THREE-DIMENSIONAL FLOPS AND NON-COMMUTATIVE RINGS (PRELIMINARY VERSION)

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Abstract. For Y, Y^+ three-dimensional smooth varieties related by a flop, Bondal and Orlov conjectured that the derived categories $D^b(\mathrm{coh}(Y))$ and $D^b(\mathrm{coh}(Y^+))$ are equivalent. This conjecture was recently proved by Bridgeland. Our aim in these notes is to give a partially new proof of Bridgeland's result using non-commutative rings. An advantage of the new proof is that it also covers some mild singular and higher dimensional situations (including the one in the recent paper by Chen: "Flops and Equivalences of derived Categories for Threefolds with only Gorenstein Singularities").

1. Introduction

Let k be an algebraically closed field of characteristic zero and assume that Y, Y^+ are 3-dimensional smooth varieties over k related by a flop. Bondal and Orlov conjectured in [4] (and proved in some special cases) that the derived categories $D^b(\mathrm{coh}(Y))$ and $D^b(\mathrm{coh}(Y^+))$ are equivalent. This conjecture was recently proved by Bridgeland [6]. Our aim in these notes is to give a partially new proof of Bridgeland's result. An advantage of the new proof is that it also covers some mild singular and higher dimensional situations (including the one in the recent paper by Chen [16], see Theorem C below). In the interest of full disclosure we should mention that Bridgeland's methods (and Chen's extension of these) also yield the existence of flops whereas in our method we have to assume this. Three-dimensional flops (in contrast to flips!) are very cheap however (see [20, Theorem 2.4]).

Saying that Y, Y^+ are related by a flop means that there is a diagram

(1.1)
$$\begin{tikzcd} Y \arrow[dr, "f"'] & & Y^+ \arrow[dl, "f^+"] \\ & X & \end{tikzcd}$$

of birational proper maps with fibers of dimension ≤ 1 such that there is an ample divisor D on Y with the property that if E is the strict transform of D on Y^+ then $-E$ is ample. According to [10, Prop. 16.2] X has Gorenstein terminal singularities.

Bridgeland's proof is based on the now familiar technique of Fourier-Mukai transform. A very important instance of this technique is the general approach to the McKay correspondence introduced in [7]. In that paper it is shown that if V is

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a 3-dimensional vector space and $G \subset \mathrm{Sl}(V)$ is a finite group then there is a derived equivalence between the skew group ring $k[V]*G$ and a suitable G-equivariant Hilbert scheme of V (see also [17] for the two-dimensional case).

An essential feature of the McKay correspondence is the appearance of the skew group algebra $k[V]*G$. This suggests that it should perhaps be possible to explain Bridgeland's proof in terms of non-commutative algebra.

Example. *The most trivial flop is the so-called "Atiyah flop". Let* $R = k[u,v,x,y]/(uv-xy)$ *and* $X = \operatorname{Spec} R$. *Let* I *be the reflexive ideal in* R *given by* (u,x). *The singular variety* X *has two resolutions* Y, Y^+ *which are obtained by blowing up either* I *or* $I^{-1} \cong (u,y)$. *Put*

$$A = \begin{pmatrix} R & I \\ I^{-1} & R \end{pmatrix} \tag{1.2}$$

It was observed by many people that there are derived equivalences

$$D^b(\mathrm{coh}(Y)) \cong D^b(\mathrm{mod}(A)) \cong D^b(\mathrm{coh}(Y^+))$$

So the question is how to construct an analogue of the ring A in general. Luckily Bridgeland's paper is very helpful: Bridgeland constructs a series of t-structures on $D^b(\mathrm{coh}(Y))$ indexed by a "perversity" $p \in \mathbb{Z}$ whose hearts are abelian categories denoted by ${}^p\operatorname{Per}(Y/X)$. Below we will always assume $p = 0, -1$. In that case ${}^p\operatorname{Per}(Y/X)$ is simply a "tilting" of $\mathrm{coh}(Y)$ in the sense of [13] (see §3 for precise definitions). We will show that if $X = \operatorname{Spec} R$ is affine then the category ${}^p\operatorname{Per}(Y/X)$ has a projective generator. Our desired non-commutative ring is the endomorphism ring of this generator.

In fact this part of the argument works in far greater generality. We prove the following result (this is a combination of Propositions 3.3.1 and 3.2.7)

Theorem A. *Let* $f : Y \to X$ *be a projective morphism between quasi-projective schemes over an affine scheme such that the fibers of* f *have dimension* ≤ 1 *and such that* $Rf_*\mathcal{O}_Y = \mathcal{O}_X$. *Then there exists a vector bundle* $\mathcal{P}$ *on* Y *with the following properties.*

1. *Let* $\mathcal{A} = f_*\mathcal{E}nd(\mathcal{P})$. *The functors* $Rf_*\,\mathrm{R}\mathcal{H}om_Y(\mathcal{P},-)$ *and* $f^{-1}(-)\overset{L}{\otimes}_{f^{-1}(\mathcal{A})}\mathcal{P}$ *define inverse equivalences between* $D^b(\mathrm{coh}(Y))$ *and* $D^b(\mathrm{coh}(\mathcal{A}))$. *These equivalences restrict to equivalences between* ${}^{-1}\operatorname{Per}(Y/X)$ *and* $\mathrm{coh}(\mathcal{A})$.
2. *The functors* $Rf_*\,\mathrm{R}\mathcal{H}om_Y(\mathcal{P}^*,-)$ *and* $f^{-1}(-)\overset{L}{\otimes}_{f^{-1}(\mathcal{A}^\circ)}\mathcal{P}^*$ *define inverse equivalences between* $D^b(\mathrm{coh}(Y))$ *and* $D^b(\mathrm{coh}(\mathcal{A}^\circ))$. *These equivalences restrict to equivalences between* ${}^0\operatorname{Per}(Y/X)$ *and* $\mathrm{coh}(\mathcal{A}^\circ)$.

The vector bundle $\mathcal{P}$ which occurs in the above theorem is a so-called *local projective generator* for ${}^{-1}\operatorname{Per}(Y/X)$. I.e. an object in ${}^{-1}\operatorname{Per}(Y/X)$ such that there exists an affine covering of X: $X = \bigcup_i U_i$ with the property that for all i: $\mathcal{P} \mid U_i$ is a projective generator for ${}^{-1}\operatorname{Per}(f^{-1}(U_i)/U_i)$.

Now let us return to the situation of diagram (1.1). The hypotheses on the dimension of the fibers yield that the categories of reflexive sheaves on Y, Y^+ and X are equivalent. Choose a local projective generator $\mathcal{P}$ for ${}^{-1}\operatorname{Per}(Y/X)$. Under the above equivalences we find a corresponding sheaf of reflexive modules $\mathcal{Q}^+$ on Y^+. Consulting Bridgeland's paper for inspiration we note that Bridgeland shows that ${}^{-1}\operatorname{Per}(Y/X)$ is equivalent to ${}^0\operatorname{Per}(Y/X)$. So this suggests we should prove that $\mathcal{Q}^+$ is a local projective generator for ${}^0\operatorname{Per}(Y^+/X)$. Indeed if this is the case then by using Theorem A for Y and Y^+ we find

$$D^b(\mathrm{coh}(Y)) \cong D^b(\mathrm{coh}(\mathcal{A})) \cong D^b(\mathrm{coh}(Y^+))$$

where $\mathcal{A} = f_*\mathcal{E}nd_Y(\mathcal{P}) = f_*\mathcal{E}nd_Y(\mathcal{Q}^+)$.

We show that the property of being a local projective generator may be verified in the completions of the closed points of X. Hence we may reduce to the case where X is the spectrum of a complete local ring. In that case we can invoke the methods of Artin and Verdier [1] to give a precise classification of the projective objects in ${}^p\operatorname{Per}(Y/X)$. Again this works in greater generality. Our result is the following (this is Theorem 3.5.5)

Theorem B. *Assume that we are in the situation of Theorem A and assume in addition that X is the spectrum of a complete local k algebra with residue field equal to k. Let $(C_i)_i$ be the irreducible components of the special fiber of f. Then the map $\mathcal{L} \mapsto \deg(\mathcal{L} \mid C_i)_{i=1,\dots,n}$ defines an isomorphism $\operatorname{Pic}(Y) \cong \mathbb{Z}^n$. Let $\mathcal{L}_i \in \operatorname{Pic}(Y)$ be such that $\mathcal{L}_i \mid C_j = \delta_{ij}$. Define $\mathcal{M}_i$ as the middle term of the exact sequence*

$$0 \to \mathcal{O}_Y^{r-1} \to \mathcal{M}_i \to \mathcal{L}_i \to 0$$

associated to a minimal set of generators of $H^1(Y, \mathcal{L}_i^{-1})$ as R-module. Then the indecomposable projective objects in ${}^{-1}\operatorname{Per}(Y/X)$ are precisely the objects $(\mathcal{M}_i)_{i=1,\dots,n}$ and $\mathcal{O}_Y$. The projective generators of ${}^{-1}\operatorname{Per}(Y/X)$ are $\mathcal{O}_Y^{\oplus a_0} \oplus \bigoplus_{i=1,\dots,n} \mathcal{M}_i^{\oplus a_i}$ with $a_i > 0$ for all i. Analogous results hold for ${}^0\operatorname{Per}(Y/X)$ if we replace $\mathcal{M}_i$ by $\mathcal{M}_i^*$.

We now return to the situation of diagram 1.1 but we assume that we are in the formal case, i.e. $X = \operatorname{Spec} R$ where R is a complete local ring. We will adorn notations with refer to f^+ by a superscript "+".

In the formal case there is a trivial description of Y^+ [20]. Recall that Gorenstein terminal singularities are hypersurface singularities of multiplicity two. Hence the equation of R may be written as $u^2 + \cdots = 0$. It follows that $u \mapsto -u$ defines an automorphism σ of X of order two. We may take $Y^+ = Y$ and $f^+ = \sigma \circ f$. We put $C_i^+ = C_i$.

From Theorem B, and the discussion preceeding it, we see that we are done if we can show that $\mathcal{M}_i$ corresponds to $(\mathcal{M}_i^+)^*$ under the equivalence of the categories of reflexive modules on Y, Y^+ and X.

Let $I_i = \Gamma(Y, \mathcal{L}_i)$. It is easy to see that I_i is a reflexive R-module of rank one Put $M_i = \Gamma(Y, \mathcal{M}_i)$. We show that the M_i are indecomposable Cohen-Macaulay modules on R which occur as middle term of an exact sequence

$$0 \to R^{r_i-1} \to M_i \to I_i \to 0$$

associated to a minimal set of generators of $\operatorname{Ext}^1_R(I_i, R)$ as R-module. In addition we also show that there are exact sequences

$$0 \to M_i^* \to R^{r_i+1} \to I_i \to 0 \tag{1.3}$$

Let $\operatorname{Cl}(R)$ be the divisor class group of R. The automorphism σ induces the operation $I \mapsto I^{-1}$ on $\operatorname{Cl}(R)$ and from this we deduce $\Gamma(Y^+, \mathcal{L}_i^+) = I_i^{-1}$. Thus M_i^+ occurs as middle term of an exact sequence

$$0 \to R^{r_i^+-1} \to M_i^+ \to I_i^{-1} \to 0 \tag{1.4}$$

associated to a minimal set of generators of $\operatorname{Ext}^1_R(I_i^{-1}, R)$ as R-module.

