ABSOLUTELY INDECOMPOSABLE REPRESENTATIONS AND KAC-MOODY LIE ALGEBRAS (WITH AN APPENDIX BY HIRAKU NAKAJIMA)

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Dedicated to Idun Reiten on the occasion of her sixtieth birthday.

Abstract. A conjecture of Kac states that the polynomial counting the number of absolutely indecomposable representations of a quiver over a finite field with given dimension vector has positive coefficients and furthermore that its constant term is equal to the multiplicity of the corresponding root in the associated Kac-Moody Lie algebra. In this paper we prove these conjectures for indivisible dimension vectors.

1. Introduction

Let Q be a finite quiver without loops with vertices I and fix $\alpha \in \mathbb{N}^I$. In [19] V. Kac showed (over an algebraically closed field) that Q has an indecomposable representation of dimension vector α if and only if α is a root of a certain Kac-Moody Lie algebra $\mathfrak{g}$ associated to Q. This was a spectacular generalization of earlier results by Gabriel [15] for the finite type case and Dlab and Ringel [13] for the tame case.

Now assume that the ground field is finite. In this case one should consider *absolutely indecomposable representions*, i.e. indecomposable representations which remain indecomposable over the algebraic closure of the ground field.

For $\alpha \in \mathbb{N}^I$ let $a_\alpha(q)$ be the number of absolutely indecomposable representations of Q with dimension vector α over $\mathbb{F}_q$. Kac has shown that $a_\alpha(q)$ is a polynomial in q with integral coefficients [20]. Regarding this polynomial Kac made the following intriguing conjectures:

Conjecture A. $a_\alpha(q) \in \mathbb{N}[q]$.

Conjecture B. If α is a root then $a_\alpha(0)$ is the multiplicity of α in $\mathfrak{g}$.

Despite our greatly increased understanding of the relationship between quivers and Kac-Moody Lie algebras (thanks to Ringel, Lusztig, Kashiwara, Nakajima and others) and despite the fact that over twenty years have passed since these conjectures were stated, virtually no progress has been made towards their proof. See [17, 26, 36] for some partial and related results.

In this paper we make the first substantial progress by proving the following result:

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Theorem 1.1. *Conjecture A and B are true if α is indivisible.*

To prove such a result it is clear that one should first find a good cohomological interpretation for the polynomial $a_\alpha(q)$. Unfortunately the equivariant cohomology of the representation space of Q (which is the obvious choice) counts representations with multiplicity (see [2, 22]) and this yields trivial results in our case.

Thus one of the main results in this paper is a new interpretation of $a_\alpha(q)$ in the case that α is indivisible. To state this new interpretation we have to introduce some notations. We assume temporarily that our base field is $\mathbb{C}$. The double $\bar{Q}$ of Q is the quiver obtained by adding a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. The preprojective algebra of Q is $\Pi^0 = \mathbb{C}\bar{Q}/(\sum[a, a^*])$ where the sum runs over the arrows in Q.

Define a bilinear form on $\mathbb{C}I$ by $i \cdot j = \delta_{ij}$ and let $\lambda \in \mathbb{Z}I$ be such that $\lambda \cdot \alpha = 0$ but $\lambda \cdot \beta \neq 0$ for $0 < \beta < \alpha$. Then we show in §2 that

$$a_\alpha(q) = \sum_{i=0}^{d} \dim H^{2d-2i}(X_s, \mathbb{C})\, q^i \tag{1.1}$$

(singular cohomology) where X_s is the (smooth) moduli-space of λ-stable Π^0-representations of dimension vector α [23] and $d = 1/2 \dim X_s$. It is clear that this formula proves Conjecture A for indivisible α.

Now let $\Lambda_\alpha = \operatorname{Rep}(\Pi^0, \alpha)^{\mathrm{nil}}$ be the nilpotent representations in the representation space of α-dimensional representations of Π^0. Lusztig has shown [27, Thm 12.9][28] that Λ_α is a Lagrangian subvariety of the affine space $\operatorname{Rep}(\bar{Q}, \alpha)$ and furthermore that the irreducible components of Λ_α index a basis of $U(\mathfrak{g}^+)_\alpha$ (see also [21]). We first observe that Conjecture B for α indivisible is equivalent to the following.

Proposition 1.2. *Let α be indivisible. The number of irreducible components of Λ_α which contain a λ-stable (or equivalently: semistable) representation is equal to* $\dim \mathfrak{g}_\alpha$.

