EXPLICIT RATIONAL FORMS FOR THE POINCARE SERIES OF THE TRACE RINGS OF GENERIC MATRICES

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ABSTRACT. In this paper we derive explicit rational forms for the Poincare series of the commutative and the non-commutative trace ring (5.3 and 5.5). To this end we use the Molien Weyl formula to reduce the question to a problem about flows in a particular graph.

1. INTRODUCTION.

For simplicity we will assume that the ground field is $\mathbb{C}$ in this paper. However it is clear that all results remain valid for an arbitrary algebraically closed field of characteristic zero.

Let m, n be natural numbers and let M_n be the variety of $n \times n$-matrices. $(M_n)^m$ will be the m-fold product $M_n \times \cdots \times M_n$. Put $G = SL_n$ and let G act on M_n by conjugation. Then one defines

$$Z_{m,n} = \{f : (M_n)^m \to \mathbb{C} \mid f \text{ polynomial and } G\text{−equivariant}\} \tag{1}$$

$$\mathbb{T}_{m,n} = \{f : (M_n)^m \to M_n \mid f \text{ polynomial and } G\text{−equivariant}\} \tag{2}$$

$Z_{m,n}$ is the commutative and $\mathbb{T}_{m,n}$ is the non-commutative trace ring of m generic $n \times n$-matrices.

Let V be an n-dimensional vector space. Define $G = SL(V)$, $W = (V \otimes V^*)^m$, $R = SW$, $\overline{R} = \mathrm{End}(V) \otimes R$. Then it is clear from (1), (2) that $Z_{m,n} = R^G$, $\mathbb{T}_{m,n} = \overline{R}^G$ for the obvious G-actions.

R and $\overline{R}$ may be $\mathbb{Z}^m$-graded by giving the elements in the i'th copy of $V \otimes V^*$ in W degree $(0, \ldots, 1, \ldots, 0)$ where the 1 occurs in the i'th place. Clearly $Z_{m,n}$ and $\mathbb{T}_{m,n}$ are graded subrings of R and $\overline{R}$ and we may therefore define their Poincare series $P(Z_{m,n}, t)$ and $P(\mathbb{T}_{m,n}, t)$ where $t = (t_i)_{i=1,\ldots,m}$ is a set of variables. Knowing the Poincare series of a graded ring can be an important first step in the determination of the actual structure of the ring. See e.g. [7].

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It is well known that $P(Z_{m,n},t)$ and $P(\mathbb{T}_{m,n},t)$ are rational functions with coefficients in $\mathbb{Z}$. However, apart from a few cases ($n=2$, $(m,n)=(2,3)$, $(2,4)$), few explicit formulas are known. See e.g. [7][3].

In this paper we will give general formulas for $P(Z_{m,n},t)$ and $P(\mathbb{T}_{m,n},t)$. Our main tool will be the Molien-Weyl formula [10] which gives the Poincaré series of a ring of invariants as an integral over a torus. The standard way to evaluate this integral is via the residue theorem. However, in all but the simplest cases, this procedure is complicated and unwieldy.

The main idea used in this paper is that, in the case of trace rings, the Molien-Weyl formula leads to an expression for the Poincaré series in terms of generating functions for flows on a certain graph. We then use elementary graph theory to compute these generating functions.

First we use a standard result in the theory of linear diophantine equations to show that the denominators involved are products of terms of the form $1-U(t)$ were $U(t)$ is a monomial of degree less than n (Thm 5.1). This result is stronger than what could be expected from the Procesi-Razmyzlow result which says that $Z_{m,n}$ is generated in degree $\leq n^2$ (see [4][5]).

A second observation we use is, that at the cost of losing some information, we may replace multiple arrows by single ones. This leads to expressions for $P(\mathbb{T}_{m,n},t)$ and $P(Z_{m,n},t)$ in terms two basic functions depending only on n (Prop. 5.3).

Finally we undertake the labour of computing the generating functions for flows on graphs in general. We obtain that such a generating function is a sum of rational functions indexed by spanning trees for the graph. Again we may apply this result to trace rings (Thm. 5.5).

Although the number of terms in the resulting expression for the Poincaré series is rather large, each of the individual terms has a simple structure. For example we could use this expression to give an almost trivial proof for the functional equation satisfied by $P(\mathbb{T}_{m,n},t)$ [2][8][9].

2. Preliminaries about graphs.

In the sequel a (finite directed) graph will be a quadruple $\mathcal{G} = (V, E, h, t)$ where V, E are finite sets and h, $t : E \to V$ are arbitrary maps. V will be the set of vertices and E will be the set of edges (arrows) in $\mathcal{G}$. If $e \in E$ then $t(e)$, $h(e)$ are resp. the beginning and the end of e. Sometimes if V, E, h, t are not specified then we will use the notation $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G}, h_\mathcal{G}, t_\mathcal{G})$.

If $\mathcal{G} = (V', E', h', t')$ then we will say that $\mathcal{G}'$ is a subgraph of $\mathcal{G}$ (notation : $\mathcal{G}' \subset \mathcal{G}$) if $V' \subset V$, $E' \subset E$ and $h' = h \mid E'$, $t' = t \mid E'$.

A path P in $\mathcal{G}$ will be a sequence $(v_1, e_1, \ldots, e_n, v_{n+1})$ where $(v_i)_i \in V$, $(e_i)_i \in E$ and $\{h(e_i), t(e_i)\} = \{v_i, v_{i+1}\}$. We say that P starts in v_1 and ends in v_{n+1}.