We show in §4.1 that the Cohen-Macaulay modules defined by (1.3) and (1.4) are the same. This finishes the proof.

As indicated above, our methods are valid in a somewhat more general setting. The precise result we prove is the following (this is Theorem 4.4.2)

Theorem C. *Let $f : Y \to X$ be a projective birational map between normal quasi-projective Gorenstein k-varieties of dimension $n \geq 3$ with fibers of dimension ≤ 1 and assume that the exceptional locus of f has codimension ≥ 2 in Y. Assume that X has canonical hypersurface singularities of multiplicity ≤ 2. Let $f^+ : Y^+ \to X$ be the flop of f. Then $D^b(\operatorname{coh}(Y))$ and $D^b(\operatorname{coh}(Y^+))$ are equivalent and we may choose this equivalence in such a way that ${}^{-1}\operatorname{Per}(Y/X)$ corresponds to ${}^{0}\operatorname{Per}(Y^+/X)$.*

To finish this introduction we make some remarks of a more philosophical nature. If we return to the McKay correspondence then we see that the singular variety $V/G = \operatorname{Spec} k[V]^G$ has two crepant "resolutions", a commutative one given by the G-equivariant Hilbert scheme and a non-commutative one given by the skew group ring $k[V]*G$. Both resolutions are derived equivalent, so in some sense it doesn't matter which one we take.

If $\dim V \geq 4$ then there are examples where V/G does not have a commutative crepant resolution. However the non-commutative resolution given by $k[V]*G$ of course always exists. So it seems that at least in some situations non-commutative resolutions are strictly more general than commutative ones.

An obvious question is whether something similar is true for a variety X with three-dimensional terminal Gorenstein singularities. I.e. does there always exists a "crepant" non-commutative resolution of X? In Appendix A I give a counter example that shows that this is not the case. I actually think that for three-dimensional terminal Gorenstein singularities the existence of commutative and non-commutative crepant resolutions are equivalent. This can presumably be proved with the same Fourier-Mukai method which was used to establish the three-dimensional McKay correspondence.

Most of the results in these notes were conceived during the 2002 OberWolfach meeting suitably called "Interactions between Algebraic Geometry and Noncommutative Algebra". During that meeting, and also at other times, the author has greatly benefited from discussions with Alexei Bondal, Tom Bridgeland, Alastair King and Aidan Schofield.

2. Notations and conventions

k will always be an algebraically closed field of characteristic zero. Schemes are *not* supposed to be k-schemes, unless this is explicitly stated. The characteristic zero hypotheses on k could be avoided at the cost of more technicality in the statement of some results (i.e. those that are related to rational singularities).

If A is a noetherian ring then $\operatorname{mod}(A)$ is the category of finitely generated right A-modules and $D^b(A) = D^b(\operatorname{mod}(A))$. If X is a noetherian scheme then $\operatorname{Qch}(X)$ and $\operatorname{coh}(X)$ are the categories of quasi-coherent and coherent $\mathcal{O}_X$-modules. If $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$ algebras then $\operatorname{coh}(\mathcal{A})$ is the category of right coherent $\mathcal{A}$-modules.

If X is a scheme and D is a closed subscheme of X then $\mathcal{O}_X(-D)$ denotes the ideal sheaf of D. If $\mathcal{O}_X(-D)$ is invertible then $\mathcal{O}_X(nD) = \mathcal{O}_X(D)^{\otimes n}$.

All Cohen-Macaulay modules in these notes are maximal Cohen-Macaulay.

3. Acyclic morphisms with one dimensional fibers

3.1. **Generalities.** Let $f : Y \to X$ be a projective map between noetherian schemes We impose the following conditions:

(1) $Rf_*\mathcal{O}_Y = \mathcal{O}_X$.
(2) The fibers of f are one-dimensional.

At this stage we do not assume that f is birational. Note however that by [15, Cor. II.11.3] the fibers of f are connected.

We consider certain categories of perverse coherent sheaves introduced by Bridgeland in [6]. We let $\mathcal{C}$ be the abelian(!) subcategory of $\operatorname{coh}(Y)$ consisting of objects E such that $Rf_*E = 0$ (this is a deviation of the notation used by Bridgeland). The following lemma was proved by Bridgeland.

Lemma 3.1.1. *For $F \in D^b(\operatorname{coh}(Y))$ one has $Rf_*F = 0$ if and only if $H^i(F) \in \mathcal{C}$ for all i.*

We define the following torsion theories on $\operatorname{coh}(Y)$.

$$\mathcal{T}_{-1} = \{T \in \operatorname{coh}(Y) \mid R^1f_*(T) = 0, \operatorname{Hom}(T, \mathcal{C}) = 0\}$$
$$\mathcal{F}_{-1} = \{F \in \operatorname{coh}(Y) \mid f_*(F) = 0\}$$

$$\mathcal{T}_0 = \{T \in \operatorname{coh}(Y) \mid R^1f_*(T) = 0\}$$
$$\mathcal{F}_0 = \{F \in \operatorname{coh}(Y) \mid f_*(F) = 0, \operatorname{Hom}(\mathcal{C}, F) = 0\}$$

Then on $D = D^b(\operatorname{coh}(Y))$ we consider the associated perverse t-structures.

$${}^pD^{\le 0} = \{E \in D^{\le 0} \mid H^0(E) \in \mathcal{T}_p\}$$
$${}^pD^{\ge 0} = \{E \in D^{\ge -1} \mid H^{-1}(E) \in \mathcal{F}_p\}$$

for $p = -1, 0$.

We denote the heart of these t-structures by ${}^p\operatorname{Per}(Y/X)$. I.e ${}^p\operatorname{Per}(Y/X)$ consists of the objects E in $D^b(\operatorname{coh}(Y))$ whose only homology lies in degree $-1, 0$ such that $H^{-1}(E) \in \mathcal{F}_p$ and $H^0(E) \in \mathcal{T}_p$.

It is not entirely clear that the categories $\mathcal{T}_{-1}$ and $\mathcal{F}_0$ are compatible with restriction since their definition involves the functor Hom instead of $\mathcal{H}om$. This defect is taken care of by the next two lemmas.

Lemma 3.1.2. *The objects in $\mathcal{T}_{-1}$ are precisely the coherent sheaves T such that the map $f^*f_*T \to T$ is surjective.*

Proof. Let $T \in \operatorname{coh}(Y)$ be such that $R^1f_*T = 0$ and let T_0 be the image of $f^*f_*T \to T$. We claim $T/T_0 \in \mathcal{C}$. Hence if $T \in \mathcal{T}_1$ then $T = T_0$ and thus the canonical map $f_*f^*T \to T$ is surjective.

We prove the claim. It is clear that $R^1f_*(T/T_0) = 0$ and that $f_*(T) \to f_*(T/T_0)$ is surjective. So if $T/T_0 \notin \mathcal{C}$ then $f_*(T/T_0) \neq 0$. This means $f^*f_*(T/T_0) \to T/T_0$ is not the zero map. Hence the composition $\phi : f^*f_*T \to f^*f_*(T/T_0) \to T/T_0$ is also not the zero map (since the first map is surjective). Now ϕ is also the composition of $f^*f_*T \to T \to T/T_0$ which was zero by hypotheses. This contradiction finishes the proof. ☐

To obtain an analogous statement for $\mathcal{F}_0$ we recall that $Rf_* : D(\operatorname{Qch}(Y)) \to D(\operatorname{Qch}(X))$ has a right adjoint $f^!$ [22, 19]. The identity $Rf_*Lf^* = \mathrm{id}$ formally implies $Rf_*f^! = \mathrm{id}$. The explicit formulas for $f^!$ in [14] show that $f^!$ maps $D(\operatorname{coh}(X))^{\ge 0}$ to $D(\operatorname{coh}(Y))^{\ge -1}$.

If $E \in \operatorname{coh}(Y)$ then the composition $E \to f^!Rf_*E \to f^!((R^1f_*E)[-1])$ yields a canonical map $\phi_E : E \to H^{-1}(f^!R^1f_*E)$.

Lemma 3.1.3. *The objects in $\mathcal{F}_0$ are precisely the coherent sheaves F such that ϕ_F is injective.*

Proof. Assume first that ϕ_F is injective. If $E \in \operatorname{coh}(X)$ then the fact that $Rf_*f^!E = E$ implies $f_*H^{-1}(f^!E) = 0$. Hence if ϕ_F is injective then $f_*F = 0$. Thus ϕ_F is the map $F \to H^0(f^!Rf_*F)$ obtained by applying H^0 to the map $F \to f^!Rf_*F$ given by adjointness. Assume that there is a non-zero homomorphism $C \to F$ for $C \in \mathcal{C}$. This yields a non-zero homomorphism $C \to H^0(f^!Rf_*F)$ and hence an non-zero homomorphism $C \to f^!Rf_*F$, but this is impossible by adjointness.

Now assume that $F \in \mathcal{F}_0$. Since $f_*F = 0$ the map ϕ_X is again the map $F \to H^0(f^!Rf_*F)$ obtained by applying H^0 to the map given by adjointness. Let U be the cone of $F \to f^!Rf_*F$. We have $Rf_*U = 0$ and hence by lemma 3.1.1 the homology of U lies in $\mathcal{C}$. Looking at the homology exact sequence we obtain

$$0 \to H^{-1}(U) \to F \xrightarrow{\phi_X} H^0(f^!Rf_*F)$$

Since $H^{-1}(U) \in \mathcal{C}$ it follows that $H^{-1}(U) = 0$ and hence that ϕ_X is injective. $\square$

We obtain the following consequence

Proposition 3.1.4. *The categories $\mathcal{T}_p$, $\mathcal{F}_p$, ${}^pD^{\geq 0}$, ${}^pD^{\leq 0}$, ${}^p\operatorname{Per}(Y/X)$ are compatible with restriction (and more generally flat base change). Furthermore, membership of these categories may be checked locally (even for the flat topology).*

One may also check that if we put for $U \subset X$ open, ${}^p\operatorname{Per}_{Y/X}(U) = {}^p\operatorname{Per}(f^{-1}(U)/U)$ then ${}^p\operatorname{Per}_{Y/X}$ is a stack of abelian categories with exact restriction functors.

3.2. **The case when the base is affine.** First we assume that $X = \operatorname{Spec} R$ with R a noetherian ring. Our aim is to show that ${}^p\operatorname{Per}(Y/X)$ is a module category. We will use the following version of Morita theory.

Lemma 3.2.1. *Let R be a noetherian commutative ring and $\mathcal{C}$ an R-linear category such that for $A, B \in \mathcal{C}$ we have that $\operatorname{Hom}_{\mathcal{C}}(A, B)$ is a finitely generated R-module. Assume that $\mathcal{P}$ is a projective object in $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(\mathcal{P}, E) = 0$ implies $E = 0$. Put $A = \operatorname{End}_{\mathcal{C}}(\mathcal{P})$. Then the functor $\operatorname{Hom}_{\mathcal{C}}(\mathcal{P}, -)$ defines an equivalence between $\mathcal{C}$ and $\operatorname{mod}(A)$ whose inverse is given by $- \otimes_A \mathcal{P}$ (with obvious notations).*

We will call an object P as in the statement of this lemma a *projective generator* for $\mathcal{C}$.