We then prove this proposition by relating the Harder-Narasimhan filtration on Π^0-representations to the PBW-theorem for $U(\mathfrak{g}^+)$. This approach was partially suggested by a talk of M. Reineke. See [33].

Let us now sketch how we prove (1.1). Unless otherwise specified our base field is now finite. We show first that $a_\alpha(q)$ counts the points of a smooth affine variety X related to a *deformed* preprojective algebra of Q [9]. Our aim is then to count the points on X using the Lefschetz fixed point formula for the Frobenius action on l-adic cohomology.

Since we are not able to extract any meaningful results directly from X, we construct a one-parameter family Ξ of smooth varieties whose general fiber is X and whose special fiber is X_s. Now it is easy to see that X_s carries a $\mathbb{G}_m$-action whose fixed point set is projective. By combining the Weil conjectures with results from [4, 5] we deduce from this that the absolute values of the eigenvalues of the Frobenius action on the cohomology of X_s are the same as those of a smooth projective variety (see Appendix A).

Since Ξ is not locally trivial we cannot directly transfer results from X_s to X. However an argument involving the hyper-Kähler structure on the representation space of $\bar{Q}$ shows that X_s and X are homeomorphic for the analytic topology in characteristic zero (see [30, Cor. 4.2]). By specialization this implies that X_s and X have isomorphic cohomology in large characteristic. Unfortunately it is not immediately clear to us that this isomorphism is compatible with Frobenius (think of the example given by elliptic curves).

Therefore we refine Nakajima's argument in such a way that it shows that the family Ξ is trivial for the analytic topology (see lemma 2.3.3 below). It follows that the cohomology of the fibers of Ξ is constant in large characteristic. Thus X and X_s have the same cohomology even when the Frobenius action is taken into account. This allows us to prove (1.1) using a simple technical lemma (see lemma A.1).

Some words on the organization of this paper. The proof of (1.1) and the equivalence of Conjecture B and Proposition 1.2 are contained in Section 2. The proof of (1.1) relies on a few basic results on l-adic cohomology and invariant theory over $\mathbb{Z}$. We have collected those in two appendices so that they don't detract from the main arguments. The proof of Proposition 1.2 is contained in Section 3. Inspired by the referees' reports we have also included the short Section 4 which discusses some natural questions raised by this paper.

We wish to thank Henning Andersen for some useful information regarding invariants over $\mathbb{Z}$. We also wish to thank Markus Reineke for communicating us the main results of [33].

At the end of the paper we include an appendix by H. Nakajima which avoids the arguments of Section 2.3 by showing directly that two varieties have the same number of points over finite fields. We have retained the original Section 2.3, however, since it shows more—the existence of a canonical isomorphism between the cohomology of $\operatorname{Rep}(\Pi^\lambda,\alpha)^\lambda /\!\!/ G(\alpha)$ and $\operatorname{Rep}(\Pi^0,\alpha)^\lambda /\!\!/ G(\alpha)$ for arbitrary λ and α (see below for notations).

2. Proof of (1.1) and the equivalence of Proposition 1.2 and conjecture B

2.1. **Notations and constructions.** Let $Q=(I,Q,h,t)$ be a finite quiver without loops with vertices I and edges Q. h,t are the maps which associate starting and ending vertex to an edge. There is a standard symmetric bilinear form on $\mathbb{Z}^I$ given by

$$(i,j)=\begin{cases}1 & \text{if } i=j\\ -\frac{1}{2}\#\{\text{arrows between } i \text{ and } j\} & \text{if } i\neq j\end{cases}$$

We let $\mathfrak{g}$ be the Kac-Moody Lie algebra whose Cartan matrix $(a_{ij})_{ij}$ is given by $a_{ij}=2(i,j)$.

An absolutely indecomposable representation of Q over a field k is an indecomposable representation V with the property that $V\otimes_k \bar{k}$ is indecomposable, or equivalently $\operatorname{End}(V)/\operatorname{rad}\operatorname{End}(V)=k$. For $\alpha\in\mathbb{N}^I$, $a_\alpha(q)$ is the number isomorphism classes of absolutely indecomposable representations of Q with dimension vector α over the finite field $\mathbb{F}_q$.

We now introduce some standard constructions related to the quiver Q. Since we want to use lifting to characteristic zero we need to define things over $\mathbb{Z}$. This makes our notations a little pedantic for which we apologize in advance. For some basic material with respect to invariants over $\mathbb{Z}$ we refer to Appendix B. The essential ingredient, *on which we will rely tacitly below*, is that all constructions are compatible with base change over an open part of $\operatorname{Spec}\mathbb{Z}$.