We call an e_i in P is correctly oriented if $v_i = t(e_i)$, $v_{i+1} = h(e_i)$. Otherwise, e_i is incorrectly oriented. If all edges in P are correctly oriented then we say that P is an oriented path.

Below we will use the following notation : assume that $a = (a_e)_{e\in E}$ are elements of $\mathbb{C}$ and $P = (v_1, e_1, \ldots, e_n, v_{n+1})$ is a path then

$$a^P = \prod_i a_{e_i}^{\epsilon(e_i)} \tag{3}$$

where $\epsilon(e_i) = 1$ if e_i is correctly oriented in P and -1 otherwise.

A graph is said to be connected if for any two vertices v, w there is a path starting in v and ending in w.

A path $P = (v_1, e_1, \ldots, e_n, v_{n+1})$ is closed if $v_{n+1} = v_1$. If P' is another closed path then we say that $P = P'$ if $P' = (v_j, e_j, \ldots, v_{j-1}, e_{j-1}, v_j)$ for some j. If $v_i \neq v_j$ except if $i = j$ or if $(i, j) = (1, n + 1)$ then we call P a cycle. If P is in addition oriented then we say that P is an oriented cycle. We will use the notation $C(\mathcal{G})$ for the set of cycles in $\mathcal{G}$. The set of oriented cycles in $\mathcal{G}$ will be denoted by $\vec{C}(\mathcal{G})$.

A tree is a connected directed graph which does not contain any non-trivial cycles. If $\mathcal{G}$ is a graph and $\mathcal{T} \subset \mathcal{G}$ is a tree then we say that $\mathcal{T}$ is a spanning tree of $\mathcal{G}$ if $V_\mathcal{T} = V_\mathcal{G}$. It is a classical fact that any connected graph contains a spanning tree. We will write $T(\mathcal{G})$ for the set of spanning trees of $\mathcal{G}$.

If $\mathcal{T} \subset \mathcal{G}$ and $\mathcal{T}$ is a spanning tree then for any $e \in E_\mathcal{G} \setminus E_\mathcal{T}$ there is a unique non-trivial cycle, with e correctly oriented, in the graph obtained by adjoining e to $\mathcal{T}$. This cycle will be denoted by $C(e, \mathcal{T})$.

Let $\mathcal{T}$ be a tree and let e be an edge in $\mathcal{T}$. Then if we remove e from $\mathcal{T}$ we obtain two trees which we will denote by $\mathcal{T}_{t,e}$ and $\mathcal{T}_{h,e}$. Here $\mathcal{T}_{h,e}$ is the tree that contains $h(e)$.

3. Flows.

In this section we assume that we are given some finite connected graph $\mathcal{G} = (V, E, h, t)$ without loops, and variables $z = (z_v)_{v \in V}$, $t = (t_e)_{e \in E}$. Let f be a laurent polynomial, homogeneous of degree 0, in $(z_v)_{v \in V}$ with rational coefficients. Define

$$\overline{\Psi}(\mathcal{G}, f, t, z) = \frac{f}{\prod_{e \in E}(1 - t_e z_{t(e)}^{-1} z_{h(e)})}$$

Now assume $|t_e| < 1$. We may then define

$$\Psi(\mathcal{G}, f, t) = \int_{T_c} \frac{f}{\prod_{e \in E}(1 - t_e z_{t(e)}^{-1} z_{h(e)})} d\nu \tag{4}$$

Here T_c is the torus given by $\{(z_v)_{v \in V} \mid |z_v| = 1 \quad \forall v \in V\}$ and

$$d\nu = \frac{1}{(2\pi i)^{|V|}} \frac{\bigwedge_{v \in V} dz_v}{\prod_{v \in V} z_v}$$

is the usual T_c-invariant volume element.

In the sequel we will be interested in evaluating (4). First we show that, at the cost of loosing some information, multiple arrows may be replaced by single ones.

Lemma 3.1. *Assume that $\mathcal{G}'$ is obtained form $\mathcal{G}$ by replacing an edge e with edges $e_1, \ldots, e_u$ having the same begin and end vertex as e. Extend t to a new set of variables $t' = (t_e)_{e \in E_{\mathcal{G}'}}$. Then*

$$\Psi(\mathcal{G}', f, t')_{[t_{e_1} = t_e, \ldots, t_{e_u} = t_e]} = \frac{1}{(u-1)!} \frac{\partial^{u-1}}{\partial t_e^{u-1}} t_e^{u-1} \Psi(\mathcal{G}, f, t) \tag{5}$$

Proof. This follows from the fact that

$$\frac{1}{(1 - t_e z_{t(e)}^{-1} z_{h(e)})^u} = \frac{1}{(u-1)!} \frac{\partial^{u-1}}{\partial t_e^{u-1}} t_e^{u-1} \frac{1}{(1 - t_e z_{t(e)}^{-1} z_{h(e)})}$$

□

We may expand $f = \sum_i a_i \chi_{\mu_i}$ where $a_i \in \mathbb{Q}$, $\mu_i : V \to \mathbb{Z}$, $\sum_{v \in V} \mu_i(v) = 0$ and $\chi_{\mu_i} = \prod_{v \in V} z_v^{\mu_i(v)}$. Then it is clear that

$$\Psi(\mathcal{G}, f, t) = \sum_i a_i \Psi(\mathcal{G}, \chi_{\mu_i}, t) \tag{6}$$

and the same holds for $\overline{\Psi}$.

