A CLASS OF COUNTER EXAMPLES TO THE GEL'FAND-KIRILLOV CONJECTURE

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Abstract. Let G be a connected non-special semisimple algebraic group and let W be a finite dimensional G-representation such that W has trivial generic stabilizer. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Then the semi-direct product $\mathfrak{g} \oplus W$ is a counter example to the Gel'fand-Kirillov conjecture.

1. Introduction and statement of the main result

In the sequel k will be an algebraically closed field of characteristic zero. All objects we consider are k-objects. For a Lie algebra L over k we denote by $U(L)$ its universal enveloping algebra and by $D(L)$ the field of fractions of $U(L)$. In [2] Gel'fand and Kirillov formulate the following conjecture :

Conjecture (GK). Assume that L is a finite dimensional algebraic Lie algebra over k. Then $D(L)$ is a Weyl skew field over a purely transcendental extension of k.

This conjecture is known to be true for L solvable [1][7][11] and for semisimple Lie algebras of type A_n [2]. It is open for the other types.

In this note we construct a class of counter examples to (GK). To the best of our knowledge, these are the first such examples. However in [10] counter examples to (GK) are given *over a non-algebraically closed base field.* These are given by the non-trivial twisted versions of $\mathfrak{sl}_2$.

Before we outline our construction let us mention that in [3] Gel'fand and Kirillov prove that for semisimple L, $D(L)$ becomes a Weyl skew field after extending the center. This suggests the following weakening of (GK).

Conjecture (GK'). Assume that L is a finite dimensional algebraic Lie algebra over k. Then there exists a field extension K of the center of $D(L)$, purely transcendental over k such that the field of fractions of $D(L) \otimes_{Z(D(L))} K$ is a Weyl skew field.

The examples we give are counter examples to (GK) but not to (GK') (see corollary 2.5). In fact the authors are rather convinced that (GK') is true.

Now we describe our examples. Let G be a connected semisimple algebraic group over k and let W be a finite dimensional G-representation. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Then

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W is also a $\mathfrak{g}$-representation. Let L be the semidirect product $\mathfrak{g} \oplus W$. That is, if $(u, w), (u', w') \in \mathfrak{g} \oplus W$ then

$$[(u, w), (u', w')] = ([u, u'], uw' - u'w)$$

Examples of this type had been considered earlier in connection with the Gel'fand Kirillov conjecture by Nghiem-Xuan Hai [5] and by the first two authors.

Now recall that G is called *special* if any principal homogeneous G-space is locally trivial for the Zariski topology. All semisimple special groups are products of copies of Sl_n and Sp_n [4]. The following is our main result :

Theorem 1.3. Assume that the generic stabilizer of W is trivial and let $L = \mathfrak{g} \oplus W$ be as above. Then the following are equivalent

(1) $D(L)$ is a Weyl skew field over some field.
(2) G is special.

The proof follows by combining corollaries 2.3 and 4.3 below.

Note that if we are given a semisimple Lie algebra $\mathfrak{g}$ and a representation W then there is *at most* one corresponding G that acts with trivial generic stabilizer on W. Hence condition (2) in the above theorem depends only on $\mathfrak{g}$ and W.

The smallest counter example to (GK) that can be made with Theorem 1.3 has dimension 9 and is given by $\mathfrak{g} = \mathfrak{sl}_2$ and W equal to two copies of the adjoint representation of $\mathfrak{g}$. In this case $G = \mathrm{PGl}_2$ which is non-special. This should be compared with work of the first two authors which shows (by explicit computation) that (GK) holds for all algebraic Lie algebras of dimension ≤ 6. An interesting example to investigate is the 8-dimensional Lie algebra L given by $\mathfrak{g} = \mathfrak{sl}_2$ and W its five dimensional irreducible representation. Here we may take again $G = \mathrm{PGl}_2$. However in this case the generic stabilizer is not trivial but equal to V_4.

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2. An alternative description of $D(L)$

We let G, W, L etc ... be as above. Let K, F be respectively the function fields of W^* and $W^* /\!/ G$. Of course $F = K^G$. Let X be the generic fiber of $W^* \to W^* /\!/ G$. If the generic stabilizer of W is trivial then it follows from [13, Appendix], the Luna Slice Theorem and lemma 2.6 below that X is a principal homogeneous G-space.

Our first step is the following observation :

Proposition 2.1. Assume that the generic stabilizer of W is finite. Then $D(L)$ is isomorphic to the field of fractions of $\mathcal{D}_F(K)$, the ring of differential operators on K/F.