Let $\bar{Q}$ be the double quiver of Q. Thus $\bar{Q}$ has the same vertices as Q but the edges are given by $\{a,a^*\mid a\in Q\}$ where $h(a^*)=t(a)$ and $t(a^*)=h(a)$.

If R is a commutative ring and $\lambda \in R^I$ then Π^λ is the corresponding deformed preprojective algebra [9]. Thus

$$\Pi^\lambda = R\bar{Q} / \left(\sum_{a \in Q} [a, a^*] - \sum_{i \in I} \lambda_i i \right) \tag{2.1}$$

For $\alpha, \beta \in \mathbb{N}$ let $M_{\alpha \times \beta}$, M_α, $\mathrm{Gl}(\alpha)$ be the $\mathbb{Z}$-schemes corresponding respectively to the $\alpha \times \beta$-matrices, the $\alpha \times \alpha$-matrices and the invertible $\alpha \times \alpha$-matrices.

For $\alpha \in \mathbb{N}^I$ we define $\mathrm{Rep}(Q, \alpha) = \prod_{e \in Q} M_{\alpha_{h(e)} \times \alpha_{t(e)}}$. We use corresponding notations for $\bar{Q}$ and Π^λ.

For $i, j \in I$ put $i \cdot j = \delta_{ij}$. This defines a bilinear form on R^I for any ring R.

Lemma 2.1.1. *If R is a field and if $\alpha \cdot \lambda \neq 0$ in R then* $\mathrm{Rep}(\Pi^\lambda, \alpha) = \emptyset$.

Proof. This follows from the standard trace argument. □

We also define $\mathrm{Gl}(\alpha) = \prod_{i \in I} \mathrm{Gl}(\alpha_i)$ and we put $G(\alpha) = \mathrm{Gl}(\alpha)/\mathbb{G}_m$.

The Lie algebra of $\mathrm{Gl}(\alpha)$ is given by $M(\alpha) = \prod_i M_{\alpha_i \times \alpha_i}$. Over a field l we may identify $\mathrm{Lie}(\mathrm{Gl}(\alpha)_l)$ with its dual via the trace pairing. Under this pairing the dual to $\mathrm{Lie}(G(\alpha)_l)$ is identified with the trace zero matrices in $M(\alpha)_l$. We denote the variety of trace zero matrices with $M(\alpha)^0$.

The algebraic group $G(\alpha)$ acts by conjugation on $\mathrm{Rep}(Q, \alpha)$ and the orbits $\mathrm{Rep}(Q, \alpha)(l)/G(\alpha)(l)$ for l a field correspond to isomorphism classes of Q-representations defined over l.

Now let $\lambda \in \mathbb{Z}^I$ such that $\lambda \cdot \alpha = 0$. Then λ defines a character χ_λ of $G(\alpha)$ given by $(x_i)_{i \in I} \mapsto \prod_i \det(x_i)_i^\lambda$. As in [23], χ defines a line bundle $\mathcal{L}$ on $\mathrm{Rep}(\bar{Q}, \alpha)$. We define $\mathrm{Rep}(\bar{Q}, \alpha)^\lambda$ as the $\mathcal{L}$-semistable part [35, §II] of $\mathrm{Rep}(\bar{Q}, \alpha)$. Using the Hilbert-Mumford criterion [23, Prop. 3.1] one finds that if k is an algebraically closed field then $V \in \mathrm{Rep}(\bar{Q}, \alpha)(k)$ lies in $\mathrm{Rep}(\bar{Q}, \alpha)(k)^\lambda$ if and only if

$$\lambda \cdot \underline{\dim} V' \geq 0 \tag{2.2}$$

for every subrepresentation $0 \neq V' \subsetneq V$. If we replace the inequality in (2.2) by a strict one then we obtain the stable representations.

Consider the map

$$\mu : \mathrm{Rep}(\bar{Q}, \alpha) \to M(\alpha)^0 : (x_a)_{a \in \bar{Q}} \mapsto \sum [x_a, x_a^*]_{a \in Q} \tag{2.3}$$

Over a field l, μ may be identified with a suitable moment map for the $G(\alpha)_l$ action on $\mathrm{Rep}(\bar{Q}, \alpha)_l$ via the identification of $\mathrm{Lie}(G(\alpha))_l^*$ with $M(\alpha)_l^0$. We will refer to (2.3) as *the* moment map. We clearly have $\mu^{-1}(\lambda) = \mathrm{Rep}(\Pi^\lambda, \alpha)$.