For simplicity's sake we will write $\Psi(\mathcal{G}, \mu, t)$ and $\overline{\Psi}(\mathcal{G}, \mu, t, z)$ for $\Psi(\mathcal{G}, \chi_\mu, t)$ and $\overline{\Psi}(\mathcal{G}, \chi_\mu, t, z)$. It should be noted that $\Psi(\mathcal{G}, \mu, t)$ is obtained from expanding $\overline{\Psi}(\mathcal{G}, \mu, t, z)$ into the t's and then dropping any term that contains z's. Using this fact one sees that $\Psi(\mathcal{G}, \mu, t)$ has a very natural interpretation in terms of graphs.

Definition 3.2. A flow in $\mathcal{G}$ associated to μ is a map $\phi : E \to \mathbb{N}$ such that for all $v \in V$

$$\sum_{h(e)=v} \phi(e) + \mu(v) = \sum_{t(e)=v} \phi(e)$$

If $\mu(v) = 0$ for all v then a flow associated to μ will just be called a flow.

We may now write

$$\Psi(\mathcal{G}, \mu, t) = \sum_{\phi} \prod_{e \in E} t_e^{\phi(e)}$$

where the summation runs over all flows in $\mathcal{G}$ associated to μ.

A flow clearly corresponds to the solution of a system of linear diophantine equations. Therefore, in analogy with [6], we will call a non-zero flow fundamental if for any decomposition $\phi = \phi_1 + \phi_2$ with flows ϕ_1, ϕ_2 either $\phi_1 = 0$ or $\phi_2 = 0$.

A non-zero flow will be called completely fundamental if $n\phi = \phi_1 + \phi_2$, for flows ϕ_1, ϕ_2 and $n \in \mathbb{N}$, implies that $\phi_1 = n_1\phi$, $\phi_2 = n_2\phi$ where n_1, $n_2 \in \mathbb{N}$.

The following lemma may be easily deduced from [1, Th. 8.2]

Lemma 3.3.

(1) *Assume that $C = (v_1, e_1, \ldots, e_u, v_1)$ is an oriented cycle in $\mathcal{G}$. Let ϕ be defined as follows : $\phi(e) = 1$ if e is in C and $\phi(e) = 0$ otherwise. Then ϕ is a completely fundamental flow.*
(2) *Any fundamental flow on $\mathcal{G}$ is of the form given in 1.*
(3) *Any fundamental flow on $\mathcal{G}$ is completely fundamental.*

Using the results in [6] we obtain :

$$\Psi(\mathcal{G}, \mu, t) = \frac{Q_\mu(t)}{\prod_{C \in \vec{C}(\mathcal{G})} (1 - t^C)} \tag{7}$$

where $Q_\mu(t) \in \mathbb{Z}[(t_e)_{e \in E}]$ (the notation t^C is as in (3)).

Combining (6) with (7) we obtain the following result.

Theorem 3.4. *With notations as above. There exists a polynomial $Q(t) \in \mathbb{Q}[(t_e)_{e \in E}]$ such that*

$$\Psi(\mathcal{G}, f, t) = \frac{Q(t)}{\prod_{C \in \vec{C}(\mathcal{G})} (1 - t^C)}$$

4. A rational expression for $\Psi(\mathcal{G}, f, t)$

In this section we retain the notations of the previous sections. We will compute $\Psi(\mathcal{G}, f, t)$ under some mild conditions (Thm. 4.13). This section is basically a series of lemmas which lead to the proof of 4.13.

Lemma 4.1. *Let $z_1, \ldots, z_n$ be a set of variables and let $b_1, \ldots, b_n$ be elements of $\mathbb{C}$, such that $b_1 \cdots b_n \neq 1$. Then*

$$\frac{z_1}{\prod_{i=1}^{n}(z_i - b_i z_{i+1})} = \sum_{j=1}^{n} \frac{B_j}{\prod_{i \neq j}(z_i - b_i z_{i+1})} \tag{8}$$

Here $z_{n+1} = z_1$, and the B_i's are elements of $\mathbb{C}$.

Proof. If we multiply (8) with $\prod_{i=1}^{n}(z_i - b_i z_{i+1})$ we see that we have to solve

$$z_1 = \sum_{j=1}^{n} B_j(z_j - b_j z_{j+1})$$

or

$$\begin{aligned} 1 &= B_1 - b_n B_n \\ 0 &= B_j - b_{j-1} B_{j-1} \qquad \text{for } j \neq 1 \end{aligned}$$

It is clear that this system has a solution if $b_1 \cdots b_n \neq 1$. □

Now let $\mathcal{G}$ be some connected, directed graph without loops and assume that there are given $t = (t_e)_{e \in E} \in \mathbb{C}$ and variables $(z_v)_{v \in V}$. Assume furthermore that for any cycle $C \in C(\mathcal{G})$, $t^C \neq 1$.

Let

$$P'(\mathcal{G}, t) = \frac{1}{\prod_{e \in E}(z_{t(e)} - t_e z_{h(e)})}$$

Lemma 4.2. *Let $C = (v_1, e_1, \ldots, v_n, e_n, v_1)$ be a cycle in $\mathcal{G}$ and let $\mathcal{G}_i$ be the graph obtained from $\mathcal{G}$ by deleting the edge e_i. Then*

$$z_{v_1} P'(\mathcal{G}, t) = \sum_{i=1}^{n} A_i P'(\mathcal{G}_i, (t_e)_{e \neq e_i})$$

where $A_i \in \mathbb{C}$.

