{(e,x)}(G \times^B W_\lambda)$$

is surjective.

Let $\theta = \pi \circ \psi$ be the composed map given by $G \times W_\lambda \to W : (g, x) \mapsto gx$. Then we have to find out when the image of

$$T_e G \times T_x W_\lambda \xrightarrow{(d\theta)_x} T_x W$$

has the right dimension.

$(d\theta)_x$ is given by $(v, \zeta) \mapsto v_x + \zeta$. If we identify $T_x W_\lambda = W_\lambda$, $T_x W = W$ then using the explicit description of the $\mathfrak{g}$-action on W given by (7.1), one sees that

$$\dim \operatorname{im}(d\theta)_x < \dim W_\lambda + 1 \iff F_x \in W_\lambda \iff x \in W_1 \quad \square$$

Corollary 7.4. $GW_\lambda - GW_1$ is smooth.

If $S \subset W$ is locally closed and smooth then $T^*_S(W)$ will be the conormal bundle of S in W. We will need the following result.

Lemma 7.5. Assume that all d_i are odd and $e \geq 2$. Let $S \subsetneq W^u$ be G-stable, locally closed and smooth. Then $T^*_S W \not\subset (T^*W)^u$

Proof. The proof parallels the proof of lemma 7.3.

The result is clear if $S = \{0\}$ so, by shrinking S if necessary, we may assume $0 \notin S$. Furthermore, without loss of generality, we may assume that S is irreducible.

Put $K = S \cap W_\lambda$. Then $GK = S \cap GW_\lambda = S$. Furthermore K is B-stable and hence there is a projective projection map

$$\pi : G \times^B K \to S \subset W$$

which is an isomorphism since it is bijection and S is smooth. In particular one deduces that K is smooth.

For our aim it is sufficient to show that there exist $x \in K$ such that

$$(T^*_S W)_x \not\subset (W^*)_\lambda \tag{7.2}$$

(see lemma 7.2).

To compute $(T^*_S W)_x$ we use the standard exact sequence

$$0 \to (T^*_S W)_x \to T^*_x W \to T^*_x S \to 0 \tag{7.3}$$

By what we have said above :

$$T^*_x S \xrightarrow{\pi^*_x} T^*_{(e,x)}(G \times^B K)$$

is an isomorphism.

Furthermore the quotient map

$$\psi : G \times K \to G \times^B K$$

is smooth and hence the corresponding

$$\psi^*_x : T^*_{(e,x)}(G \times^B K) \to T^*_{(e,x)}(G \times K) = T^*_e G \times T^*_x K$$

is injective. Therefore (7.3) becomes an exact sequence

$$0 \to (T^*_S W)_x \to T^*_x W \xrightarrow{\theta^*_x} T^*_e G \times T^*_x K$$

Here $\theta = \pi \circ \psi$ is the map $G \times K \to W : (g, t) \mapsto gt$

θ^*_x is the dual of

$$(d\theta)_x : T_e G \times T_x K \to T_x W : (v, \zeta) \mapsto v_x + \zeta$$

If we identify $T_x(W)$ with W^* then

$$(W^*)_\lambda = \{f \in W^* \mid f(W_\lambda) = 0\}$$
$$(T_S^*W)_x = \{f \in W^* \mid f \circ (d\theta)_x = 0\}$$

Hence (7.2) is equivalent with $\mathrm{im}(d\theta)_x \not\supset W_\lambda$.

Obviously $\dim \mathrm{im}(d\theta)_x \le \dim K + 1$. Hence if $\dim W_\lambda - \dim K \ge 2$ then we are done. So we assume $\dim W_\lambda - \dim K = 1$. In this case we need a slightly more refined analysis.

If we would have $\mathrm{im}(d\theta)_x \supset W_\lambda$ then by comparing dimensions $\mathrm{im}(d\theta)_x = W_\lambda$. Hence to contradict this it is sufficient to find $x \in K$, $v \in \mathfrak{g}$ such that $v_x \notin W_\lambda$. Therefore we assume that such a pair (x, v) does not exist. That is for all $x \in K$ and for all $v \in \mathfrak{g}$ one has $v_x \in W_\lambda$.