Let L be the line in the affine space in $M(\alpha)^0$ spanned by 0 and λ and let $W = \mu^{-1}(L) \cap \mathrm{Rep}(\bar{Q}, \alpha)^\lambda$. Put $\Xi = W /\!\!/ G(\alpha)$ and let $f : \Xi \to L$ be the induced map. We put $X = f^{-1}(\lambda) = \mathrm{Rep}(\Pi^\lambda, \alpha)^\lambda /\!\!/ G(\alpha)$ and $X_s = f^{-1}(0) = \mathrm{Rep}(\Pi^0, \alpha)^\lambda /\!\!/ G(\alpha)$.

Definition 2.1.2. We say that $\lambda \in \mathbb{Z}^I$ is *generic* with respect to $\alpha \in \mathbb{N}^I$ if $\lambda \cdot \alpha = 0$ but $\lambda \cdot \beta \neq 0$ for all $0 < \beta < \alpha$ (note that such a λ exists if and only if α is indivisible).

If λ is generic for α then it follows from (2.2) that over an algebraically closed field the notions of λ-semistability and λ-stability coincide.

Lemma 2.1.3. *Assume that λ is generic with respect to α. Then there exists a non-empty open $U \subset \mathrm{Spec}\,\mathbb{Z}$ such that* $\mathrm{Rep}(\Pi^\lambda, \alpha)_U^\lambda = \mathrm{Rep}(\Pi^\lambda, \alpha)_U$.

Proof. It is sufficient to prove this over $k = \bar{\mathbb{Q}}$. In that case every $x \in \operatorname{Rep}(\Pi^\lambda, \alpha)(k)$ is simple by lemma 2.1.1. Then by (2.2) it follows that x is semistable (in fact stable) for λ. □

Since we are only interested in large characteristics we will commit a slight abuse of notation by identifying X with $\operatorname{Rep}(\Pi^\lambda, \alpha)/\!\!/G(\alpha)$ in the case that λ is generic. This is justified by the last lemma.

Lemma 2.1.4. *Assume that λ is generic with respect to α. Then there exists a non-empty open $U \subset \operatorname{Spec}\mathbb{Z}$ such that the map $f : \Xi_U \to L_U$ is smooth.*

Proof. Again it is sufficient to do this over $k = \bar{\mathbb{Q}}$.

First we note that if $x \in \operatorname{Rep}(\bar{Q}, \alpha)^\lambda(k)$ then by (2.2) $\operatorname{End}(x) = k$ and in particular $G(\alpha)_k$ acts freely on $\operatorname{Rep}(\bar{Q}, \alpha)^\lambda_k$.

By lemma 2.1.5 below μ is smooth at x. Thus the restriction of μ to $\operatorname{Rep}(\bar{Q}, \alpha)^\lambda_k$ is smooth. It follows that the induced map $W_k \to L_k$ is also smooth.

Since $G(\alpha)_k$ acts freely on W_k we deduce that $W_k \to W_k/G(\alpha)_k = \Xi_k$ is also smooth. This then yields that $\Xi_k \to L_k$ is surjective on tangent spaces and hence smooth. □

We have used the following standard lemma.

Lemma 2.1.5. *Let X be a smooth symplectic variety over an algebraically closed field k and assume that G is a linear algebraic group acting symplectically on X. Assume that in addition there is a moment map $\mu : X \mapsto \mathfrak{g}^*$ where $\mathfrak{g} = \operatorname{Lie}(G)$. Let $x \in X$. If the differential in x of the G-action $\mathfrak{g} \to T_x(X)$ is injective then μ is smooth at x.*

Proof. Since we don't have a reference where this lemma is stated in the current generality (i.e. arbitrary characteristic) we include the easy proof for the convenience of the reader.

Let $\omega \in \Gamma(X, \wedge^2\Omega_X)$ be the symplectic form. The defining property for a moment map is that for all $x \in X$, $v \in T_x(X)$ and $w \in \mathfrak{g}$ we have

$$d\mu_x(v)(w) = \omega(w_x, v)$$

where w_x is the image of w in $T_x(X)$.