Proof. Let l be a field. If $\mathfrak{h}$ is an l-Lie algebra acting on an l-algebra A by derivations then by definition $A \#_l U(\mathfrak{h})$ is the algebra which equals $A \otimes_l U(\mathfrak{h})$ as l-vector space and whose multiplication is defined by

$$(a \# u)(a' \# u') = a(ua') \# uu'$$

where $a, a' \in A$, $u, u' \in \mathfrak{h}$.

If A is commutative then there is a canonical l-algebra homomorphism

$$A\#_l U(\mathfrak{h}) \to \mathcal{D}_l(A) \tag{2.1}$$

and we have the following easy lemma :

Sublemma. Suppose that A is commutative and smooth over l. Assume furthermore that the canonical map $A \otimes_l \mathfrak{h} \to \mathrm{Der}_l(A)$ is an isomorphism. Then the map given by (2.1) is an isomorphism.

Proof. The PBW-filtration on $U(\mathfrak{h})$ extends to a filtration on $A\#_l U(\mathfrak{h})$ such that $\operatorname{gr} A\#_l U(\mathfrak{h})$ is equal to the symmetric algebra $S_A(A \otimes_l \mathfrak{h})$.

Similarly $\mathcal{D}_l(A)$ is filtered by order of differential operators and since A/l is assumed to be smooth, $\operatorname{gr} \mathcal{D}_l(A) = S_A(\mathrm{Der}_l(A))$. This finishes the proof since (2.1) is compatible with these two filtrations. $\square$

It is easily verified that one has the identity

$$U(L) = SW\#_k U(\mathfrak{g})$$

So $D(L)$ is the field of fractions of $K\#_k U(\mathfrak{g}) = K\#_F U(\mathfrak{g}_F)$ where $\mathfrak{g}_F = F \otimes_k \mathfrak{g}$. The proposition now follows from the above sublemma and the next one. $\square$

Sublemma. The canonical map

$$K \otimes_F \mathfrak{g}_F \to \mathrm{Der}_F(K) \tag{2.2}$$

is an isomorphism.

Proof. First note that the generic stabilizer of W^* is also finite (lemma 2.6 below) and hence $\operatorname{tr.deg} K/F = \dim G = \dim \mathfrak{g}_F$. Thus the left and right hand side of (2.2) are K-vector spaces of the same dimension. So it suffices to prove that (2.2) is injective. To this end it suffices to prove that

$$K \otimes_k \mathfrak{g} \to \mathrm{Der}_k(K) \tag{2.3}$$

is injective. Let $u_1, \dots, u_n$ be a basis for $\mathfrak{g}$ and let $u = \sum_i f_i u_i$ be an non-zero element in the kernel of (2.3). Choose $x \in W^*$ in such a way that

- the stabilizer of x is finite;
- all f_i are regular in x;
- not all f_i are zero in x.

Then $u(x)$ defines a non-zero element of the kernel of the differential in e of the map $G \to W^* : g \mapsto gx$. Now the stabilizer of x is finite and so this map an immersion locally around e. Therefore the differential is injective in e which yields a contradiction. $\square$

Corollary 2.2. Assume that the generic stabilizer of W is finite. Then the center of $D(L)$ is equal to F.

Proof. One easily verifies from Proposition 2.1 that the center of $D(L)$ is given by those elements of K that are constant for all F-derivations of K. Because $\mathfrak{g}$ maps to $\mathrm{Der}_F(K)$ it is sufficient to show that $K^{\mathfrak{g}} = F$. Now the fact that G is connected implies that $K^{\mathfrak{g}} = K^G = F$. $\square$

We are now in a position to prove half of Theorem 1.3.

Corollary 2.3. $D(L)$ is a Weyl skew field over F in the following cases.

(1) The generic stabilizer of W is finite and K/F is purely transcendental.
(2) The generic stabilizer of W is trivial and G is special.

Proof. (1) is a direct application of Proposition 2.1. In case (2), X is a principal homogeneous G-space and hence the fact that G is special implies that $X \cong G_F$. This shows what we want since G is rational. □

Example 2.4. Let $\mathfrak{g} = \mathfrak{sl}_2$ and let W be the 4-dimensional irreducible representation of $\mathfrak{g}$. Then the generic stabilizer of W is non-trivial so corollary 2.3(2) does not apply. However Kraft has shown by explicit computation that in this case K/F is rational [8]. So 2.3(1) yields that $D(L)$ is a Weyl skew field.

Corollary 2.5. Assume that the generic stabilizer of W is trivial. Then the field of fractions of $D(L) \otimes_F K$ is a Weyl skew field over K.