Proof. This is a direct consequence of lemma 4.1. Note however that, since C is not necessarily oriented, the A_i's are not necessarily the same as the B_i's in lemma 4.1. □

If $\mathcal{T} \in T(\mathcal{G})$ then we define $M'_{\mathcal{T}} = \prod_{e \in E \setminus E_{\mathcal{T}}} z_{t(e)}$. Now choose some $\mathcal{T}_0 \in T(\mathcal{G})$ and $S \subset E \setminus E_{\mathcal{T}_0}$. Define $U = \prod_{e \in S} z_{t(e)}^{-1} z_{h(e)}$

Lemma 4.3.

$$UM'_{\mathcal{T}_0}P'(\mathcal{G},t)=\sum_{\mathcal{T}\in T(\mathcal{G})}A_{\mathcal{T}}P'(\mathcal{T},(t_e)_{e\in E_{\mathcal{T}}})$$

where the $A_{\mathcal{T}}$'s are elements of $\mathbb{C}$ (depending on the choice of S).

Proof. The proof is by induction on $|E|$. Let $e_1\in E\setminus E_{\mathcal{T}_0}$ and assume that $C(e_1,\mathcal{T}_0)=(v_1,e_1,\ldots,v_n,e_n,v_1)$. Denote with $\mathcal{T}_0^{(i)}$ $(i=1,\ldots,n)$ the graph obtained from $\mathcal{T}_0$ by first adding e_1 and then deleting e_i. Hence $\mathcal{T}_0^{(1)}=\mathcal{T}_0$. It is well known that $\mathcal{T}_0^{(i)}$ is still a tree, and hence it is a spanning tree for $\mathcal{G}_i$ (with $\mathcal{G}_i$ as in lemma 4.2). Also define $S_i=S\setminus\{e_i\}$ and $U_i=\prod_{e\in S_i}z_{t(e)}^{-1}z_{h(e)}$. From lemma 4.2 it now follows that

$$UM'_{\mathcal{T}_0}P'(\mathcal{G},t)=\sum_{i=1}^{n}A_iU_iM'_{\mathcal{T}_0^{(i)}}P'(\mathcal{G}_i,(t_e)_{e\neq e_i})$$

The proof follows now by induction since lemma 4.3 is trivially correct if $\mathcal{G}$ is a tree. $\square$

Now define

$$P(\mathcal{G},t)=\frac{1}{\prod_{e\in E}(1-t_ez_{t(e)}^{-1}z_{h(e)})}$$

and if $\mathcal{T}\in T(\mathcal{G}):\ M_{\mathcal{T}}=\prod_{e\in E_{\mathcal{T}}}z_{t(e)}$.

Then lemma 4.3 implies that

$$P(\mathcal{G},t)=\sum_{\mathcal{T}\in T(\mathcal{G})}A_{\mathcal{T}}U^{-1}M_{\mathcal{T}_0}M_{\mathcal{T}}^{-1}P(\mathcal{T},(t_e)_{e\in E_{\mathcal{T}}})\tag{9}$$

Our next aim will be to determine the value of the $A_{\mathcal{T}}$'s.

We associate with $\mathcal{T}$ the equations

$$z_{t(e)}=t_ez_{h(e)}$$

where $e\in E_{\mathcal{T}}$. Fix $v_0\in V$. We then use those equations to express $(z_v)_{v\in V}$ in z_{v_0}. If we substitute the expressions for $(z_v)_{v\in V}$ into a rational function f in $(z_v)_{v\in V}$ we obtain a rational function in z_{v_0}. We will denote this new function by $f|\mathcal{T}$. This construction is particularly interesting if f is actually a function of $(z_v^{-1}z_w)_{v,w\in V}$ since then $f|\mathcal{T}$ will be a scalar. I.e. $f|\mathcal{T}$ will not depend on z_{v_0}.

Lemma 4.4. *With notations as above*

$$A_{\mathcal{T}}=\frac{(UM_{\mathcal{T}}M_{\mathcal{T}_0}^{-1})|\mathcal{T}}{\prod_{e\in E\setminus E_{\mathcal{T}}}(1-t^{C(e,\mathcal{T})})}$$

Proof. Multiply (9) with $\prod_{e\in E}(1-t_e z_{t(e)}^{-1} z_{h(e)})$ and apply $\mid \mathcal{T}$. From the fact that

$$(1-t_e z_{t(e)}^{-1} z_{h(e)}) \mid \mathcal{T} = \begin{matrix} 0 & \text{if} \quad e\in E_{\mathcal{T}} \\ 1-t^{C(e,\mathcal{T})} & \text{if} \quad e\notin E_{\mathcal{T}} \end{matrix}$$

we deduce that

$$\begin{aligned} 1 &= A_{\mathcal{T}}(U^{-1}M_{\mathcal{T}_0}M_{\mathcal{T}}^{-1}) \mid \mathcal{T} \prod_{e\in E\setminus E_{\mathcal{T}}} (1-t_e z_{h(e)}^{-1} z_{t(e)}) \mid \mathcal{T} \\ &= A_{\mathcal{T}}(U^{-1}M_{\mathcal{T}_0}M_{\mathcal{T}}^{-1}) \mid \mathcal{T} \prod_{e\in E\setminus E_{\mathcal{T}}} (1-t^{C(e,\mathcal{T})}) \end{aligned}$$

This proves lemma 4.4. □

From now on, we assume as above $|t_e|<1$ and if $C\in C(\mathcal{G})$ then $t^C \neq 1$. This last condition excludes an additional set of measure 0. We will use the above computations to derive a rational expression for $\Psi(\mathcal{G},\mu,t)$.