We need to show that $d\mu_x$ is surjective. In other words for all $\phi \in \mathfrak{g}^*$ we need to find $v \in T_xX$ such that for all $w \in \mathfrak{g}$ one has $d\mu_x(v)(w) = \phi(w)$ which is equivalent to $\omega(w_x, v) = \phi(w)$.

Since $\mathfrak{g} \to T_xX : w \to w_x$ is injective we may extend ϕ linearly to an element $\phi' \in T_x(X)^*$ such that $\phi(w) = \phi'(w_x)$. Since ω is non-degenerate we may find $v \in T_xX$ such that $\omega(-, v) = \phi'$. This finishes the proof. □

2.2. **Reformulation of Kac's conjectures for indivisible dimension vectors.** We assume throughout that $\alpha \in \mathbb{N}^I$ is indivisible. We put $k = \bar{\mathbb{F}}_p$ and we let q be a power of p. We prove the following result.

Proposition 2.2.1. *Assume that $\lambda \in \mathbb{Z}^I$ is generic for $\alpha \in \mathbb{N}^I$ and let $X = \operatorname{Rep}(\Pi^\lambda, \alpha)/\!\!/G(\alpha)$ be as in §2.1. Then for $p \gg 0$ we have*

$$a_\alpha(q) = q^{-d}|X(\mathbb{F}_q)|$$

with $d = 1 - (\alpha, \alpha)$

Proof. We consider the projection map

$$\pi : \operatorname{Rep}(\Pi^\lambda, \alpha) \to \operatorname{Rep}(Q, \alpha)$$

According to [7, Thm 3.3] the image of $\pi(\mathbb{F}_q)$ consists of indecomposable representations. Since α is indivisible, representations of dimension vector α are absolutely indecomposable if and only if they are indecomposable. Thus the image of $\pi(\mathbb{F}_q)$ consists of absolutely indecomposable representations.

Let $\operatorname{Rep}(Q,\alpha)^{a.i}$ denote the constructible subset of absolutely indecomposable representations in the affine space $\operatorname{Rep}(Q,\alpha)$. It is also shown in loc. cit. that the elements of $\operatorname{Rep}(Q,\alpha)^{a.i.}(\mathbb{F}_q)$ lift to $\operatorname{Rep}(\Pi^\lambda,\alpha)$. More precisely the inverse image of $x \in \operatorname{Rep}(Q,\alpha)^{a.i.}(\mathbb{F}_q)$ can be identified with $\operatorname{Ext}^1(x,x)^*$.

Starting from a variant of the Burnside formula we compute

$$\begin{aligned}
\left|\operatorname{Rep}(Q,\alpha)^{a.i.}(\mathbb{F}_q)/G(\alpha)(\mathbb{F}_q)\right| &= \frac{1}{|G(\alpha)(\mathbb{F}_q)|} \sum_{x \in \operatorname{Rep}(Q,\alpha)^{a.i}(\mathbb{F}_q)} |\operatorname{Stab}_{G(\alpha)}(x)| \\
&= q^{-1} \frac{1}{|G(\alpha)(\mathbb{F}_q)|} \sum_{x \in \operatorname{Rep}(Q,\alpha)^{a.i}(\mathbb{F}_q)} |\operatorname{End}(x)| \\
&= q^{-1} \frac{1}{|G(\alpha)(\mathbb{F}_q)|} \sum_{x \in \operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)} \frac{|\operatorname{End}(\pi(x))|}{|\operatorname{Ext}^1(\pi(x),\pi(x))|} \\
&= q^{(\alpha,\alpha)-1} \frac{|\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)|}{|G(\alpha)(\mathbb{F}_q)|}
\end{aligned}$$

where we have used that $(-,-)$ is the symmetrization of the Euler form on $K_0(\operatorname{mod}(kQ))$.

Since $p \gg 0$ the inequalities defining genericity also hold in $\mathbb{F}_p$. Hence we will assume this. By lemma 2.1.1 our choice of λ insures that $\operatorname{Rep}(\Pi^\lambda,\alpha)(k)$ contains only simple representations. Thus if $x \in \operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)$ then $\operatorname{End}(x) = \mathbb{F}_q$ and hence x has trivial stabilizer in $G(\alpha)(k)$.