We use the following result.

Lemma 3.2.2. *Let $f : Y \to X$ be a projective morphism between noetherian schemes with fibers of dimension $\leq n$. Assume that X is affine. Let $\mathcal{L}$ be an ample line bundle generated by global sections and let $a \in \mathbb{Z}$.*

If M in $D^b(\operatorname{coh}(Y))$ is such that $\operatorname{Ext}^i_Y(\mathcal{L}^{a+j}, M) = 0$ for all i and for $0 \leq j \leq n$ then $M = 0$.

Proof. This is basically [5, Lemma 4.2.4]. For the convenience of the reader we repeat the proof in the current setting. Without loss of generality we may assume $a = -n$.

Use $\mathcal{L}$ to construct a finite map $Y \to \mathbb{P}^N_X$. The Koszul complex of a polynomial ring in $N+1$ variables leads to a long exact sequence on $\mathbb{P}^N_X$:

$$0 \to \mathcal{O}_{\mathbb{P}^N_X}(-N-1) \to \cdots \to \mathcal{O}_{\mathbb{P}^N_X}(-u)^{\binom{N+1}{u}} \to \cdots \to \mathcal{O}_{\mathbb{P}^N_X} \to 0$$

The inverse image on Y yields an exact sequence

$$0 \to \mathcal{L}^{-N-1} \to \cdots \to (\mathcal{L}^{-u})^{\binom{N+1}{u}} \to \cdots \to \mathcal{O}_Y \to 0$$

Let U be the kernel at $(\mathcal{L}^{-n-1})^{\binom{N+1}{n+1}}$. Then the long exact sequence

$$0 \to U \to (\mathcal{L}^{-n-1})^{\binom{N+1}{n+1}} \to \cdots \to (\mathcal{L}^{-1})^{N+1} \to \mathcal{O}_Y \to 0$$

represents an element of $\operatorname{Ext}^{n+1}_Y(\mathcal{O}_Y, U)$ which must be zero. It follows that $\mathcal{O}_Y$ is a direct summand in $D^b(\operatorname{coh}(Y))$ of the complex

$$(\mathcal{L}^{-n-1})^{\binom{N+1}{n+1}} \to \cdots \to (\mathcal{L}^{-1})^{N+1}$$

Dualizing and tensoring with $\mathcal{L}^{-n-p-1}$ for $p \geq 0$ we deduce that $\mathcal{L}^{-n-p-1}$ is a direct summand of

$$\mathcal{L}^{-n-p} \to \cdots \to (\mathcal{L}^{-p})^{\binom{N+1}{n+1}}$$

By induction on p we now easily deduce from the hypotheses that $\operatorname{Ext}^i_Y(\mathcal{L}^j, M) =$ for all i and $j \leq 0$. Applying this with $i = 0$ and using the fact that $\mathcal{L}$ is ample this implies $M = 0$. □

Now we revert to our blanket assumptions (but we assume $X = \operatorname{Spec} R$ is affine).

Let $\mathfrak{V}$ be the category of vector bundles $\mathcal{M}$ on Y generated by global sections such that $H^1(Y, \mathcal{M}^*) = 0$ and let $\mathfrak{V}^* = \{\mathcal{M}^* \mid \mathcal{M} \in \mathfrak{V}\}$.

Lemma 3.2.3. *If $\mathcal{M} \in \mathfrak{V}$ then for all $E \in {}^{-1}\operatorname{Per}(Y/X)$ and for all $i > 0$ we have $\operatorname{Ext}^i_Y(\mathcal{M}, E) = 0$. In particular $\mathcal{M}$ is a projective object in ${}^{-1}\operatorname{Per}(Y/X)$.*

Proof. By lemma 3.1.2 we know that $\mathcal{M} \in \mathcal{T}_{-1}$. It is also clear that if $F \in \mathcal{F}_{-1}$ then for $i > 0$: $\operatorname{Ext}^i_Y(\mathcal{M}, F[1]) = \operatorname{Ext}^{i+1}_Y(\mathcal{M}, F) = 0$.

So we need to show that $\operatorname{Ext}^i_Y(\mathcal{M}, T) = 0$ for $T \in \mathcal{T}_{-1}$ and for $i > 0$. The case $i > 1$ follows from the hypotheses on the fibers of f so we assume $i = 1$. Since by lemma 3.1.2 T is generated by global sections, T is a quotient of some $\mathcal{O}^l_Y$. Let $\mathcal{K}$ be the kernel. Then $\operatorname{Ext}^1_Y(\mathcal{M}, T) = \operatorname{Ext}^2_Y(\mathcal{M}, \mathcal{K}) = 0$. Hence we are done. □

Lemma 3.2.4. *If $\mathcal{N} \in \mathfrak{V}^*$ then for all $E \in {}^{0}\operatorname{Per}(Y/X)$ and for all $i > 0$ we have $\operatorname{Ext}^i_Y(\mathcal{N}, E) = 0$. In particular $\mathcal{N}$ is a projective object in ${}^{0}\operatorname{Per}(Y/X)$.*

Proof. The fact that $H^1(Y, \mathcal{N}) = 0$ implies that $\mathcal{N} \in \mathcal{T}_0$. It is also clear that if $F \in \mathcal{F}_{-1}$ then for $i > 0$: $\operatorname{Ext}^i_Y(\mathcal{M}, F[1]) = \operatorname{Ext}^{i+1}_Y(\mathcal{M}, F) = 0$. So we need to show that $\operatorname{Ext}^i_Y(\mathcal{N}, T) = 0$ for $T \in \mathcal{T}_0$ and for $i > 0$. The case $i > 1$ follows from the hypotheses on the fibers of f so we assume $i = 1$. Now $\operatorname{Ext}^1_Y(\mathcal{N}, T) = H^1(Y, \mathcal{N}^* \otimes_Y T)$. Since $\mathcal{N}^*$ is generated by global sections it follows that $\mathcal{N}^* \otimes_Y T$ is a quotient of a number of copies of T. Hence $\mathcal{N}^* \otimes T$ has vanishing cohomology. □

The following is our main result.

Proposition 3.2.5. *There exists a vector bundle $\mathcal{P} \in \mathfrak{V}$ which is a projective generator in ${}^{-1}\operatorname{Per}(Y/X)$ and whose dual $\mathcal{P}^*$ is a projective generator in ${}^{0}\operatorname{Per}(Y/X)$.*

Proof. Let $\mathcal{L}$ be an ample line bundle on Y generated by global sections.

Let $\mathcal{P}_0$ be given by the extension

$$0 \to \mathcal{O}^{r-1}_Y \to \mathcal{P}_0 \to \mathcal{L} \to 0 \tag{3.1}$$

associated to a set of generators of $H^1(Y, \mathcal{L}^{-1})$ as R-module and put $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{O}_Y$.

Clearly $\mathcal{P}_0 \in \mathfrak{V}$ and hence by lemma 3.2.3: $\operatorname{Ext}^i_Y(\mathcal{P}, -) = 0$ for $i \neq 0$. Hence we only need to show that $\mathcal{P}$ is a generator.

Assume that $E \in D^b(\operatorname{coh}(Y))$ is such that $\operatorname{Ext}^i_Y(\mathcal{P}, E) = 0$ for all i. Then we deduce from this that $\operatorname{Ext}^i_Y(\mathcal{O}_X, E) = 0$ for all i and $\operatorname{Ext}^i_Y(\mathcal{L}, E) = 0$ for all i. The conclusion now follows from lemma 3.2.2.

The proof for $p = 0$ is similar. □

Now we give a more detailed discussion of the projective objects and the projective generators in ${}^p\operatorname{Per}(Y/X)$.

Proposition 3.2.6. *The projective objects in ${}^{-1}\operatorname{Per}(Y/X)$ are precisely the objects in $\mathfrak{V}$. The projective objects in ${}^0\operatorname{Per}(Y/X)$ are precisely the objects in $\mathfrak{V}^*$.*

Proof. Assume $p = -1$. By lemma 3.2.3 we already know that the objects in $\mathfrak{V}$ are projective. Since $\mathfrak{V}$ is clearly closed under direct sums and direct summands, the converse follows from the fact that $\mathfrak{V}$ contains a projective generator for ${}^{-1}\operatorname{Per}(Y/X)$. The case $p = 0$ is identical. □

If $\mathcal{M}$ is a vector bundle of rank r on Y then by $c_1(\mathcal{M})$ we denote the class of $\wedge^r \mathcal{M}$ in $\operatorname{Pic}(Y)$.

Proposition 3.2.7. *The projective generators in ${}^{-1}\operatorname{Per}(Y/X)$ are the objects $\mathcal{M}$ in $\mathfrak{V}$ such that $c_1(\mathcal{M})$ is ample and such that $\mathcal{O}_Y$ is a direct summand of some $\mathcal{M}^{\oplus a}$. The projective generators in ${}^0\operatorname{Per}(Y/X)$ are the objects in $\mathfrak{V}^*$ which are dual to projective generators in ${}^{-1}\operatorname{Per}(Y/X)$*

Proof. Assume $p = -1$. We first prove that every projective generator is of the form stated. Let $\mathcal{M}$ be a projective generator of ${}^{-1}\operatorname{Per}(Y/X)$. Then since $\mathcal{O}_Y$ is a projective object in ${}^{-1}\operatorname{Per}(Y/X)$, $\mathcal{O}_Y$ must be a direct summand of some $\mathcal{M}^{\oplus a}$.

Let $\mathcal{P}$ be the projective generator constructed in the proof of Proposition 3.2.5. Then there exists a $b \in \mathbb{N}$ such that $\mathcal{M}^{\oplus b} = \mathcal{P} \oplus \mathcal{P}'$ with $\mathcal{P}' \in \mathfrak{V}$. Hence $c_1(\mathcal{M})^b = c_1(\mathcal{P})c_1(\mathcal{P}')$. Since $c_1(\mathcal{P})$ is ample by construction and $c_1(\mathcal{P}')$ is generated by global sections we deduce from this that $c_1(\mathcal{M})$ is ample.

Now we prove the converse. Let $\mathcal{M}$ be as in the statement of the proposition. By Proposition 3.2.6 we already know that $\mathcal{M}$ is projective. So we only have to show that $\mathcal{M}$ is a generator.

As in the proof of Propositions 3.2.5 we will show that if $E \in D^b(\operatorname{coh}(Y)) = 0$ is such that if $\operatorname{Ext}^i_Y(\mathcal{M}, E) = 0$ for all i then $E = 0$. Looking at the appropriate spectral sequence we see that if $\operatorname{Ext}^i_Y(\mathcal{M}, E) = 0$ for all i then $\operatorname{Ext}^i_Y(\mathcal{M}, H^j(E)) = 0$ for all i, j (this is similar to lemma 3.1.1). Thus without loss of generality we may assume $E \in \operatorname{coh}(Y)$. To prove that $E = 0$ it is sufficient to prove that for every closed point x in X with defining ideal m we have $E/mE = 0$. Consider mE. Since this is a subobject of E we have $\operatorname{Hom}_Y(\mathcal{M}, mE) = 0$. However it is also a quotient of some $E^{\oplus c}$ so we have in addition $\operatorname{Ext}^1_Y(\mathcal{M}, mE) = 0$. Thus $\operatorname{Ext}^i_Y(\mathcal{M}, E/mE) = 0$ for all i. Let C be the fiber of x. We have $0 = \operatorname{Ext}^i_Y(\mathcal{M}, E/mE) = \operatorname{Ext}^i_C(\mathcal{M}/m\mathcal{M}, E/mE)$. Hence we are now reduced to the case where X is the spectrum of a field and without loss of generality we may assume that this field is algebraically closed. According to lemma 3.5.1 below we now have an exact sequence

$$0 \to \mathcal{O}_Y^{r-1} \to \mathcal{M} \to \mathcal{L} \to 0$$

with $\mathcal{L} = c_1(\mathcal{M})$ ample and generated by global sections. Since $\mathcal{O}_Y$ is a direct summand of some $\mathcal{M}^{\oplus a}$ we also have $\operatorname{Ext}^i_Y(\mathcal{O}_Y, E) = 0$ for all i. We now conclude as in the proof of Proposition 3.2.5 that $E = 0$.