Now by the description of the $\mathfrak{g}$-action on W (7.1) we know that $F_x \in W_\lambda \iff x \in W_1$. Hence $K \subset W_1$. However our hypotheses imply that $\mathrm{codim}(W_1, W) \ge 2$ which yields a contradiction. $\square$

If $S \subset W$ is a closed subset then we denote by $\overline{T_S^*W}$ the closure in T^*W of the cotangent bundle of the smooth part of S.

Lemma 7.6. (1) The category of holonomic $\mathcal{D}_W$-modules $\mathcal{M}$ such that $\mathrm{Ch}(\mathcal{M}) \subset (T^*W)^u$ is closed under subquotients and extensions.

(2) Assume that W and $\mathcal{M}$ are as in Theorem D. Then

$$\mathrm{Ch}(\mathcal{M}) = \overline{T^*_{W^u}(W)}$$

(3) Let W and $\mathcal{M}$ be as in Theorem D and let $s : W - GW_1 \to W$ be the inclusion map. Then $s_{!+}(\mathcal{M} \mid W - GW_1) = \mathcal{M}$.

Proof. (1) Clear.

(2) Let $\bigcup_i \Lambda_i = \mathrm{Ch}(\mathcal{M})$ be the decomposition in irreducible components of $\mathrm{Ch}(\mathcal{M})$. Since $\mathrm{Ch}(\mathcal{M})$ is conical and Lagrangian one has

$$\mathrm{Ch}(\mathcal{M}) = \bigcup_i \overline{T^*_{p(\Lambda_i)}W}$$

By lemma 7.5 we necessarily have $p(\Lambda_i) = W^u$.

(3) Let $\mathcal{K} = \ker(\mathcal{H}^0(s_!(\mathcal{M} \mid W - GW_1)) \to \mathcal{H}^0(s_+(\mathcal{M} \mid W - GW_1)))$. Clearly $\mathrm{Supp}\,\mathcal{K} \subset GW_1$. We have the following diagram

$$(7.4)\qquad \begin{array}{ccccccc} & & & & \mathcal{M} & & \\ & & & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{H}_0(s_!(\mathcal{M} \mid W - GW_1)) & \longrightarrow & \mathcal{H}_0(s_+(\mathcal{M} \mid W - GW_1)) \end{array}$$

By (1)(2) the composition

$$\mathcal{K} \to \mathcal{H}_0(s_!(\mathcal{M} \mid W - GW_1)) \to \mathcal{M}$$

is zero. Hence (7.4) yields a map

$$(7.5)\qquad s_{!+}(\mathcal{M} \mid W - GW_1) \to \mathcal{M}$$

whose kernel and cokernel have support in GW_1.

So by (1)(2) the cokernel of (7.4) is zero. Furthermore $s_{!+}(\mathcal{M} \mid W - GW_1)$, being a submodule of $s_*(\mathcal{M} \mid W - GW_1)$ has no submodules with support contained in GW_1. Therefore the kernel of (7.5) is also zero. $\square$

Proof of Theorem D. Let $r : G \times^B (W_\lambda - W_1) \to G \times^B W_\lambda$ and $s : W - GW_1 \to W$ be the inclusion maps and let $\pi : G \times^B W_\lambda \to W$ be given by $(g, x) \mapsto gx$.

Note that by lemma 7.3, $\pi \mid G \times^B (W_\lambda - W_1)$ is a closed immersion. Hence by Kashiwara's theorem

$$(\pi_+\pi^!\mathcal{M}) \mid W - GW_1 = \mathcal{M} \mid W - GW_1$$

By lemma 7.6(2) and corollary 7.4 we know

$$\mathrm{Ch}(\mathcal{M} \mid W - GW_1) = T^*_{W^u - GW_1}W$$

By [3, p. 260] this implies that

$$\mathrm{Ch}(\pi^!\mathcal{M} \mid G\times^B(W_\lambda - W_1)) = T^*_{G\times^B(W_\lambda-W_1)}(G\times^B(W_\lambda - W_1))$$

In particular $\pi^!\mathcal{M} \mid G\times^B (W_\lambda - W_1)$ is a coherent $\mathcal{O}_{G\times^B(W_\lambda-W_1)}$ module and since it is a $\mathcal{D}_{G\times^B(W_\lambda-W_1)}$-module, it is automatically a vector bundle.