As before

$$\begin{aligned} \Psi(\mathcal{G},\mu,t) &= \int_{T_c} \overline{\Psi}(\mathcal{G},\mu,t,z)d\nu \\ &= \int_{T_c} \chi_\mu P(\mathcal{G},t)d\nu \\ &= \int_{T_c} \chi_\mu \sum_{\mathcal{T}\in T(\mathcal{G})} A_{\mathcal{T}} U^{-1}M_{\mathcal{T}_0}M_{\mathcal{T}}^{-1} \frac{1}{\prod_{e\in E_{\mathcal{T}}}(1-t_e z_{t(e)}^{-1} z_{h(e)})} d\nu \\ &= \sum_{\mathcal{T}\in T(\mathcal{G})} A_{\mathcal{T}} \int_{T_c} \frac{\chi_\mu U^{-1}M_{\mathcal{T}_0}M_{\mathcal{T}}^{-1}}{\prod_{e\in E_{\mathcal{T}}}(1-t_e z_{t(e)}^{-1} z_{h(e)})} d\nu \qquad (10) \end{aligned}$$

Lemma 4.5. *With notations and assumptions as above. Let $\eta : V\to \mathbb{Z}$ be some function with $\sum_{v\in V}\eta(v)=0$ and assume that $\chi_\eta \mid \mathcal{T} = \prod_{e\in E_{\mathcal{T}}} t_e^{b_e}$. Then*

$$\int_{T_c} \frac{\chi_\eta}{\prod_{e\in E_{\mathcal{T}}}(1-t_e z_{t(e)}^{-1} z_{h(e)})} d\nu = \begin{matrix} 0 & \text{if } \exists e\in E_{\mathcal{T}} : b_e<0 \\ \chi_\eta \mid \mathcal{T} & \text{otherwise} \end{matrix} \qquad (11)$$

Proof.

$$\frac{\chi_\eta}{\prod_{e\in E_{\mathcal{T}}}(1-t_e z_{t(e)}^{-1} z_{h(e)})} = \sum_{(u_e\geq 0)} \prod_e t_e^{u_e} \prod_e (z_{t(e)}^{-1} z_{h(e)})^{u_e} \chi_\eta \qquad (12)$$

To compute the left hand side of (11) we have to pick those terms of (12) that do not depend on z. I.e. those terms where

$$1 = \prod_{e\in E_{\mathcal{T}}} (z_{t(e)}^{-1} z_{h(e)})^{u_e} \chi_\eta$$

Applying $\mid \mathcal{T}$ we obtain

$$1 = \prod_{e \in E_{\mathcal{T}}} t_e^{-u_e} (\chi_\eta \mid \mathcal{T})$$

or $u_e = b_e$. Hence if there exists an e such that $b_e < 0$ then there is no suitable term in (12). Otherwise there is exactly one and it has the form

$$\prod_e t_e^{u_e} \prod_e (z_{t(e)}^{-1} z_{h(e)})^{u_e} \chi_\eta = \prod_e t_e^{b_e} = \chi_\eta \mid \mathcal{T}$$

□

Below we will need a way to determine the numbers $(b_e)_{e \in E_{\mathcal{T}}}$ which were introduced in the proof of the previous lemma. This may be done as follows.

Lemma 4.6. *Assume that* $\chi_\eta \mid \mathcal{T} = \prod_{e \in E_{\mathcal{T}}} (t_e)^{b_e}$. *Then for* $e \in E_{\mathcal{T}}$: $b_e = \sum_{v \in V_{\mathcal{T}_{t,e}}} \eta(v) = -\sum_{v \in V_{\mathcal{T}_{h,e}}} \eta(v)$.

It now becomes natural to introduce the following definition.

Definition 4.7. $\mathcal{T} \in T(\mathcal{G})$ will be $(S, \mathcal{T}_0, \mu)$ admissible if $\chi_\mu U^{-1} M_{\mathcal{T}_0} M_{\mathcal{T}}^{-1} \mid \mathcal{T}$ is of the form $\prod_{e \in E_{\mathcal{T}}} t_e^{b_e}$ with $b_e \geq 0$. (Recall that $U = \prod_{e \in S} z_{t(e)}^{-1} z_{h(e)}$).

We define

$$T_{S,\mathcal{T}_0,\mu}(\mathcal{G}) = \{\mathcal{T} \in T(\mathcal{G}) \mid \mathcal{T} \text{ is } (S, \mathcal{T}_0, \mu) \text{ admissible } \}$$

We are now ready to prove the following result :

Theorem 4.8. *With notations as above :*

$$\Psi(\mathcal{G}, \mu, t) = \sum_{\mathcal{T} \in T_{S,\mathcal{T}_0,\mu}(\mathcal{G})} \frac{\chi_\mu \mid \mathcal{T}}{\prod_{e \in E \setminus E_{\mathcal{T}}} (1 - t^{C(e,\mathcal{T})})}$$

Proof. This is a direct consequence of (10), lemma 4.4 and lemma 4.5. □

Our next aim will now be to analyze $T_{S,\mathcal{T}_0,\mu}(\mathcal{G})$. This seems to be hard to do in general. However, if $\mathcal{G}$ satisfies condition 4.9 below, then the problem is tractable.