The case $p = 0$ is similar. $\square$

Now we employ our projective generators to construct a derived equivalence.

Corollary 3.2.8. *Assume that $\mathcal{P}$ is a projective generator for ${}^p\operatorname{Per}(Y/X)$. Put $A = \operatorname{End}_Y(\mathcal{P})$ and write ${}_A\mathcal{P}$ to emphasize the left A-structure on $\mathcal{P}$ Then the functors $\operatorname{RHom}_Y({}_A\mathcal{P}, -)$ and $- \overset{L}{\otimes}_A {}_A\mathcal{P}$ define inverse equivalences between $D^b(\operatorname{coh}(Y))$ and $D^b(A)$. These equivalences restrict to equivalences between ${}^p\operatorname{Per}(Y/X)$ and $\operatorname{mod}(A)$.*

Proof. It is sufficient to show that the indicated functors define equivalences between ${}^p\operatorname{Per}(Y/X)$ and $\operatorname{mod}(A)$ since the statement about derived categories then follows by induction over triangles.

Since we compute $\operatorname{RHom}_Y({}_A\mathcal{P}, -)$ with injective resolutions in the second argument, we have that the composition of $\operatorname{RHom}_Y({}_A\mathcal{P}, -)$ with the forgetful functor $D^b(A) \to D^b(R)$ coincides with $\operatorname{RHom}_Y({}_R\mathcal{P}, -)$. So below we ignore the difference between these functors.

If $E \in {}^p\operatorname{Per}(Y/X)$ then by lemmas 3.2.3 and 3.2.4 and Proposition 3.2.6 we have $\operatorname{Ext}^i_Y(\mathcal{P}, E) = 0$ for $i \neq 0$. Hence on ${}^p\operatorname{Per}(Y/X)$ we have $\operatorname{RHom}_Y(\mathcal{P}, -) = \operatorname{Hom}_Y(\mathcal{P}, -)$ and the latter sends ${}^p\operatorname{Per}(Y/X)$ to $\operatorname{mod}(R)$.

Now we consider $- \overset{L}{\otimes}_A {}_A\mathcal{P}$. Let $M \in \operatorname{mod}(A)$. Replacing M by a projective resolution and using the fact that $\mathcal{P} \in \mathcal{T}_p$ we easily deduce that for $i \neq 0$: ${}^pH^i(M \overset{L}{\otimes}_A \mathcal{P}) = 0$ and ${}^pH^0(M \overset{L}{\otimes}_A \mathcal{P})$ coincides with the functor which would be denoted by $M \otimes_A \mathcal{P}$ in the notation of lemma 3.2.1. Thus $- \overset{L}{\otimes}_A {}_A\mathcal{P}$ restricted to $\operatorname{mod}(A)$ coincides with $- \otimes_A \mathcal{P}$. Since $\operatorname{Hom}_Y(\mathcal{P}, -)$ and $- \otimes_A \mathcal{P}$ define inverse equivalences between ${}^p\operatorname{Per}(Y/X)$ and $\operatorname{mod}(A)$ we are done my lemma 3.2.1 $\square$

As a final result in this section we show that in favourable cases there is a relation between the projective objects in ${}^p\operatorname{Per}(Y/X)$ and R-Cohen-Macaulay modules.

Lemma 3.2.9. *Assume that R is finitely generated over a field or else that it is a complete local ring containing a copy of its residue field. Besides our blanket hypotheses assume in addition that X and Y are Gorenstein of pure dimension n and that f is birational and crepant (meaning: $f^*\omega_X = \omega_Y$). If $\mathcal{M}$ is a vector bundle on Y then for every maximal ideal of R we have $\operatorname{depth}_m \Gamma(Y, \mathcal{M}) \geq n - 1$. If in addition $H^1(Y, \mathcal{M}) = H^1(Y, \mathcal{M}^*) = 0$ then $\Gamma(Y, \mathcal{M})$ is a Cohen-Macaulay R-module.*

Proof. Let $\mathcal{D}_X, \mathcal{D}_Y$ be the Grothendieck dualizing complexes on X and Y. One has $\mathcal{D}_Y = f^!\mathcal{D}_X$ and hence by the Gorenstein/dimension hypotheses $\omega_Y = \mathcal{D}_Y[-n] = f^!\mathcal{D}_X[-n] = f^!\omega_X = f^!\mathcal{O}_X \otimes_Y f^*\omega_X$. Using the crepant hypothesis we obtain $\mathcal{O}_Y = f^!\mathcal{O}_X$. We compute for $\mathcal{M} \in \operatorname{coh}(Y)$: $\operatorname{Ext}^i_Y(\mathcal{M}, \mathcal{O}_Y) = \operatorname{Ext}^i_Y(\mathcal{M}, f^!\mathcal{O}_X) = \operatorname{Ext}^i_R(R\Gamma(Y, \mathcal{M}), R)$. From this one easily deduces the assertions about $\Gamma(Y, \mathcal{M})$. $\square$

Proposition 3.2.10. *Let the hypotheses be as in lemma 3.2.9 and assume in addition that X is normal. Then there exists a Cohen-Macaulay module M over R*

such that if $A = \mathrm{End}_R(M)$ then $D^b(\mathrm{coh}(X))$ is derived equivalent to $D^b(A)$. In addition A itself is Cohen-Macaulay.

Proof. Let $\mathcal{M}$ be a projective generator for ${}^{-1}\operatorname{Per}(Y/X)$. We have $H^1(Y, \mathcal{M}^*) = 0$ by definition and since $\mathcal{M}$ is generated by global sections we also have $H^1(Y, \mathcal{M}) = 0$. So $M = \Gamma(Y, \mathcal{M})$ is Cohen-Macaulay by lemma 3.2.9. Let $\mathcal{A} = \mathcal{E}nd_Y(\mathcal{M})$. As a vectorbundle $\mathcal{A}$ is self dual. We have $\mathrm{Ext}^i_Y(\mathcal{M}, \mathcal{M}) = H^i(Y, \mathcal{A})$. Hence by lemmas 3.2.3 and 3.2.9 and Proposition 3.2.6 $A = \Gamma(Y, \mathcal{A})$ is Cohen-Macaulay, and hence reflexive since $\dim R \geq 2$.

There is an obvious map $A \to \mathrm{End}_R(M)$. This is an isomorphism outside the exeptional locus in X and this locus has codimension ≥ 2 in X. Since both objects are reflexive as R-modules this implies that in fact $A = \mathrm{End}_R(M)$. $\square$

Let R be as in lemma 3.2.9. If A is an R-algebra which is finite as an R-module then A is said to be homologically homogeneous if all simple A modules have the same projective dimension (necessarily n). It is shown in [9] that this implies that A has finite global dimension and is Cohen-Macaulay. In fact the converse is also true and we leave it as a pleasant exercise in homological algebra for the reader to check this.

The homological behaviour of homologically homogeneous rings closely ressembles that of commutative rings of finite global dimension [9].

Corollary 3.2.11. *Let the hypotheses be as in Proposition 3.2.10 but assume in addition that Y is regular. Then there is a Cohen-Macaulay R-module M such that $D^b(\mathrm{coh}(Y))$ is equivalent to $D^b(A)$ and such that $A = \mathrm{End}_R(M)$ homologically homogeneous,*

Proof. We already know that A is Cohen-Macaulay. The fact that $D^b(\mathrm{coh}(Y))$ is equivalent to $D^b(A)$ implies that A has finite global dimension. Hence by the above discussion A is homologically homogeneous. $\square$

3.3. **General base.** Now we assume that X is a general noetherian scheme. We call an object $\mathcal{P}$ in ${}^p\operatorname{Per}(Y/X)$ a *local projective generator* if there exists an open affine covering $X = \bigcup_i U_i$ of X such that for all i $\mathcal{P} \mid f^{-1}(U_i)$ is a projective generator in ${}^p\operatorname{Per}(f^{-1}(U_i)/U_i)$.

Proposition 3.3.1. *Assume that $\mathcal{P}$ is a local projective generator for ${}^p\operatorname{Per}(Y/X)$. Put $\mathcal{A} = f_*\mathcal{E}nd_Y(\mathcal{P})$. Then the functors $Rf_*\,\mathrm{R}\mathcal{H}om_Y(\mathcal{P}, -)$ and $f^{-1}(-)\overset{L}{\otimes}_{f^{-1}(\mathcal{A})}\mathcal{P}$ define inverse equivalences between $D^b(\mathrm{coh}(Y))$ and $D^b(\mathrm{coh}(\mathcal{A}))$. These equivalences restrict to equivalences between ${}^p\operatorname{Per}(Y/X)$ and $\mathrm{coh}(\mathcal{A})$.*

Proof. This is easily proved by restricting to affine opens and invoking Corollary 3.2.8. $\square$

Under mild hypotheses we may construct a local projective generator for ${}^p\operatorname{Per}(Y/X)$.

Proposition 3.3.2. *Assume that X is quasi-projective over a noetherian ring S. Then there exists a local projective generator $\mathcal{P}$ for ${}^{-1}\operatorname{Per}(Y/X)$ such that $\mathcal{P}^*$ is a local projective generator for ${}^{-1}\operatorname{Per}(Y/X)$.*

Proof. Let $\bar{X}$ be a projective S-scheme containing X as an open subset. Let $\bar{Y}$ be the closure of Y under a locally closed embedding $Y \to \mathbb{P}^N_X \to \mathbb{P}^N_{\bar{X}}$ and let $\bar{f} : \bar{Y} \to \bar{X}$ be the corresponding projection morphism.

Let $\bar{\mathcal{L}}$ be an $\bar{f}$-ample line bundle generated by global sections and choose an epimorphism $\phi : \bar{\mathcal{E}} \to R^1 f_*(\bar{\mathcal{L}}^{-1})$ with $\bar{\mathcal{E}} = \mathcal{O}_{\bar{X}}(-a)^b$. By enlarging a we may assume

$$\operatorname{Ext}^i_{\bar{X}}(\bar{\mathcal{E}}, \bar{f}_*\bar{\mathcal{L}}^{-1}) = 0 \qquad \text{for } i > 0 \tag{3.2}$$

We have the following chain of equalities: $\operatorname{Ext}^1_{\bar{Y}}(\bar{\mathcal{L}}, \bar{f}^*(\bar{\mathcal{E}}^*)) = \operatorname{Ext}^1_{\bar{Y}}(\bar{f}^*\bar{\mathcal{E}}, \bar{\mathcal{L}}^{-1}) = \operatorname{Ext}^1_{\bar{X}}(\bar{\mathcal{E}}, R\bar{f}_*\bar{\mathcal{L}}^{-1}) = \operatorname{Hom}_{\bar{X}}(\bar{\mathcal{E}}, R^1\bar{f}_*\bar{\mathcal{L}}^{-1})$ where the last equality follows from (3.2).