Since, by our choice of W, $\mathrm{codim}(GW_1, W^u)\geq 2$, we find that

$$\mathcal{N} = \mathcal{H}^0(r_+(\pi^!\mathcal{M} \mid G\times^B(W_\lambda - W_1))) = r_*(\pi^!\mathcal{M} \mid G\times^B(W_\lambda - W_1))$$

is a coherent $\mathcal{O}_{G\times^B W_\lambda}$-module.

Since $\mathcal{N}\in (G,\mathcal{D}_{G\times^B W_\lambda})$-qch we have by lemma 6.2.5 that $\mathcal{N} = \widetilde{\mathcal{N}_{[e]}}$ where $\mathcal{N}_{[e]}\in(B,\mathcal{D}_{W_\lambda})$-qch is coherent as $\mathcal{O}_{W_\lambda}$-module.

Since $\mathcal{N}_{[e]}$ is in particular in $(T,\mathcal{D}_{W_\lambda})$-qch we may use Theorem 7.1 to deduce that $\mathcal{N}_{[e]} = \mathcal{O}_{W_\lambda}^{\oplus r}$. Since $\mathcal{O}_{W_\lambda}^{\oplus r}$ is also in $(B,\mathcal{D}_{W_\lambda})$-qch we also have by Theorem 6.2.4 that $\mathcal{N}_{[e]} = \mathcal{O}_{W_\lambda}^{\oplus r}$ in $(B,\mathcal{D}_{W_\lambda})$-qch.

So we obtain that $\mathcal{N} = (\mathcal{O}_{W_\lambda}^{\oplus r})\tilde{} = \mathcal{O}_{G\times^B W_\lambda}^{\oplus r}$ and hence $\mathcal{M}\mid W - GW_1 = \pi_+\mathcal{N}\mid W - GW_1 = \mathcal{H}^{\mathrm{codim}(W^u,W)}_{W^u - GW_1}(W - GW_1, \mathcal{O}_{W-GW_1})^{\oplus r} = \mathcal{L}(W^u - GW_1, W - GW_1)^{\oplus r}$

So finally by lemma 7.6(3) : $\mathcal{M} = s_{!+}(\mathcal{M}\mid W - GW_1) = s_{!+}\mathcal{L}(W^u - GW_1, W - GW_1)^{\oplus r} = \mathcal{L}(W^u, W)^{\oplus r}$ $\square$

8. Finite dimensional representations

We use the same notations as in §7, in particular $G = \mathrm{Sl}_2(\mathbb{C})$, $W = \oplus_{i=1}^e S^{d_1}\mathbb{C}^2$.

We associate to the pair (G,W) the following magic number [6][21][22] : $s = \sum_{i=1}^e s^{(d_i)}$ where

$$s^{(n)} = \begin{cases} n + (n-2) + \cdots + 1 = \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd} \\ n + (n-2) + \cdots + 2 = \dfrac{n(n+2)}{4} & \text{if } n \text{ is even} \end{cases}$$

Let $U = S^\mu\mathbb{C}^2$ be an irreducible finite dimensional G-representation, and let $\mathcal{U}$ be the corresponding trivial G-vector bundle on W.

In this section we prove the following result.

Theorem 8.1. Assume that all d_i are odd and $e\geq 2$. Then $\mathcal{D}(X,\mathcal{U})^G/(\mathcal{D}(X,\mathcal{U})\mathfrak{g})^G$ will have no two-sided ideal of finite codimension if $\mu < s-2$. If $\mu\geq s-2$ then there will be a unique such an ideal.