Condition 4.9. *There exists a vertex* v_0 *in* $\mathcal{G}$ *such that for any* $w \in V$ *there is an oriented path in* $\mathcal{G}$, *starting in* w *and ending in* v_0.

Hence from now on we will assume that $\mathcal{G}$ satisfies 4.9. We will need the following lemma below.

Lemma 4.10. *Let* $\mathcal{T}$ *be a tree containing a vertex* v_0. *Then the following are equivalent :*

(1) $M_{\mathcal{T}} = \prod_{v \in V_{\mathcal{T}} \setminus \{v_0\}} z_v$
(2) *No edge in $\mathcal{T}$ starts in v_0. For any other vertex v in $\mathcal{T}$ there is at most one edge leaving v.*
(3) *Let $e \in E_{\mathcal{T}}$. Then $v_0 \in V_{\mathcal{T}_{h,e}}$.*

Proof. (2)⇒(1) is easy using the fact that for a tree $|V_{\mathcal{T}}| = 1 + |E_{\mathcal{T}}|$. (3)⇒(2) is also easy, so we will prove (1)⇒(3).

Let $e \in E_{\mathcal{T}}$ and assume that $v_0 \in V_{\mathcal{T}_{t,e}}$. On the one hand we have

$$M_{\mathcal{T}} = M_{\mathcal{T}_{h,e}} M_{\mathcal{T}_{t,e}} z_{t(e)}$$

and on the other hand

$$M_{\mathcal{T}} = \prod_{v \in V_{\mathcal{T}_{h,e}}} z_v \prod_{v \in V_{\mathcal{T}_{t,e}} \setminus \{v_0\}} z_v$$

Comparing these two expressions leads to

$$M_{\mathcal{T}_{h,e}} = \prod_{v \in V_{\mathcal{T}_{h,e}}} z_v$$

which is impossible by the fact that

$$|E_{\mathcal{T}_{h,e}}| = |V_{\mathcal{T}_{h,e}}| - 1$$

□

Lemma 4.11. *With notations and assumptions as above. There exist a $\mathcal{T} \in T(\mathcal{G})$ satisfying one of the conditions in lemma 4.10 (and hence all off them).*

Proof. We will show that there is a $\mathcal{T}$ satisfying condition (2) of lemma 4.10. Suppose $\mathcal{T}'$ is a maximal subtree of $\mathcal{G}$ containing v_0 and satisfying condition (2) of lemma 4.10. Assume $V_{\mathcal{T}'} \neq V$. Let $v \in V \setminus V_{\mathcal{T}'}$. Then there is an oriented path P starting in v and ending in v_0. Suppose v_1 is the vertex on P nearest to v that is also contained in $V_{\mathcal{T}'}$. Let P_1 be the subpath of P starting in v and ending in v_1. If we adjoin the vertices and edges of P_1 to $\mathcal{T}'$ we obtain a bigger tree, still satisfying 4.10(2). This is a contradiction. □

Now let $T_{v_0}(\mathcal{G})$ be the set of all spanning trees of $\mathcal{G}$, satisfying one of the conditions of lemma 4.10. By lemma 4.11 this is a non-empty set and hence we may pick a particular $\mathcal{T}_0 \in T_{v_0}(\mathcal{G})$.

We will now be interested in when $T_{S,\mathcal{T}_0,\mu}(\mathcal{G}) = T_{v_0}(\mathcal{G})$. It clearly suffices to treat the case $S = \emptyset$ since we may always write $U^{-1}\chi_\mu = \chi_\eta$. For simplicity we define $T_{\mathcal{T}_0,\mu}(\mathcal{G}) = T_{1,\mathcal{T}_0,\mu}(\mathcal{G})$.

Lemma 4.12. *With notations and assumptions as above.*

$$\forall v \neq v_0 : \mu(v) \geq 0 \Rightarrow T_{\mathcal{T}_0,\mu}(\mathcal{G}) = T_{v_0}(\mathcal{G})$$

Proof. Assume that μ is such that $\forall v \neq v_0 : \mu(v) \geq 0$. We will first show that $T_{v_0}(\mathcal{G}) \subset T_{\mathcal{T}_0,\mu}(\mathcal{G})$ and then $T_{\mathcal{T}_0,\mu}(\mathcal{G}) \subset T_{v_0}(\mathcal{G})$.

First choose $\mathcal{T} \in T_{v_0}(\mathcal{G})$. Then $M_{\mathcal{T}_0} M_{\mathcal{T}}^{-1} = 1$ by lemma 4.10. Let $\chi_\mu | \mathcal{T} = \prod_{e \in E_{\mathcal{T}}} t_e^{b_e}$. If $e \in E_{\mathcal{T}}$ then by lemma 4.6 $b_e = \sum_{v \in V_{\mathcal{T}_{t,e}}} \mu(v)$ and since by lemma 4.10 $v_0 \notin V_{\mathcal{T}_{t,e}}$ we deduce that $b_e \geq 0$. Hence $\mathcal{T} \in T_{\mathcal{T}_0,\mu}(\mathcal{G})$. This proves half of lemma 4.12.

To prove the other half, let $\mathcal{T} \in T_{\mathcal{T}_0,\mu}(\mathcal{G})$ and define $\chi_\eta = \chi_\mu M_{\mathcal{T}_0} M_{\mathcal{T}}^{-1}$. If $\chi_\eta | \mathcal{T} = \prod_{e \in E_{\mathcal{T}}} t_e^{b_e}$ then by assumption $b_e \geq 0$.