Using [24, Cor. 5.3.b] we obtain

$$\begin{aligned}
|\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)|/|G(\alpha)(\mathbb{F}_q)| &= |\operatorname{Rep}(\Pi^\lambda,\alpha)(\mathbb{F}_q)/G(\alpha)(\mathbb{F}_q)| \\
&= |(\operatorname{Rep}(\Pi^\lambda,\alpha)(k)/G(\alpha)(k))^{\operatorname{Gal}(k/\mathbb{F}_q)}| \\
&= |X(k)^{\operatorname{Gal}(k/\mathbb{F}_q)}| = |X(\mathbb{F}_q)| \quad \square
\end{aligned}$$

2.3. **Cohomological triviality.** According to the program outlined in the introduction we want to compare the cohomology of X and X_s (see §2.1). One way to do this is to show that $R^i f_!(\mathbb{Q}_l)$ is constant, at least over an open part of the base $\operatorname{Spec}\mathbb{Z}$. This is the content of the next proposition. Note that we do not assume that λ is generic with respect to α.

Proposition 2.3.1. *There exists a non empty open $U \subset \operatorname{Spec}\mathbb{Z}$ such that for every i, $R^i f_!(\mathbb{Q}_l)_U$ is the pullback of a sheaf on U.*

Corollary 2.3.2. *Let $k = \overline{\mathbb{F}}_p$. For $p \gg 0$ there is an isomorphism between $H^i_c(X_{s,k},\mathbb{Q}_l)$ and $H^i_c(X_k,\mathbb{Q}_l)$ which is compatible with the Frobenius action.*

Proof. Let f_s, f_g be the restrictions of f to X_s and X.

Using the previous proposition and the fact that $R^i f_!$ commutes with base change we find for $p \gg 0$: $R^i f_{s!,\mathbb{F}_p}(\mathbb{Q}_l) \cong R^i f_{g!,\mathbb{F}_p}(\mathbb{Q}_l)$ on $\operatorname{Spec}\mathbb{F}_p$. We may consider $R^i f_{s!,\mathbb{F}_p}(\mathbb{Q}_l)$ and $R^i f_{g!,\mathbb{F}_p}(\mathbb{Q}_l)$ as the $\operatorname{Gal}(k/\mathbb{F}_p)$-modules given by $H^i_c(X_{s,k},\mathbb{Q}_l)$ and

$H_c^i(X_k, \mathbb{Q}_l)$ respectively. Since the Frobenius action is determined by the action of $\mathrm{Gal}(k/\mathbb{F}_p)$ [10, §1.8] this proves what we want. □

Proof of Proposition 2.3.1. We use Deligne's generic base change result for direct images [11, Thm 1.9]. This result was only stated for torsion sheaves, but the corresponding result for l-adic sheaves is an easy consequence.

Since f is of finite type there are only a finite number of i for which $R^i f_!(\mathbb{Q}_l)$ is non-zero. So we may treat each i separately. Put $\mathcal{F} = R^i f_!(\mathbb{Q}_l)$. Let $g : L \to \mathbb{Z}$ be the structure map and let $\epsilon : g^* g_* \mathcal{F} \to \mathcal{F}$ be the map given by adjointness. Let $\mathcal{A}, \mathcal{B}$ be the kernel and cokernel of ϵ. By [11, Thm 1.9] $g^* g_* \mathcal{F}$ and hence $\mathcal{A}, \mathcal{B}$ will be constructible over an open subset $V \subset \operatorname{Spec} \mathbb{Z}$.

Below we show that $\epsilon_{\mathbb{C}} : g_{\mathbb{C}}^* g_{\mathbb{C},*} \mathcal{F}_{\mathbb{C}} \to \mathcal{F}_{\mathbb{C}}$ is an isomorphism. By [11, Thm 1.9] we have $g_{\mathbb{C}}^* g_{\mathbb{C},*} \mathcal{F}_{\mathbb{C}} = (g^* g_* \mathcal{F})_{\mathbb{C}}$. Hence $\mathcal{A}_{\mathbb{C}} = \mathcal{B}_{\mathbb{C}} = 0$. From the fact that $\mathcal{A}_V$ and $\mathcal{B}_V$ are constructible it follows that $\mathrm{Supp}(\mathcal{A}_V)$ and $\mathrm{Supp}(\mathcal{B}_V)$ are constructible subsets of Ξ whose image in $\operatorname{Spec} \mathbb{Z}$ does not contain the generic point. Hence we find $\mathcal{A}_U = \mathcal{B}_U = 0$ for a suitable open $U \subset V$.