Hence the map ϕ provides us with an extension

$$0 \to \bar{f}^*(\bar{\mathcal{E}})^* \to \bar{\mathcal{P}}_0 \to \bar{\mathcal{L}} \to 0$$

and restricting to Y we obtain a corresponding extension

$$0 \to f^*(\mathcal{E})^* \to \mathcal{P}_0 \to \mathcal{L} \to 0$$

and it is clear from the construction that on small open affines the extension restricts to (3.1). Thus $\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{O}_Y$ is our required local projective generator. $\square$

3.4. **The formal case.** We now discuss the case $X = \operatorname{Spec} R$ with R a noetherian complete local ring with maximal ideal m such that $k = R/m$ is algebraically closed and $k \subset R$ (at the cost of some technicalities one can get by with less). It is not necessary to assume $\operatorname{char} k = 0$. The formal case is interesting since it contains some features not present in the general case. Also our main result in §4.4 is proved by reduction to the formal case.

Let x be the unique closed point of X (i.e the defining ideal of x is m) and let $C = f^{-1}(x)$. C is either scheme-theoretically a point (if f is an isomorphism) or else it has dimension one.

First we discuss the structure of C_{red}. This is of course well-known.

Lemma 3.4.1. *C_{red} is either a point or a tree of $\mathbb{P}^1$'s with normal crossings.*

Proof. Assume that C is not a point. Since C is connected, it is clear that the only global sections of $\mathcal{O}_{C_{\mathrm{red}}}$ are scalar. Furthermore since $\mathcal{O}_{C_{\mathrm{red}}}$ is a quotient of $\mathcal{O}_Y$ we deduce that $H^1(C_{\mathrm{red}}, \mathcal{O}_{C_{\mathrm{red}}}) = 0$. Thus C_{red} is a reduced projective curve of arithmetic genus zero. It is well-known that this must be a tree of $\mathbb{P}^1$'s. $\square$

We also have the following.

Lemma 3.4.2. *C is Cohen-Macaulay (i.e. C has no embedded components) and $H^0(C, \mathcal{O}_C) = k$.*

Proof. This clear if C is a point. Assume that this is not the case. We have an exact sequence

$$0 \to m\mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_C \to 0$$

Since $m\mathcal{O}_Y$ is generated by global sections we have $H^1(Y, m\mathcal{O}_Y) = 0$. Hence $H^0(C, \mathcal{O}_C)$ is a quotient of $H^0(Y, \mathcal{O}_Y)$ by an ideal containing m. It follows that $H^0(C, \mathcal{O}_C) = k$. Any embedded component of C would be zero-dimensional and hence would give rise to extra sections. So such embedded components cannot exist. $\square$

We will denote the irreducible components of C by $(C_i)_{i=1,\ldots,n}$. If C is a point then we set $n = 0$.

In each of the C_i we choose a point y_i not lying on the other components and not lying on the possible finite number of embedded zero dimensional components of Y.

The following result is well-known.

Lemma 3.4.3. *The map* $\mathcal{L} \mapsto \deg(\mathcal{L} \mid C_i)_{i=1,\dots,n}$ *defines an isomorphism* $\operatorname{Pic}(Y) \cong \mathbb{Z}^n$.

Proof. Let $\hat{Y}$ be the formal scheme associated to (Y, C). We have $H^1(\hat{Y}, \mathcal{O}_{\hat{Y}}) = H^1(Y, \mathcal{O}_Y) = 0$ [12, §4] and by Grothendieck's existence theorem we also have $\operatorname{Pic}(X) = \operatorname{Pic}(\hat{X})$. Consider the exact sequence

$$0 \to m\mathcal{O}_{\hat{Y}} \to \mathcal{O}^*_{\hat{Y}} \to \mathcal{O}^*_C \to 0 \tag{3.3}$$

As in the proof of the previous lemma we have $H^1(\hat{Y}, m\mathcal{O}_{\hat{Y}}) = 0$. Hence from the long exact sequence associated to (3.3) we deduce $\operatorname{Pic}(\hat{Y}) = \operatorname{Pic}(C)$. Now C is either a point or a projective curve with $H^1(C, \mathcal{O}_C) = 0$. It is well known that this implies $\operatorname{Pic}(C) = \mathbb{Z}^n$. It is easy to see that the isomorphism has the indicated form. $\square$

The previous lemma implies the existence of line bundles $\mathcal{L}_i$ on Y such that $\deg(\mathcal{L}_i \mid C_j) = \delta_{ij}$. One may improve slightly on this fact.

Lemma 3.4.4. *There exist a closed subscheme D_i in Y with $\mathcal{O}_Y(-D_i)$ invertible such that scheme theoretically we have*

$$D_i \cap C_j = \begin{cases} \{y_i\} & \textit{if } i = j \\ \emptyset & \textit{otherwise} \end{cases}$$

Proof. If D is an arbitrary subscheme of Y then the connected components of D coincide with the connected components of $D \cap C$.

It is easy to see that on an affine neighborhood U_i of y_i we can choose a non-zero divisor $z \in \Gamma(U_i, \mathcal{O}_{U_i})$ such that $V(z) \cap C_i = \{y_i\}$ scheme-theoretically. Let D' be the closure of $V(z)$ in Y. We let D_i be the component of D' containing y_i. $\square$

Let $\operatorname{Pic}^+(Y)$ and $\operatorname{Pic}^{++}(Y)$ be respectively the line bundles generated by gobal sections and the ample line bundles. The following is well-known:

Lemma 3.4.5. *We have*

$$\operatorname{Pic}^+(Y) = \{\mathcal{L} \in \operatorname{Pic}(Y) \mid \deg(\mathcal{L} \mid C_i) \geq 0 \textit{ for all } i\} \tag{3.4}$$

$$\operatorname{Pic}^{++}(Y) = \{\mathcal{L} \in \operatorname{Pic}(Y) \mid \deg(\mathcal{L} \mid C_i) > 0 \textit{ for all } i\} \tag{3.5}$$

Proof. The statement about $\operatorname{Pic}^{++}(Y)$ follows from [18, Prop. 2.7]. For the statement about $\operatorname{Pic}^+(Y)$ we use the fact that if $\mathcal{L}$ is as in (3.4) then by lemma 3.4.4 $\mathcal{L} = \mathcal{O}_Y(E)$ with $E = \sum n_i D_i$, $n_i \geq 0$. The support of $\mathcal{O}_E(E)$ is finite over X and hence affine. From the exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(E) \to \mathcal{O}_E(E) \to 0$$

and the fact that $H^1(Y, \mathcal{O}_Y) = 0$ we obtain that $\mathcal{O}_Y(E)$ is generated by global sections. $\square$

3.5. **Artin-Verdier theory.** In [1] Artin and Verdier show that non-trivial indecomposable Cohen-Macaulay modules on a rational double point are in one-one correspondence with the irreducible components of the exceptional divisor in a minimal resolution. In this section we show that part of that theory generalizes to the setting introduced in §3.4.

Lemma 3.5.1. *Let $\mathcal{M}$ be a vector bundle of rank r on Y generated by global sections. Then $\mathcal{M}$ occurs in exact sequences of the form*

$$0 \to \mathcal{O}_Y^{r-1} \to \mathcal{M} \to \mathcal{L} \to 0 \tag{3.6}$$

$$0 \to \mathcal{L}^{-1} \to \mathcal{O}_Y^{r+1} \to \mathcal{M} \to 0 \tag{3.7}$$

where $\mathcal{L} = c_1(\mathcal{M})$.

Proof. This is proved in a similar way as [1, Lemma 1.2,1.5]. The case where C is a point is clear so we assume this is not the case.

Let $\phi : \mathcal{O}_Y^{r-1} \to \mathcal{M}$, $\theta : \mathcal{O}_Y^{r+1} \to \mathcal{M}$ be obtained from choosing respectively $r-1$ and $r+1$ generic sections of $\mathcal{M}$.

Let J_i be the ideal sheaf of C_i. From the exact sequence

$$0 \to J_i \to \mathcal{O}_Y \to \mathcal{O}_{C_i} \to 0$$

we deduce that $H^1(Y, J_i) = 0$. Since $\mathcal{M}$ is generated by global sections we also have $H^1(Y, J_i\mathcal{M}) = 0$. It follows that $H^0(Y, \mathcal{M}) \to H^0(Y, \mathcal{M} \otimes_Y \mathcal{O}_{C_i})$ is surjective for all i. In particular generic sections of $\mathcal{M}$ correspond to generic sections of $H^0(Y, \mathcal{M} \otimes_Y \mathcal{O}_{C_i})$.

We claim that ϕ and θ have maximal rank everywhere. This may be checked on closed points and hence it may be checked by restricting to C_i. By lemma 3.4.1 $C_i = \mathbb{P}^1$ and on $\mathbb{P}^1$ the statement is elementary.

Hence $\mathcal{L} = \operatorname{coker} \phi$ and $\mathcal{L}' = \ker \theta$ are line bundles. That $\mathcal{L} = \mathcal{L}'^{-1} = c_1(\mathcal{M})$ is clear. □

Lemma 3.5.2. *The vector bundles $\mathcal{M}$ in $\mathfrak{V}$ are precisely the ones which occur as middle term in an exact sequence with $\mathcal{L} \in \operatorname{Pic}^+(Y)$*

$$0 \to \mathcal{O}_Y^{r-1} \to \mathcal{M} \to \mathcal{L} \to 0 \tag{3.8}$$

which is determined by a set of $r-1$ generators of $H^1(Y, \mathcal{L}^{-1})$. $\mathcal{M}$ is uniquely determined by $\mathcal{L} = c_1(\mathcal{M})$, up to the addition of copies of $\mathcal{O}_Y$.

The vector bundles $\mathcal{N}$ in $\mathfrak{V}^*$ are precisely the ones which occur in an exact sequence with $\mathcal{L} \in \operatorname{Pic}^+(Y)$

$$0 \to \mathcal{N} \to \mathcal{O}_Y^{r+1} \to \mathcal{L} \to 0 \tag{3.9}$$

which is determined by a set of $r+1$ generators of $H^0(Y, \mathcal{L})$. $\mathcal{N}$ is uniquely determined by $\mathcal{L} = c_1(\mathcal{N})^{-1}$, up to the addition of copies of $\mathcal{O}_Y$.

Proof. That every $\mathcal{M}$ which occurs in an exact sequence (3.8) is in $\mathfrak{V}$ is clear so we concentrate on the converse. According to lemma 3.5.1 $\mathcal{M}$ occurs in an exact sequence

$$0 \to \mathcal{O}_Y^{r-1} \to \mathcal{M} \to \mathcal{L} \to 0$$

and the fact that $H^1(Y, \mathcal{M}^*) = 0$ implies that this exact sequence is determined by a set of $r-1$-generators of $H^1(Y, \mathcal{L}^{-1})$.