Let $e \in E_{\mathcal{T}}$ and suppose that $v_0 \in V_{\mathcal{T}_{t,e}}$. Then by lemma 4.6

$$0 \geq -b_e \geq \sum_{v \in \mathcal{T}_{h,e}} \eta(v) \geq \sum_{v \in \mathcal{T}_{h,e}} \eta(v) - \mu(v)$$

$$= \sum_{v \in \mathcal{T}_{h,e}} (1 - \#\text{of times } v \text{ is a tail in } \mathcal{T}) = |V_{\mathcal{T}_{h,e}}| - |E_{\mathcal{T}_{h,e}}|$$

which implies $|V_{\mathcal{T}_{h,e}}| \leq |E_{\mathcal{T}_{h,e}}|$. This is of course impossible. Therefore $v_0 \in E_{\mathcal{T}_{t,e}}$ and by lemma 4.10 we deduce that $\mathcal{T} \in T_{v_0}(\mathcal{G})$. □

Now let us call μ good (with respect to $(v_0, \mathcal{T}_0)$) if there exists an $S \subset E \setminus E_{\mathcal{T}_0}$ with $U = \prod_{e \in S} z_{t(e)}^{-1} z_{h(e)}$ such that if $\chi_\eta = U^{-1} \chi_\mu$ then $\eta(v) \geq 0$ for all $v \in V \setminus \{v_0\}$.

We now we use the above to prove the final theorem of this section.

Theorem 4.13. *Assume that $\mathcal{G} = (V, E, h, t)$ is a finite, directed and connected graph satisfying 4.9 with respect to some $v_0 \in V$. Choose a fixed $\mathcal{T}_0 \in T_{v_0}(\mathcal{G})$. Let $f = \sum_i a_i \chi_{\mu_i}$ where all the μ_i's are good with respect to $(v_0, \mathcal{T}_0)$. Then*

$$\Psi(\mathcal{G}, f, t) = \sum_{\mathcal{T} \in T_{v_0}(\mathcal{G})} \frac{f | \mathcal{T}}{\prod_{e \in E \setminus E_{\mathcal{T}}} (1 - t^{C(e, \mathcal{T})})}$$

5. Trace rings.

Let m, n be positive integers and let V be an n-dimensional vector space. Define $G = SL(V)$, $W = (V \otimes V^*)^m$, $R = SW$, $\overline{R} = \mathrm{End}(V) \otimes R$. Here $\overline{R}$ will be considered as a non-commutative R-algebra.

Both R and $\overline{R}$ may be $\mathbb{Z}^m$-graded by giving the elements in the i'th copy of $V \otimes V^*$ in W degree $(0, \dots, 1, \dots, 0)$ where the 1 occurs in the i'th place.

Both R and $\overline{R}$ admit a G-action, and it is classical that $Z_{m,n} = R^G$, $\mathbb{T}_{m,n} = \overline{R}^G$ where $Z_{m,n}$ and $\mathbb{T}_{m,n}$ are respectively the commutative trace ring and the noncommutative trace ring of m generic $n \times n$-matrices.

Let $t = (t_i)_{i=1,\dots,m}$ be indeterminates. Then it is possible to give an expression for the Poincare series $P(Z_{m,n}, t)$, $P(\mathbb{T}_{m,n}, t)$ using the Molien Weyl formula [10]. Assume that $T \subset G$ is a maximal algebraic torus in G. We will identify T with $(\mathbb{C}^*)^n$ and write its elements as $(z_1, \dots, z_n)$ where $(z_i)_i \in \mathbb{C}^*$. Let T_c consists of the elements $(z_1, \dots, z_n)$ in T such that $|z_i| = 1$ for all i. The Molien Weyl formula states that

$$P(Z_{m,n}, t) = \frac{1}{n!} \int_{T_c} \frac{\prod_{i \neq j} (1 - z_i^{-1} z_j)}{\prod_{k=1}^m \prod_{i,j=1}^n (1 - z_i^{-1} z_j t_k)} d\nu$$

where as usual

$$d\nu = \frac{1}{(2\pi i)^n} \frac{dz_1 \wedge \dots \wedge dz_n}{z_1 \cdots z_n}.$$

Now let $\mathcal{K}_{m,n}$ be the following graph : $\mathcal{K}$ has n vertices, labeled $[1], \dots, [n]$ and $mn(n-1)$ edges labeled $[i,j,k]$ for $1 \leq i, j \leq n$, $i \neq j$, $1 \leq k \leq m$ where $[i,j,k]$ starts in $[i]$ and ends in $[j]$.

We now choose a set of variables where $t^\circ = (t_{ij,k})_{i,j=1,\dots,n}^{k=1,\dots,m}$ where $t_{ij,k}$ will be associated to the edge $[i,j,k]$ in $\mathcal{K}_{m,n}$. We then obtain

$$\begin{aligned} P(Z_{m,n}, t) &= \frac{1}{n!} \left[\frac{1}{\prod_{i,k} (1 - t_{ii,k})} \int_{T_c} \frac{\prod_{i \neq j} (1 - z_i^{-1} z_j)}{\prod_{\substack{i,j,k \\ i \neq j}} (1 - z_i^{-1} z_j t_{ij,k})} d\nu \right]_{t_{ij,k} = t_k} \\ &= \frac{1}{n!} \frac{1}{\prod_k (1 - t_k)^n} \Psi(\mathcal{K}_{m,n}, f, t^\circ) \mid_{t_{ij,k} = t_k} \end{aligned}$$

where $f = \prod_{i \neq j} (1 - z_i^{-1} z_j)$.