Proof. Let V be a finite-dimensional representation of $\mathcal{D}(X,\mathcal{U})^G/(\mathcal{D}(X,\mathcal{U})\mathfrak{g})^G$. We will show that $V=0$ if $\mu < s-2$ and otherwise that it is a direct sum of copies of a fixed simple representation of $\mathcal{D}(X,\mathcal{U})^G/(\mathcal{D}(X,\mathcal{U})\mathfrak{g})^G$.

It follows that lemma 2.3 that V is the quotient of a graded finite dimensional representation (for the natural grading on $\mathcal{D}(X,\mathcal{U})^G/(\mathcal{D}(X,\mathcal{U})\mathfrak{g})^G$). Hence we may, and we will, assume that V itself is graded.

To V corresponds a $\mathcal{D}_X(\mathcal{U})^G/(\mathcal{D}_X(\mathcal{U})\mathfrak{g})^G$-module which we denote by the same letter. Then by lemmas 6.2.3 and 6.2.6(4)

$$\mathcal{N} = \mathcal{D}_X(\mathcal{U},\mathcal{O}_X)\otimes_{\mathcal{D}_X(\mathcal{U})}\mathcal{D}_X(\mathcal{U})/\mathcal{D}_X(\mathcal{U})\mathfrak{g}\otimes_{\phi^{-1}\mathcal{D}_X(\mathcal{U})^G/(\mathcal{D}_X(\mathcal{U})\mathfrak{g})^G}\phi^{-1}V$$

lies in $(G,\mathcal{D}_X)$-qch and

$$V = (\mathcal{D}_X(\mathcal{O}_X,\mathcal{U})\otimes_{\mathcal{D}_X}\mathcal{N})^G = (U\otimes_{\mathbb{C}}\mathcal{N})^G$$

Since $\mathfrak{g}\subset\mathcal{D}_W$, the elements of $\mathfrak{g}$ may be interpreted as functions on T^*W. Denote by $V(\mathfrak{g})$ their common zeroes. Clearly

$$\mathrm{Ch}(\mathcal{N})\subset (T^*W)^u\cap V(\mathfrak{g})$$

In lemma 8.2 below we will show that $\dim((T^*W)^u\cap V(\mathfrak{g}))\leq\dim W$. Then we may invoke Theorem D to deduce $\mathcal{N} = \mathcal{L}(W^u,W)^{\oplus r}$ and hence $V = V_0^{\oplus r}$ where $V_0 = (U\otimes_{\mathbb{C}}\mathcal{L}(W^u,W))^G$. Now $\mathcal{L}(W^u,W)$ is simple by definition and from this one easily deduces that V_0 is either simple or zero.

The G-character of $\mathcal{L}(W^u,W)$ is computed in [23]. The computation is based upon the fact that under our current hypotheses $G\times^B W_\lambda\to GW_\lambda$ is a small birational map [10]. See also [6][22].

From this computation one obtains

$$(U \otimes \mathcal{L}(W^u, W))^G = 0 \iff \mu < s - 2$$

which finishes our proof. $\square$

Lemma 8.2. One always has

$$\dim((T^*W)^u \cap V(\mathfrak{g})) \leq \dim W$$

Proof. We identify T^*W with $W \oplus W^*$. Recall that $(T^*W)^u = G(W_\lambda \oplus (W^*)_\lambda)$. We suppose that all d_i are odd, since otherwise the lemma follows by simple dimension counting.

$V(\mathfrak{g})$ is G-stable and hence

$$T^*W \cap V(\mathfrak{g}) = G((W_\lambda \oplus (W^*)_\lambda) \cap V(\mathfrak{g}))$$

It therefore suffices to show that

$$\dim((W_\lambda \oplus (W^*)_\lambda) \cap V(\mathfrak{g})) \leq \dim W - 1 \tag{8.1}$$

If $v \in \mathfrak{g}$ then the corresponding function f_v on the point (x, ω_x) of T^*W is given by the contraction $\langle v_x, \omega_x \rangle$.