Any set of generators $H^1(Y, \mathcal{L}^{-1})$ contains a minimal set of generators. Adding extra generators corresponds to adding free summands to $\mathcal{M}$. It is easy to see that two minimal sets of generators yield isomorphic $\mathcal{M}$.

The proof for $\mathcal{N}$ is similar ((3.9) is obtained by dualizing (3.7)). □

From the previous lemmas we extract the following corollary.

Proposition 3.5.3. *The map*

$$\phi : \mathfrak{V} \to \mathbb{Z} \times \operatorname{Pic}^+(Y) : \mathcal{M} \to (\operatorname{rk}(\mathcal{M}), c_1(\mathcal{M}))$$

is an injection on isomorphism classes of objects.

In order to understand the image of ϕ we have to investigate the indecomposable objects in $\mathfrak{V}$.

Let $\mathcal{M}_i \in \mathfrak{V}$ be the extension

$$0 \to \mathcal{O}_Y^{r_i - 1} \to \mathcal{M}_i \to \mathcal{L}_i \to 0 \tag{3.10}$$

associated to a minimal set $r_i - 1$ of generators of $H^1(Y, \mathcal{L}_i^{-1})$ where $\mathcal{L}_i$ is as in §3.4. We also define $\mathcal{M}_0 = \mathcal{O}_Y$.

Proposition 3.5.4. *The $\mathcal{M}_i$ are indecomposable and furthermore every indecomposable object in $\mathfrak{V}$ is isomorphic to one of the $\mathcal{M}_i$. If $i > 0$ then $\operatorname{rk} \mathcal{M}_i$ is equal to the multiplicity of C_i in C.*

Proof. In the proof below we assume $i > 0$. It is clear from Proposition 3.5.3 that if $\mathcal{M}_i$ is decomposable then it is of the form $\mathcal{O}_Y^a \oplus \mathcal{M}'$ with $\mathcal{M}'$ indecomposable (this corresponds to the only possible way of decomposing $(\operatorname{rk}(\mathcal{M}_i), c_i(\mathcal{M}_i))$ in $\mathbb{Z} \times \operatorname{Pic}^+(Y)$). Let r' be the rank of $\mathcal{M}'$. Thus $r' \le r_i$. According to lemmas 3.5.2 $\mathcal{M}'$ appears in an exact sequence

$$0 \to \mathcal{O}_Y^{r'-1} \to \mathcal{M}' \to \mathcal{L}_i \to 0$$

corresponding to a set of $r' - 1$ generators of $H^1(Y, \mathcal{L}_i^{-1})$. But the minimal number of generators of $H^1(Y, \mathcal{L}_i^{-1})$ is $r_i - 1$. Thus $r' \ge r_i$ and hence $r' = r_i$. Thus $\mathcal{M}_i = \mathcal{M}'_i$. In other words: $\mathcal{M}_i$ is indecomposable.

Now let $\mathcal{M}$ be an arbitrary object in $\mathfrak{V}$. We may find an object $\mathcal{R}' = \mathcal{M}_1^{\oplus a_1} \oplus \cdots \oplus \mathcal{M}_n^{\oplus a_n}$ such that $c_1(\mathcal{M}) = c_1(\mathcal{R}')$. So by Proposition 3.5.3 $\mathcal{M}$ and $\mathcal{R}'$ differ only by copies of $\mathcal{O}_Y$. Since $\mathcal{R}'$ can have no summands isomorphic to $\mathcal{O}_Y$ these must be in $\mathcal{M}$. So there is some $\mathcal{R} = \mathcal{O}_Y^{\oplus a_0} \oplus \mathcal{R}'$ such that $\mathcal{M} \cong \mathcal{R}$. This shows that every indecomposable object in $\mathfrak{V}$ is isomorphic to one of the $\mathcal{M}_i$.

To prove the statement on the rank of $\mathcal{M}_i$ we need to compute the minimal number of generators of $H^1(Y, \mathcal{L}_i^{-1})$. By lemma 3.4.3 we have $\mathcal{L}_i \cong \mathcal{O}_Y(-D_i)$. From the exact sequence

$$0 \to \mathcal{O}_Y(-D_i) \to \mathcal{O}_Y \to \mathcal{O}_{D_i} \to 0$$

we obtain an exact sequence

$$R \to \Gamma(Y, \mathcal{O}_{D_i}) \to H^1(Y, \mathcal{O}_Y(-D_i)) \to 0$$

The first map is a ring map so its image cannot be in $m\Gamma(Y, \mathcal{O}_{D_i})$. Thus the minimal number of generators of $H^1(Y, \mathcal{O}_Y(-D_i))$ is one less than the minimal number of generators of $\Gamma(Y, \mathcal{O}_{D_i})$. The latter is equal to the dimension of the vector space $\Gamma(Y, \mathcal{O}_{D_i}/m\mathcal{O}_{D_i})$ which is precisely $C \cdot D_i$. Given the definition of D_i this is also the multiplicity of C_i in C. $\square$

In the sequel we put $\mathcal{N}_i = \mathcal{M}_i^*$. By lemma 3.5.2 $\mathcal{M}_i$ occurs in an exact sequence

$$0 \to \mathcal{N}_i \to \mathcal{O}_Y^{r_i + 1} \to \mathcal{L}_i \to 0 \tag{3.11}$$

Theorem 3.5.5. *The indecomposable projective objects in* ${}^{-1}\operatorname{Per}(Y/X)$ *are the* $(\mathcal{M}_i)_i$. *The indecomposable projective objects in* ${}^{0}\operatorname{Per}(Y/X)$ *are the* $(\mathcal{N}_i)_i$. *The projective generators in* ${}^{-1}\operatorname{Per}(Y/X)$ *are of the form* $\oplus_i \mathcal{M}_i^{\oplus a_i}$, $a_i > 0$. *The projective generators in* ${}^{0}\operatorname{Per}(Y/X)$ *are of the form* $\oplus_i \mathcal{N}_i^{\oplus b_i}$, $b_i > 0$.

Proof. This is a combination of Propositions 3.2.6, 3.2.7 and 3.5.4. □

Summarizing we have proved the following result.

Theorem 3.5.6. *We have* ${}^{-1}\operatorname{Per}(Y/X) \cong \operatorname{mod}(A)$, ${}^{0}\operatorname{Per}(Y/X) \cong \operatorname{mod}(A^\circ)$ *where* A *is finitely generated as a module over* R *and* $A/\operatorname{rad}(A) \cong k^{n+1}$.

Proof. It suffices to take as projective generator $\oplus_i \mathcal{M}_i$ for ${}^{-1}\operatorname{Per}(Y/X)$ and $\oplus_i \mathcal{N}_i$ for ${}^{0}\operatorname{Per}(Y/X)$ □

It is possible to give explicitly the $n+1$ simple objects in ${}^{p}\operatorname{Per}(Y/X)$. This was independently observed by Bridgeland. This result will not be used afterwards.

Proposition 3.5.7. *The* $n+1$ *simple objects in* ${}^{-1}\operatorname{Per}(Y/X)$ *are* $\mathcal{S}_0 = \mathcal{O}_C$ *and* $\mathcal{S}_i = \mathcal{O}_{C_i}(-1)[1]$, $i = 1, \dots, n$.

Proof. By Theorem 3.5.6 it is sufficient to prove $\operatorname{Hom}(\mathcal{M}_i, \mathcal{S}_j) = \delta_{ij} \cdot k$.

The case $i = 0$ is clear so we assume $i \neq 0$. If $j \neq 0$ then we have $\operatorname{Hom}_Y(\mathcal{M}_i, \mathcal{O}_{C_j}) = \operatorname{Ext}^1_Y(\mathcal{L}_i, \mathcal{O}_{C_j}) = \operatorname{Ext}^1_{C_j}(\mathcal{L}_i \mid C_j, \mathcal{O}_{C_j})$. Now $\mathcal{L}_i \mid C_j$ is either $\mathcal{O}_{C_j}(1)$ or $\mathcal{O}_{C_j}$ depending on whether $i = j$ or $i \neq j$. This finishes the proof in the case $i, j \neq 0$.

The only remaining case is $i \neq 0$, $j = 0$. We need to compute $\operatorname{Hom}_Y(\mathcal{M}_i, \mathcal{O}_C) = \operatorname{Hom}_C(\mathcal{M}_i \otimes_Y \mathcal{O}_C, \mathcal{O}_C)$ for $i \neq 0$. From the right exactness of $H^1(Y, -)$ we deduce for $M \in \operatorname{mod}(R)$: $H^1(Y, \mathcal{L}_i^{-1} \otimes_R M) = H^1(Y, \mathcal{L}_i^{-1}) \otimes_R M$. Applying this with $M = R/m$ we see that a set of minimal generators of $H^1(Y, \mathcal{L}_i^{-1})$ as R-module yield a basis of $H^1(Y, \mathcal{L}_i^{-1} \otimes_Y \mathcal{O}_C)$ as k-vector space.

The long exact sequence for $\operatorname{Hom}_Y(-, \mathcal{O}_C)$ applied to (3.10) yields $\operatorname{Hom}_Y(\mathcal{M}, \mathcal{O}_C) = \Gamma(C, \mathcal{L}_i^{-1} \otimes_Y \mathcal{O}_C)$. Now $\Gamma(C, \mathcal{L}_i^{-1} \otimes_Y \mathcal{O}_C) = \Gamma(C, \mathcal{O}_C(-D_i))$ and this is zero since $\Gamma(C, \mathcal{O}_C(-D_i) \subset \Gamma(C, \mathcal{O}_C) = k$, and the generator 1 for $\Gamma(C, \mathcal{O}_C)$ is nowhere vanishing. □

The following result is proved in a similar way.

Proposition 3.5.8. *The simple objects in* ${}^{0}\operatorname{Per}(Y/X)$ *are* $\omega_C[1]$ *and* $\mathcal{O}_{C_i}(-1)$. *If* $(-)^D$ *denotes the Grothendieck duality functor on* $D^b(\operatorname{coh}(Y))$ [14] *then* $(-)^D$ *defines a duality between the categories of finite length objects in* ${}^{p}\operatorname{Per}(Y/X)$ *and* ${}^{-1-p}\operatorname{Per}(Y/X)$.

4. Flops

4.1. **Some special Cohen-Macaulay modules.** Below P is a $n \geq 3$-dimensional regular complete local ring and R is a normal integral Gorenstein ring which is of rank two over P. In particular R is free over P. By $\operatorname{Cl}(R)$ we denote the *class group* of R [11]. Its elements are isomorphism classes of rank one reflexive R-modules and the multiplication is given by $I \cdot J = (I \otimes_R J)^{**}$.

Let I be a reflexive rank one module over P of depth $\geq n-1$. Thus $\operatorname{Ext}^i_R(I, R) = 0$ for $i \geq 2$.

We consider exact sequences of R-modules of the form

$$0 \to R^{r-1} \to M \to I \to 0 \tag{4.1}$$

$$0 \to N \to R^{s+1} \to I \to 0 \tag{4.2}$$

where we assume that (4.1) is given by a set of $r-1$ generators of $\mathrm{Ext}^1_R(I,R)$. Looking at the long exact sequences for $\mathrm{Hom}_R(-,R)$ we find $\mathrm{Ext}^i_R(M,R) = \mathrm{Ext}^i_R(N,R) = 0$ for $i > 0$. Hence M, N are (maximal) Cohen-Macaulay modules.