Proof. In this situation X is principal homogeneous G-space. Now the generic point of X is by definition a K-rational point and so we have $X_K \cong G_K$. In particular the field of fractions of $K \otimes_F K$ is purely transcendental over K. The corollary now follows from Proposition 2.1, and the fact that differential operators are compatible with base field extension. □

In Theorem 1.3 and Proposition 2.1 we have stated, for elegance, the results in terms of W, although it would have been more natural to work with W^*. We therefore recall the following well-known result.

Lemma 2.6. Let G be a connected semisimple algebraic group over k. Then there exists a (non-unique) automorphism θ of G such that for any finite dimensional representation W of G one has $W^* = {}_\theta W$.

Proof. There are various constructions for θ. One possibility is the following. Let $T \subset G$ be a maximal torus and let θ' be the automorphism of T which sends t to t^{-1}. θ' sends the roots of (G,T) to their inverses and hence defines an automorphism of the root system of (G,T). Therefore θ' extends to an automorphism θ of G [6, Ch. 32]. This θ is the one we need since it sends the character of W to that of W^*. □

3. Subfields of Weyl skew fields

We now need a criterion for certain division algebras *not* to be Weyl skew fields. We use the following observation :

Proposition 3.1. Assume that D is a Weyl skew field with center F. Let $K \subset D$ be a subfield containing F and let H be a reductive group over F. Then the canonical map

$$H^1_{\text{et}}(F,H) \to H^1_{\text{et}}(K,H)$$

is injective.

(Recall that for any field F, $H^1_{\text{et}}(F,H)$ parametrizes the principal homogeneous H-spaces over F. See e.g. [12])

Proof. By definition D is the field of fractions of some Weyl algebra A_n. If we filter A_n with the Bernstein filtration then $\operatorname{gr} A_n$ is a graded polynomial ring in $2n$ variables, each of degree one.

The filtration on A_n extends to a filtration on D whose associated graded ring is the *graded* field of fractions [14] of $\operatorname{gr} A_n$. It is easy to see that this implies that $\operatorname{gr} D = P[X, X^{-1}]$ where P is a purely transcendental field extension of F and $\deg X = 1$.

One easily verifies that the filtration on D is obtained from a discrete valuation with residue field P [9]. Then the restriction of this discrete valuation to K is either trivial or also a discrete valuation. In the first case $K \subset P$ and in the second case the residue field of the valuation on K is contained in P.

It now suffices to invoke the following lemma.

Sublemma. Assume that K/F is a field extension which satisfies one of the following conditions.

(1) K/F is unirational.
(2) K is the field of fractions of a discrete valuation ring R containing F whose residue class field is unirational.

Then $H^1_{\mathrm{et}}(F, H) \to H^1_{\mathrm{et}}(K, H)$ is injective.

Proof. We only prove (2) since there is an extra point involved.

Let l be the residue class field of R and let P/F be a purely transcendental extension containing l.

Let $X \in \ker(H^1_{\mathrm{et}}(F, H) \to H^1_{\mathrm{et}}(K, H))$. By [15], $H^1_{\mathrm{et}}(R, H) \to H^1_{\mathrm{et}}(K, H)$ is injective, so

$$X \in \ker(H^1_{\mathrm{et}}(F, H) \to H^1_{\mathrm{et}}(R, H))$$

But then X is also in the kernel of the composition

$$H^1_{\mathrm{et}}(F, H) \to H^1_{\mathrm{et}}(R, H) \to H^1_{\mathrm{et}}(l, H) \to H^1_{\mathrm{et}}(P, H)$$

Now the map $H^1_{\mathrm{et}}(F, H) \to H^1_{\mathrm{et}}(P, H)$ is injective since P/F is purely transcendental and hence X must be trivial. □

This finishes the proof of Proposition 3.1. □

4. Non-special groups

To conclude we need the following result which is probably well known.

Proposition 4.1. Let G be a non-special semisimple algebraic group and let V be a G-representation with trivial generic stabilizer. Then

$$V \to V/\!\!/G$$

is generically a principal homogeneous G-space which is not trivial.

Proof. Let G be a semisimple, not necessarily non-special, algebraic group. According to [13, Appendix], the generic orbit in V is closed, so by the Luna Slice Theorem there exists a non-empty open affine G-invariant subset U of V such that the action of G on U is free.

Now recall that a principal homogeneous G-space $\phi : Y \to X$ is called *versal* if for all local rings A and for all principal homogeneous G-spaces $\theta : Z \to \operatorname{Spec} A$

there is a morphism $\operatorname{Spec} A \to X$ such that the pullback of ϕ along this map is equal to θ.