In a similar way we deduce from the Molien Weyl formula that

$$P(\mathbb{T}_{m,n}, t) = \frac{1}{n!} \frac{1}{\prod_k (1 - t_k)^n} \Psi(\mathcal{K}_{m,n}, g, t^\circ) \mid_{t_{ij,k} = t_k}$$

where $g = \left[\prod_{i \neq j} (1 - z_i^{-1} z_j) \right] \left(\sum_{i,j=1}^n z_i^{-1} z_j \right)$.

Hence we have to determine what the $\Psi(\mathcal{K}_{m,n}, h, t^\circ)$ are, where h is some polynomial function of $(z_i^{-1}z_j)_{i,j}$. To this end we will use the formulas derived in the previous sections.

As a direct application of Th. 3.4 we deduce the following

Proposition 5.1. *Both $P(Z_{m,n}, t)$ and $P(\mathbb{T}_{m,n}, t)$ may be written in the form P/Q, P, $Q \in \mathbb{Z}[(t_k)_k]$ where Q is a product of terms of the form $(1 - U(t))$ with U a monomial in $(t_k)_k$ of degree $\leq n$.*

Remark 5.2. Prop. 5.1 seems to suggest that $Z_{m,n}$ is generated in degrees $\leq n$. This is completely false however, even in the case $(m, n) = (2, 2)$. In general $Z_{m,n}$ is conjectured to be generated in degrees $\leq \frac{n(n+1)}{2}$ and this bound is supposed to be optimal.

Using lemma 3.3 we obtain a general formula for $P(Z_{m,n}, t)$ and $P(\mathbb{T}_{m,n}, t)$ in single grading. Let s be a variable and write

$$P(Z_{m,n}, s) = P(Z_{m,n}, t) \mid_{t_k = s}$$

$$P(\mathbb{T}_{m,n}, s) = P(\mathbb{T}_{m,n}, t) \mid_{t_k = s}$$

Now we introduce another set of variables $u = (u_{ij})_{\substack{i,j=1,\dots n \\ i \neq j}}$ and we let

$$H(h, u) = \Psi(\mathcal{K}_{1,n}, h, t^\circ) \mid_{t_{ij,1} = u_{ij}} = \int_{T_c} \frac{h}{(1 - \prod_{i \neq j} u_{ij} z_i^{-1} z_j)} d\nu$$

Then we obtain

Proposition 5.3.

$$P(Z_{m,n}, t) = \frac{1}{n!} \frac{1}{(m-1)!} \frac{1}{(1-s)^{mn}} \frac{\partial^{(m-1)n(n-1)}}{\partial u_{12}^{m-1} \partial u_{13}^{m-1} \cdots \partial u_{n\,n-1}^{m-1}} (\prod_{i \neq j} u_{ij}^{m-1}) H(f, u) \mid_{u_{ij} = s}$$

$$P(\mathbb{T}_{m,n}, t) = \frac{1}{n!} \frac{1}{(m-1)!} \frac{1}{(1-s)^{mn}} \frac{\partial^{(m-1)n(n-1)}}{\partial u_{12}^{m-1} \partial u_{13}^{m-1} \cdots \partial u_{n\,n-1}^{m-1}} (\prod_{i \neq j} u_{ij}^{m-1}) H(g, u) \mid_{u_{ij} = s}$$

To apply Theorem 4.13, note that $\mathcal{K}_{m,n}$ obviously satisfies 4.9. We will put $v_0 = [n]$ and $\mathcal{T}_0$ will be the tree consisting of the edges $[i, i+1, 1]$ for $i = 1, \dots, n-1$.

If we then decompose f and g into χ_μ's, we have to determine which μ's are good. It is easy to see that the result is given by the following lemma

Lemma 5.4. *The μ's occurring in the expansion of f and g are all good (with respect to the chosen $(v_0, \mathcal{T}_0)$) if $(m, n) \geq (2, 3)$ or $(m, n) \geq (3, 2)$.*

Hence we have proved the following theorem :

Theorem 5.5. *With notations as above. Assume that* $(m,n) \geq (2,3)$ *or* $(m,n) \geq (3,2)$*. Then Poincare series of* $Z_{m,n}$ *and* $\mathbb{T}_{m,n}$ *are given by*

$$P(Z_{m,n},t) = \frac{1}{n!}\frac{1}{\prod_k (1-t_k)^n}\left[\sum_{\mathcal{T}\in T_{v_0}(\mathcal{K}_{m,n})} \frac{f \mid \mathcal{T}}{\prod_{e\in E\setminus E_{\mathcal{T}}}(1-t^{\circ C(e,\mathcal{T})})}\right]_{t_{ij,k}=t_k}$$

$$P(\mathbb{T}_{m,n},t) = \frac{1}{n!}\frac{1}{\prod_k (1-t_k)^n}\left[\sum_{\mathcal{T}\in T_{v_0}(\mathcal{K}_{m,n})} \frac{g \mid \mathcal{T}}{\prod_{e\in E\setminus E_{\mathcal{T}}}(1-t^{\circ C(e,\mathcal{T})})}\right]_{t_{ij,k}=t_k}$$

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