Now M, N are determined by I up to adding of free summands. We denote by $M(I)$, $N(I)$ the Cohen-Macaulay modules obtained from M, N by deleting the free summands.

Remark 4.1.1. Indecomposable Cohen-Macaulay modules are not always of the form $M(I)$, $N(I)$. Recall that if R is Cohen-Macaulay but non regular there exist always non-free indecomposable Cohen-Macaulay modules (for example given by a summand of a suitable syzygy of R/m). But if R is in addition factorial then the only Cohen-Macaulay module of the form $M(I)$, $N(I)$ are trivial which yields a contradiction. A concrete counter example is given by $k[[x,y,z,t]]/(x^2+y^2+z^2+t^3)$.

The author thinks it is an intriguing question to understand precisely which indecomposable Cohen-Macaulay modules are of the form $M(I)$ or $N(I)$. It seems possible to solve this problem for complete local rings with a three-dimensional isolated Gorenstein terminal singularity. We hope to come back on this afterwards.

In the sequel we need the following technical result.

Proposition 4.1.2. *We have $N(I) \cong M(I^{-1})$.*

Proof. Note that for an R-module M we have $\mathrm{depth}_R\, M = \mathrm{depth}_P\, M$ and furthermore M is reflexive as R module if and only if it is reflexive as P-module.

Hence since R is free of rank two over P, $R \otimes_P I$ is reflexive and has depth $\geq n-1$. Its rank over R is two.

Since P is regular local the projective dimension of I over P is one. Since R is flat as P module it follows that $R \otimes_P I$ also has projective dimension one.

Let K be the kernel of $R \otimes_P I \to I$. K is also reflexive of rank one and it has depth $\geq n-1$. To compute K we note that if we assign to a reflexive R-module M the element of $\mathrm{Cl}(R)$ given by $(\wedge^{\mathrm{rk}\, M} M)^{**}$ then c_1 is multiplicative on short exact sequences.

Applying this to the exact sequence

$$0 \to K \to R \otimes_P I \to I \to 0$$

and using the fact that $c(R \otimes_P I) = R$ since $R \otimes_P I$ has finite projective dimension, we find $K \cong I^{-1}$.

Now suppose that we have an exact sequence as in (4.2). After possibly adding free summands to N and R^{s+1} we may construct a diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & R^{s+1} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, equal] & 0 \\
 & & R \otimes_P I \arrow[r] \arrow[d] & I \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

Using the fact that $R \otimes_P I$ has projective dimension one this diagram may be completed to

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & R^{r-1} \arrow[r, equal] \arrow[d] & R^{r-1} \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & R^{s+1} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^{-1} \arrow[r] \arrow[d] & R \otimes_P I \arrow[r] \arrow[d] & I \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By construction N is Cohen-Macaulay so $\operatorname{Ext}^1_R(N, R) = 0$. It follows that the left most vertical exact sequence is obtained from a set of generators of $\operatorname{Ext}^1_R(I^{-1}, R)$. So up to free summands we have $N(I) = N = M(I^{-1})$. $\square$

4.2. **Application to hypersurface singularities.** The following lemma will be used.

Lemma 4.2.1. *Let $f : Y \to X$ be a projective birational map between normal noetherian schemes such that the exceptional locus of f has codimension ≥ 2 in Y. The functor f_* restricts to an equivalence between the category of reflexive $\mathcal{O}_Y$-modules and the category of reflexive R-modules.*

Proof. We may assume that $X = \operatorname{Spec} R$ is affine. Let $\mathcal{M}$ be a reflexive $\mathcal{O}_Y$ module. We need to show that for every prime ideal p in the exceptional locus of f in X we have $\operatorname{Hom}_X(p, f_*\mathcal{M}) = \Gamma(Y, \mathcal{M})$. Now $\operatorname{Hom}_X(p, f_*\mathcal{M}) = \operatorname{Hom}_Y(f^*p, \mathcal{M})$. By hypotheses the exceptional locus of f has codimension 2, and hence f^*p is equal to $\mathcal{O}_Y$ in codimension one. Since $\mathcal{M}$ is reflexive this implies $\operatorname{Hom}_Y(f^*p, \mathcal{M}) = \operatorname{Hom}_Y(\mathcal{O}_Y, \mathcal{M}) = \Gamma(Y, \mathcal{M})$.

That f_* yields an equivalence between the category of reflexive $\mathcal{O}_Y$-modules and the category of reflexive R-modules also follows from the hypotheses on the codimension of the exceptional locus. $\square$

Let $\operatorname{Cl}(Y)$ be the group of Weil divisors on Y modulo the principal divisors. The previous lemma gives us a canonical identification between $\operatorname{Cl}(Y)$ and $\operatorname{Cl}(R)$.

We specialize the situation of §3.4. Now R is a normal complete local k-algebra of dimension $n \geq 3$ with residue field k and with a canonical hypersurface singularity of multiplicity two. According to [21, Cor 5.24] X has rational singularities.

Let $f : Y \to X$ be a birational projective map such that Y is normal Gorenstein. Assume in addition that the exceptional locus of f has codimension ≥ 2 in Y. It is easy to see that Y also has canonical and hence rational singularities (again by [21, Cor 5.24]) and therefore $Rf_*\mathcal{O}_Y = \mathcal{O}_X$

Note that by lemma 3.2.9 $\operatorname{Pic}(Y)$ under the identification $\operatorname{Cl}(Y) \cong \operatorname{Cl}(R)$ is sent to reflexive ideals of depth $\geq n - 1$.

Let $\mathcal{M}_i$, $\mathcal{N}_i$ be as in the previous section. Then we deduce from lemma 3.2.9 that $\Gamma(Y, \mathcal{M}_i)$ and $\Gamma(Y, \mathcal{N}_i)$ are Cohen-Macaulay.

We let $\mathcal{L}_i$ be as in §3.4 and we put $I_i = \Gamma(Y, \mathcal{L})$. Thus the I_i are reflexive R-modules of rank one and depth $\geq n-1$. Put $M_i = M(I_i)$, $N_i = N(I_i)$ for $i > 0$ and $M_0 = N_0 = R$.

The following observation is crucial.

Lemma 4.2.2. *We have* $\Gamma(Y, \mathcal{M}_i) = M_i$, $\Gamma(Y, \mathcal{N}_i) = N_i$.

Proof. The case $i = 0$ is trivial so we assume $i > 0$. We consider the first equality. Applying $\Gamma(Y, -)$ to (3.10) we obtain an exact sequence

$$0 \to R^{r_i - 1} \to \Gamma(Y, \mathcal{M}_i) \to I_i \to 0 \tag{4.3}$$

According to Proposition 3.5.4, lemma 4.2.1 and the above discussion we have that $\Gamma(Y, \mathcal{M}_i)$ is an indecomposable Cohen-Macaulay R-module. Applying $\operatorname{Hom}(-, R)$ to (4.3) we see that (4.3) is associated to a set $r_i - 1$ generators of $\operatorname{Ext}^1_R(I_i, R)$. Thus $\Gamma(Y, \mathcal{M}_i)$ is obtained from M_i by adding free summands. Since $\Gamma(Y, \mathcal{M}_i)$ is indecomposable we obtain $\Gamma(Y, \mathcal{M}_i) = M_i$.

Now we consider the second equality. Applying $\Gamma(Y, -)$ to (3.11) we obtain an exact sequence

$$0 \to \Gamma(Y, \mathcal{N}_i) \to R^{r_i + 1} \to I_i \to 0 \tag{4.4}$$

Now since $\mathcal{N}_i = \mathcal{M}_i^*$ it follows from Proposition 3.5.4, lemma 4.2.1, and the above discussion that $\Gamma(Y, \mathcal{N}_i)$ is an indecomposable Cohen-Macaulay R-module.

Thus $\Gamma(Y, \mathcal{N}_i)$ is obtained from N_i by adding free summands. Since $\Gamma(Y, \mathcal{N}_i)$ is indecomposable we obtain $\Gamma(Y, \mathcal{N}_i) = N_i$. □

Remark 4.2.3. It follows from the above proof that the M_i, N_i are indecomposable and furthermore that they occur in exact sequences

$$0 \to R^{r_i - 1} \to M_i \to I_i \to 0$$
$$0 \to N_i \to R^{r_i + 1} \to I_i \to 0$$

In addition lemma 4.2.1 together with the fact that $\mathcal{N}_i = \mathcal{M}_i^*$ implies $N_i = M_i^*$.

4.3. **Formal flops.** We recycle the notations of the previous section. According to [20, Prop. 2.3] the map $f : Y \to X$ has a flop $f^+ : Y^+ \to X$.

The construction of f^+ is as follows. Write $R = k[[x_1, \dots, x_{n+1}]]/(x_1^2 + f(x_2, \dots, x_{n+1}))$ and let $\sigma : X \to X$ be given by $(x_1, \dots, x_{n+1}) \to (-x_1, \dots, x_{n+1})$. Then $Y^+ = Y$ and $f^+ = \sigma \circ f$.

Below we use the same notations for Y as for Y^+ but we adorn the latter with a superscript "+". To fix the numbering of the C_i^+ we use the identification $Y^+ = Y$ to put $C_i^+ = C_i$, $D_i^+ = D_i$.

Lemma 4.2.1 gives us a canonical identification $\operatorname{Cl}(Y) = \operatorname{Cl}(R) = \operatorname{Cl}(Y^+)$. Since σ induces the operation $I \mapsto I^{-1}$ on $\operatorname{Cl}(R)$ (see [20, Example 2.3]) we find that under this identification $\mathcal{L}_i^+$ on Y^+ is equivalent to $\mathcal{L}_i^{-1}$ on Y. In particular $I_i^+ = I_i^{-1}$.

We now obtain our key result.

Proposition 4.3.1. *One has* $M_i^+ \cong N_i$, $N_i^+ \cong M_i$.

Proof. This is a direct application of Proposition 4.1.2. Indeed we have $M_i^+ = M(I_i^+) = M(I_i^{-1}) = N(I_i) = N_i$. The argumentation for N_i^+ is identical. □

4.4. **Global flops.**

Lemma 4.4.1. *Let $f : Y \to X$ be a projective birational map between normal varieties over k of dimension $n \geq 3$ such that the exceptional locus of f has codimension ≥ 2 in Y. Assume that X has hypersurface singularities (not necessarily isolated) of multiplicity ≤ 2.*

Under these conditions the flop of f exists and is unique. More precisely there exists a unique morphism $f^+ : Y^+ \to X$ with the following properties

(1) *f^+ is projective and birational and Y^+ is a normal variety. The fibers of f^+ have dimension ≤ 1 and the exceptional locus of f^+ has codimension 2 in Y^+. If Y is Gorenstein then so is Y^+.*
(2) *Under the identifications obtained from lemma 4.2.1: $\mathrm{Cl}(Y) \cong \mathrm{Cl}(X) \cong \mathrm{Cl}(Y^+)$, $\mathrm{Pic}(Y)$ corresponds to $\mathrm{Pic}(Y^+)$.*
(3) *If E is an f-nef (resp. f-ample) divisor on Y then $-E$ corresponds to an f-nef (resp. f-ample) divisor on Y^+.*

Proof. Choose an f-ample Cartier divisor D on Y and identify it with a Weil divisor on X, also denoted by D. Then Y^+, if it exists, is the D-flop of f (in the terminology of [20, §2]). Hence Y^+ is unique if it exists.