To prove the proposition it now suffices to prove the following stronger result which was suggested to us by Colliot-Thélène. □

Theorem 4.2. Let G be a linear algebraic group over k and let V be a non-empty finite dimensional G-representation. Assume that there is a non-empty open $U \subset V$ such that there exists a geometric quotient $U \to U/G$ which is a principal homogeneous G-space. Then $U \to U/G$ is versal.

Proof. It is shown by Serre [16] that if $G \subset \mathrm{Gl}_n$ is an embedding then $\mathrm{Gl}_n \to \mathrm{Gl}_n/G$ is versal. We now consider the following diagram

$$\begin{array}{ccccc} U & \longleftarrow & \mathrm{Gl}_n \times U & \longrightarrow & \mathrm{Gl}_n \\ \downarrow & & \downarrow & & \downarrow \\ U/G & \longleftarrow & \mathrm{Gl}_n \times^G U & \longrightarrow & \mathrm{Gl}_n/G \end{array}$$

Here the left and right squares are pull backs. Furthermore $\mathrm{Gl}_n \times^G U$ is open in $\mathrm{Gl}_n \times^G V$ which is a vector bundle on Gl_n/G.

It is easy to deduce from this last fact that the map $\mathrm{Gl}_n \times^G U \to \mathrm{Gl}_n/G$ has sections locally and is therefore surjective on A-points (where, as above, A denotes a local ring over k). Hence an A-point of Gl_n/G may be lifted to $\mathrm{Gl}_n \times^G U$ and then projected to U/G. This shows that any pullback of Gl_n for a map $\operatorname{Spec} A \to \mathrm{Gl}_n/G$ can be realized as a pullback of U for a map $\operatorname{Spec} A \to U/G$. This is what we had to show. □

Corollary 4.3. Let the notation be as in the statement of Theorem 1.3. If G is not special then $D(L)$ is not a Weyl skew field.

Proof. By Proposition 4.1, X is a non-trivial principal homogeneous G-space. Now X is trivially split by K and hence X represents a non-trivial element of

$$\ker(H^1_{\mathrm{et}}(F, G) \to H^1_{\mathrm{et}}(K, G))$$

By corollary 2.2 the center of $D(L)$ is F. Then, by Proposition 3.1, $D(L)$ cannot be a Weyl skew field over F since it contains K. □

References

1. W. Borho, P. Gabriel, and R. Rentschler, *Primideale in Einhüllenden auflösbarer Lie-Algebren*, Lecture Notes in Mathematics **357** (1973).
2. I. M. Gel'fand and A. A. Kirillov, *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie*, Inst. Hautes Études Sci. Publ. Math. **31** (1966), 5–19.
3. ———, *The structure of the Lie field connected with a split semisimple Lie algebra*, Funktsional. Anal. i Prilozhen. (1969), 6–21.
4. A. Grothendieck, *Torsion homologique et sections rationelles*, Anneaux de Chow et applications, Séminaire Chevalley, vol. 2, Secrétariat mathématiques, 1958.
5. N.-X. Hai, *Réduction de produits semi-directs et conjecture de Gel'fand et Kirillov*, Bull. Soc. Math. France (1979), 241–267.
6. J. E. Humpreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Springer Verlag, Berlin Heidelberg New York, 1975.
7. A. Joseph, *Proof of the Gel'fand-Kirillov conjecture for solvable Lie algebras*, Proc. Amer. Math. Soc. **45** (1974), 1–10.
8. H.-P. Kraft, June 1993, (personal communication to Jacques Alev).

9. H. Li and F. Van Oystaeyen, *Filtrations on simple artinian rings*, J. Algebra **132** (1990), no. 2, 361–376.
10. P. Malcolmson, *Enveloping algebras of simple three-dimensional Lie algebras*, J. Algebra **146** (1992), no. 1, 210–218.
11. J. C. McConnell, *Representations of solvable Lie algebras and the Gel'fand-Kirillov conjecture*, Proc. London Math. Soc. (3) (1974), 453–484.
12. J. S. Milne, *Etale cohomology*, Princeton Series in Mathematics, vol. 33, Princeton University Press, Princeton, New Jersey, 1980.
13. D. Mumford and J. Fogarty, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer Verlag, Berlin, Heidelberg, New-York, 1982.
14. C. Nastacescu and F. Van Oystaeyen, *Graded ring theory*, North-Holland, 1982.
15. Y. A. Nisnevic, *Espaces homogènes principaux rationellement triviaux et arithmétique des schémas en groupes réductifs sur les anneaux de Dedekind*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), 5–8.
16. J.-P. Serre, *Espaces fibrés algébriques*, Anneaux de Chow et applications, Séminaire Chevalley, vol. 2, Secrétariat mathématiques, 1958.

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