Now we prove our claim that $\epsilon_{\mathbb{C}}$ is an isomorphism. To do this we replace the etale topology on $\Xi_{\mathbb{C}}, L_{\mathbb{C}}$ with the analytic topology. Then the claim follows from the comparison theorem [3, §6.1.2], lemma 2.3.3 below and the fact that $L_{\mathbb{C}}$ is connected. □

In the rest of this subsection our base field will be $\mathbb{C}$ *so we drop the corresponding subscript.*

Lemma 2.3.3. *$f : \Xi \to L$ is a trivial (topological) family.*

Proof. Let $V = \mathrm{Rep}(\bar{Q}, \alpha)$. We will use the hyper-Kähler structure on V which was introduced by Kronheimer [25]. For the benefit of the reader we recall the basic facts. First we define a Riemannian metric on V via the trace form:

$$(x, y) = \mathrm{Re} \sum_{a \in \bar{Q}} \mathrm{Tr}(x_a y_a^\dagger) \tag{2.4}$$

where $z^\dagger$ is the conjugate transpose to z.

Let $\mathbb{H} = \mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ be the quaternions. We define an action of $\mathbb{H}$ on V via

$$I(x_a)_{a \in \bar{Q}} = (ix_a)_{a \in \bar{Q}}$$
$$J(x_a, x_{a^*})_{a \in Q} = (-x_{a^*}^\dagger, x_a^\dagger)_{a \in Q}$$
$$K(x_a, x_{a^*})_{a \in Q} = (-ix_{a^*}^\dagger, ix_a^\dagger)_{a \in Q}$$

It is clear that with respect to this quaternionic structure the metric (2.4) is hyper-Kähler. Let $\mathbb{H}^0$ be the kernel of the reduced trace map on $\mathbb{H}$. If $\beta \in \mathbb{H}^0$ then there is an associated real symplectic form on V defined by $\omega_\beta(v, w) = (v, \beta w)$.

Let us write $\mathfrak{gl} = \mathrm{Lie}(\mathrm{Gl}(\alpha))$ and $\mathfrak{u} = \mathrm{Lie}(U(\alpha))$ where $U(\alpha)$ is the maximal compact subgroup of $\mathrm{Gl}(\alpha)$ given by the product of unitary groups $\prod_{i \in I} U(\alpha_i)$. The hyper-Kähler structure on V is clearly $U(\alpha)$-invariant and it is a standard fact that the symplectic form ω_β has an associated moment map $\mu_\beta : V \to \mathfrak{u}^*$ given by $\mu_\beta(v)(u) = -\frac{1}{2}\omega_\beta(v, uv)$ for $v \in V$, $u \in \mathfrak{u}$. Below we will write $\mu_{\mathbb{R}}$ for μ_I.

The three moment maps μ_I, μ_J, μ_K may be combined into a so-called hyper-Kähler moment map

$$\mu : V \to \mathbb{H}^0 \otimes_{\mathbb{R}} \mathfrak{u}^* : x \mapsto I \otimes \mu_I(x) + J \otimes \mu_J(x) + K \otimes \mu_K(x) \tag{2.5}$$

From the explicit description of μ_β we deduce for $h \in \mathbb{H}$:

$$\mu_\beta(hx) = \mu_{\bar{h}\beta h}(x) \tag{2.6}$$

where $\bar{h}$ is the conjugate of h in $\mathbb{H}$. From (2.6) we deduce that (2.5) is $\mathbb{H}^*$-invariant if we let $\mathbb{H}^*$ act on $\mathbb{H}^0$ by $h \cdot \beta = h\beta\bar{h}$.

For this action $\mathbb{H}^0 - \{0\}$ is a homogeneous space and hence if we choose $\beta \in \mathbb{H}^0 - \{0\}$ and a contractible subset $S \subset \mathbb{H}^0 - \{0\}$ containing β then there is a continuous map $\theta_{\beta,S} : S \to \mathbb{H}^*$ which is a section (above S) for the map $h \mapsto h \cdot \beta$.

Choose a $U(\alpha)$-invariant $\lambda \in \mathfrak{u}^*$ and let $V' = \mu^{-1}(S \times \lambda)$, $V'' = \mu^{-1}(\beta \times \lambda)$. Then $V'' \times S \to V' : (x,s) \mapsto \theta_{\beta,S}(s)x$ defines a trivialization of $\mu \mid V'$. Thus we have proved that above $S \times \lambda$, μ is a trivial bundle. Moreover this trivialization is clearly $U(\alpha)$-equivariant.