As before let $p : T^*W \to W$ be the projection map. A dimension count shows that to prove (8.1) it is sufficient to show that for all $x \in W_\lambda - W_1$

$$p^{-1}(x) \cap (W_\lambda \oplus (W^*)_\lambda) \cap V(\mathfrak{g}) \subsetneq x \oplus (W^*)_\lambda$$

To prove this it is enough to verify that $\langle F_x, (W^*)_\lambda \rangle \neq 0$, or equivalently $F_x \notin W_\lambda$. This follows from (7.1). $\square$

9. Simplicity of differential operators for $G = \mathrm{Sl}_2$

We follow the notations of §7. That is $G = \mathrm{Sl}_2$ and $W = \oplus_{i=1}^e S^{d_i}\mathbb{C}^2$ is a finite dimensional representation of G. In this section we give a proof of Theorem C.

Proof of Theorem C. Let $f \in \mathcal{O}(W)^G$ be such that $f(\bar{0}) = 0$. Then every point in W_f has a finite stabilizer which acts without pseudo reflections since it is contained in Sl_2. By lemma 2.4 one may localize at f and hence it follows from [19] that $(\mathcal{D}(W)^G/(\mathcal{D}(W)\mathfrak{g})_f = \mathcal{D}(W_f)^G/(\mathcal{D}(W_f)\mathfrak{g})^G \to \mathcal{D}((W/\!\!/G)_f) = \mathcal{D}(W/\!\!/G)_f$ is an isomorphism.

So $\mathcal{D}(W/\!\!/G)$ and $\mathcal{D}(W)^G/(\mathcal{D}(W)\mathfrak{g})^G$ will have the same ring of fractions and hence a non-zero two-sided ideal in $\mathcal{D}(W/\!\!/G)$ will have a non-zero inverse image in $\mathcal{D}(W)^G/(\mathcal{D}(W)\mathfrak{g})^G$. Therefore it suffices to show that $\mathcal{D}(W)^G/(\mathcal{D}(W)\mathfrak{g})^G$ is simple.

Let $I \subset \mathcal{D}(W)^G/(\mathcal{D}(W)\mathfrak{g})^G$ be a non-trivial two-sided ideal with inverse image $J \subset \mathcal{D}(W)^G$. By lemma 2.3, I and J are graded.

Since by lemma 2.4, I_f defines a two-sided ideal in $(\mathcal{D}(W)^G/(\mathcal{D}(W)\mathfrak{g})^G)_f = \mathcal{D}((W/\!\!/G)_f)$ and since by Theorem A $\mathcal{D}((W/\!\!/G)_f)$ is simple for all such f, we find that $\mathrm{gr}\, J \cap \mathcal{O}(W)^G$ has finite codimension. By interchanging the roles of W and W^* we find likewise that $\mathrm{gr}\, J \cap \mathcal{O}(W^*)$ has finite codimension.

Our hypotheses imply that the generic stabilizer of $T^*W = W \oplus W^*$ is trivial. Let $(T^*W)_{\mathrm{n.pr}}$ be the non-principal locus. Put $K = \mathcal{D}(X)J$. Then the above discussion plus Theorem B implies that

$$V(\mathrm{gr}\, K) \subset (W^u \times (W^*)^u) \cap (T^*W)_{\mathrm{n.pr}}$$

Now a trite computation shows that $(W^u \times (W^*)^u) \cap (T^*W)_{\mathrm{n.pr}}$ is the union of $(T^*W)^u$ and some irreducible components whose dimension is smaller than $\dim W$. Since by Theorem 2.2 $\mathrm{rad}(\mathrm{gr}\, K)$ is involutive, we find that $V(\mathrm{gr}\, K) \subset (T^*W)^u$ and hence I has finite codimension in $\mathcal{D}(W)^G/(\mathcal{D}(W)\mathfrak{g})^G$ which is excluded by Theorem 8.1. $\square$

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