According to [21, Cor. 6.4] for Y^+ to exist at least the sheaf of graded ring $\mathcal{S} = \oplus_n \mathcal{O}_X(-nD)$ should be a sheaf of finitely generated $\mathcal{O}_X$-algebras. In that case $Y^+ = \underline{\mathrm{Proj}}\, \mathcal{S}$.

According to [21, Prop. 6.6, Cor. 6.7] the finite generation of $\mathcal{S}$ can be checked in the completions of the closed points of X.

In the formal case we have $X = \operatorname{Spec} R$ where R is a hypersurface singularity of multiplicity one or two. In the case of multiplicity one there is nothing to prove and in the case of multiplicity two the flop exists by [20, Example 2.3]. From the construction of f^+ (see §4.3) it easily follows that $f^+ : Y^+ \to X$ has the properties (1)(2)(3) listed in the statement of the theorem. Since these properties may also be checked in the completions of the closed points they hold in the general case. $\square$

Now we give our main theorem.

Theorem 4.4.2. *Let $f : Y \to X$ be a projective birational map between normal quasi-projective Gorenstein k-varieties of dimension $n \geq 3$ with fibers of dimension ≤ 1 and assume that the exceptional locus of f has codimension ≥ 2. Assume that X has canonical hypersurface singularities of multiplicity ≤ 2. Let $f^+ : Y^+ \to X$ be the flop of f. Then $D^b(\mathrm{coh}(Y))$ and $D^b(\mathrm{coh}(Y^+))$ are equivalent and we may choose this equivalence in such a way that ${}^{-1}\operatorname{Per}(Y/X)$ corresponds to ${}^{0}\operatorname{Per}(Y^+/X)$.*

Proof. Let $\mathcal{P}$ be a local projective generator of ${}^{-1}\operatorname{Per}(Y/X)$ ($\mathcal{P}$ exists by Proposition 3.3.2).

Then according to lemma 3.2.9 $f_*\mathcal{P}$ is a local Cohen-Macauly sheaf on X. Hence by lemma 4.2.1 $f_*\mathcal{P}$ corresponds to a reflexive $\mathcal{O}_{Y^+}$-module $\mathcal{Q}^+$. We claim that $\mathcal{Q}^+$ is a local projective generator for ${}^{0}\operatorname{Per}(Y^+/X)$. By the characterization given in Proposition 3.2.7 it follows that it is sufficient to check this in the formal case, i.e. in the situation of §4.3.

So let us for a moment assume that we are in the formal situation. Then according to Theorem 3.5.5 and lemma 4.2.2 $\mathcal{P} = \oplus_i \mathcal{M}_i^{\oplus a_i}$ with $a_i > 0$ and $f_*\mathcal{P} = \oplus_i M_i^{\oplus a_i}$ and by Proposition 4.3.1 this is equal to $\oplus_i (N_i^+)^{\oplus a_i}$ and using lemma 4.2.2 for f^+ this corresponds to $\oplus_i = (\mathcal{N}_i^+)^{\oplus a_i}$. Using lemma 4.2.1 for f^+

we obtain $\mathcal{Q}^+ = (\mathcal{N}_i^+)^{\oplus a_i}$ and using Theorem 3.5.5 for f^+ we obtain that $\mathcal{Q}^+$ is a projective generator for ${}^0\operatorname{Per}(Y^+/X)$.

Now we revert to the global case. Put $\mathcal{A} = f_*\mathcal{E}nd_Y(\mathcal{P})$. According to lemma 4.2.1 we also have $\mathcal{A} = f_*\mathcal{E}nd_Y(\mathcal{Q}^+)$.

According to Proposition 3.3.1 we have now equivalences of categories

(4.5)

$$D^b(\operatorname{coh}(Y)) \xrightarrow{Rf_*\operatorname{R}\mathcal{H}om_Y(\mathcal{P},-)} D^b(\operatorname{coh}(\mathcal{A})) \xrightarrow{(f^+)^{-1}(-)\overset{L}{\otimes}_{(f^+)^{-1}(\mathcal{A})}\mathcal{Q}^+} D^b(\operatorname{coh}(Y^+))$$

and these equivalences restrict to equivalences

$$ {}^{-1}\operatorname{Per}(Y/X) \to \operatorname{coh}(\mathcal{A}) \to {}^0\operatorname{Per}(Y^+/X)$$

This finishes the proof. □

Remark 4.4.3. It is easy to see that the constructed equivalence is independent of the choice of $\mathcal{P}$. This is expected since Bridgeland constructs a canonical equivalence between $D^b(\operatorname{coh}(Y))$ and $D^b(\operatorname{coh}(Y^+))$ [6].

Since our equivalence is independent of choices it should be possible to get rid of the quasi-projectivity hypothesis on X. This works of the level of the categories ${}^{-1}\operatorname{Per}(Y/X)$ and ${}^0\operatorname{Per}(Y^+/X)$ but on the derived level there are some technical problems with gluing which I haven't satisfactorally resolved yet.

Appendix A. Non-commutative crepant resolutions?

Let $X = \operatorname{Spec} R$ where (R, m) is a complete local k-algebra with $R/m = k$. Assume that R has a three-dimensional isolated Gorenstein terminal singularity. According to [23, Cor. 3.12] R is a deformation of a Du Val singularity. Let t be the deformation parameter.

If $f : Y \to X$ is a crepant resolution of singularities of X then according to Corollary 3.2.11 there exists a Cohen-Macaulay module M over R such that $A = \operatorname{End}_R(M)$ is homologically homogeneous. To simplify the discussion below let us call an R-algebra which is finitely generated and torsion free as R-module an R-*order*.

Now assume that X does not necessarily have a crepant resolution of singularities. A natural question is: does there always exist a homologically homogeneous R-order A? The answer to this question is yes for trivial reasons. If we put $R' = R[t^{1/n}]$ and $X' = \operatorname{Spec} R'$ then according to [8] for some n, X' has a crepant resolution of singularities. So there is a homologically homogeneous R'-order A'. It now suffices to take $A = A'_R$.

Hence to make the question meaningful we have to establish some rules. The most obvious condition to impose is that A should be *crepant* over R. By this we mean that $\omega_A = \operatorname{Hom}_R(A, \omega_R)$ is equal to $A \otimes_R \omega_R$ where the inclusion $A \otimes_R \omega_R \to \operatorname{Hom}_R(A, \omega_R) \cong \operatorname{Hom}_R(A, R) \otimes_R \omega_R$ is obtained from the inclusion $A \to \operatorname{Hom}_R(A, R) : a \mapsto \operatorname{Tr}(a-)$ where Tr denotes the reduced trace map. So the reduced trace map must be non-degenerate which is the same as saying that A should be Azumaya in codimension one [24]. Let $\eta \in X$ be the generic point. We also want A and R to be "birational" in some sense. If we take birational to mean that $k(\eta)$ and A_η should have the same module category then by Morita theory it follows that A_η should be a full matrix ring over $k(\eta)$. By [2, Prop 4.2] it follows that $A = \operatorname{End}_R(M)$ for a reflexive R-module M.

So the more restrictive question is: does there exist a reflexive R-module M such that $A = \operatorname{End}_R(M)$ is homologically homogeneous. Let us consider a concrete example.

Example A.1. Let $R = k[[x, y, z, t]]/(x^2 + y^2 + z^2 - t^{2b+1})$. Then there is no reflexive R-module M such that $A = \operatorname{End}_R(M)$ is homologically homogeneous.

Proof. Assume that M exists. Let x be the unique closed point in X. $X - \{x\}$ is regular so it follows from [2, Thm 4.3] that M is projective in codimension two.

We apply $\operatorname{Hom}_R(M, -)$ to the short exact sequence

$$0 \to M \xrightarrow{\times t} M \to M/tM \to 0$$

This yields an exact sequence

$$0 \to A/tA \to \operatorname{End}_{R/tR}(M/tM) \to \operatorname{Ext}^1_R(M, M)$$

M is projective in codimension two so $\operatorname{Ext}^1_R(M, M)$ has finite length.

Since M is reflexive, M/tM is torsion free and furthermore the cokernel of $(M/tM) \to (M/tM)^{**}$ is finite dimensional. So we have an inclusion

$$\operatorname{End}_{R/tR}(M/tM) \hookrightarrow \operatorname{End}_{R/tR}((M/tM)^{**})$$

whose cokernel has finite length.

We conclude that the cokernel $A/tA \to \operatorname{End}_{R/tR}((M/tM)^{**})$ has finite length. Since A is Cohen-Macaulay, the same holds for A/tA. So A/tA is reflexive and hence $A/tA = \operatorname{End}_{R/tR}((M/tM)^{**})$.

Now write $\bar{M} = (M/tM)^{**}$, $\bar{R} = R/tR$. Since $\bar{R}$ is integrally closed of dimension two, the maximal Cohen-Macaulay modules are precisely the reflexive modules. So $\bar{M}$ is Cohen-Macaulay. The ring $\bar{R}$ has a simple singularity so its indecomposable maximal Cohen-Macaulay modules are known. The only non-trivial one is the ideal $\bar{I} = (z, x + iy)$. So if $n = \operatorname{rk} M$ then $\bar{M}$ is the direct sum of n rank one Cohen-Macaulay modules over $\bar{R}$. The corresponding idempotents in A/tA may be lifted to A. It follows that M is also the sum of n rank one reflexive modules. But since R is factorial this implies that M is free.

So we conclude that $A \cong M_n(R)$. But then A is Morita equivalent to R and so it has infinite global dimension. It follows that A can not be homologically homogeneous. □

Remark A.2. I think that for three-dimensional terminal Gorenstein singularities the existence of commutative and non-commutative crepant resolutions are equivalent. This can presumably be shown with the same Fourier-Mukai method which was used to establish the three-dimensional McKay correspondence.

Remark A.3. If $R = k[[x, y, z, t]]/(x^2 + y^2 + z^2 - t^{2b+1})$ is as in example A.1 then there exist a homologically homogeneous R-order A *with center* R. However A is not Azumaya in codimension one so it is not crepant in the above sense. This example was independently discovered by Schofield. It can probably be generalized to arbitrary Gorenstein terminal singularities.

Here is the construction of A: let $S = k[[x, y, z, s]]/(x^2+y^2+z^2-s^{4b+2})$. Sending $s \mapsto -s$ defines an automorphism σ of order two of S and if we put $t = s^2$ then $R = S^\sigma$.

Let $I = (x+iy, z-s^{2b+1}) \subset S$ and put $A' = \begin{pmatrix} S & I \\ I^{-1} & S \end{pmatrix}$. We have $\sigma(I) = (x+iy)I^{-1}$ and hence if $M = \begin{pmatrix} 0 & x+iy \\ 1 & 0 \end{pmatrix}$ then $M\sigma(-)M^{-1}$ defines an automorphism τ of A' of order two. We take $A = A' * G$ where $G = \{1, \tau\} \cong \mathbb{Z}/2\mathbb{Z}$.

Remark A.4. If R is as in lemma A.1 then it is still possible that there exists some saturated (see [3, 5]) triangulated category $\mathcal{A}$ which in some sense serves as a crepant resolution of R. Tom Bridgeland has shown me a heuristic argument which seems to indicate that such $\mathcal{A}$ does not exist. It would be interesting to settle this matter.

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