Put $\omega_{\mathbb{C}} = \omega_J + i\omega_K$. This is a complex $\mathrm{Gl}(\alpha)$-invariant symplectic form on V and it is easy to see that the associated moment map $V \to \mathfrak{gl}^*$ is given by $\mu_{\mathbb{C}}(x) = \mu_J(x) + i\mu_K(x)$ where we have extended $\mu_J(x), \mu_K(x)$ to linear maps $\mathfrak{gl} \to \mathbb{C}$. A straightforward computation shows that

$$\mu_{\mathbb{R}}(x) = \frac{i}{2}\sum_a [x_a, x_a^\dagger]$$
$$\mu_{\mathbb{C}}(x) = \sum_{a \in Q}[x_a, x_{a^*}]$$

where we have identified $\mathfrak{u}$, $\mathfrak{gl}$ with their duals via the trace form $(g,h) = -\operatorname{Tr}(gh)$ (the minus sign makes the form positive definite on $\mathfrak{u}$).

From the description $\mu_{\mathbb{C}} = \mu_J + i\mu_K$ we obtain:

$$\mu_{\mathbb{C}}^{-1}(a) = \mu_J^{-1}\left(\frac{a - a^\dagger}{2}\right) \cap \mu_K^{-1}\left(\frac{a + a^\dagger}{2i}\right)$$

which yields

$$\mu_{\mathbb{C}}^{-1}(\mathbb{C}\lambda) \cap \mu_{\mathbb{R}}^{-1}(i\lambda) \cong \mu^{-1}((I + \mathbb{R}J + \mathbb{R}K) \times i\lambda)$$
$$\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(i\lambda) \cong \mu^{-1}(I \times i\lambda)$$

From the fact that $I + \mathbb{R}J + \mathbb{R}K$ is contractible we deduce as explained above that μ is trivial above $(I + \mathbb{R}J + \mathbb{R}K) \times i\lambda$. Since on the inverse image of $(I + \mathbb{R}J + \mathbb{R}K) \times i\lambda$, μ and $\mu_{\mathbb{C}}$ are basically the same we deduce that $\mu_{\mathbb{C}} : \mu_{\mathbb{C}}^{-1}(\mathbb{C}\lambda) \cap \mu_{\mathbb{R}}^{-1}(i\lambda) \to \mathbb{C}\lambda$ is a trivial family in a way that is compatible with the $U(\alpha)$-action.

We now use this to construct the following commutative diagram of continuous maps:

$$\begin{array}{ccc}
X_s \times L & \xrightarrow{\mathrm{pr}} & L \\
{\scriptstyle r}\uparrow & & \| \\
\left(\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(i\lambda)\right)/U(\alpha) \times L & \xrightarrow{\mathrm{pr}} & L \\
{\scriptstyle p}\downarrow & & \| \\
\left(\mu_{\mathbb{C}}^{-1}(L) \cap \mu_{\mathbb{R}}^{-1}(i\lambda)\right)/U(\alpha) & \xrightarrow{\bar{\mu}_{\mathbb{C}}} & L \\
{\scriptstyle r'}\downarrow & & \| \\
\Xi & \xrightarrow{f} & L
\end{array}$$

Here p is obtained from the trivialization of $\mu_{\mathbb{C}}$ we have constructed above (recall that $L = \mathbb{C}\lambda$) and r, r' are obtained from the inclusion $\mu_{\mathbb{R}}^{-1}(i\lambda) \subset \operatorname{Rep}(\bar{Q}, \alpha)^{\lambda}$ [23, Prop. 6.5].

To prove the lemma it is now sufficient to show that the vertical maps on the left are homeomorphisms. This is true by construction for p. We claim that it is also true for r, r'. It suffices to consider r' since r is obtained from r' by restricting to a fiber.

By [23, Prop. 6.5] r' is a bijection. Hence it suffices to show that r' is proper. Clearly r' is the restriction to $\left(\mu_{\mathbb{C}}^{-1}(L) \cap \mu_{\mathbb{R}}^{-1}(i\lambda)\right)/U(\alpha)$ of the first map in the following diagram

$$\mu_{\mathbb{R}}^{-1}(i\lambda)/U(\alpha) \to \operatorname{Rep}(\bar{Q}, \alpha)^{\lambda} /\!\!/ G(\alpha) \to \operatorname{Rep}(\bar{Q}, \alpha) /\!\!/ G(\alpha)$$

By Theorem 2.3.4 below the composition of these two maps is proper. It follows that the first map is also proper. This finishes the proof. $\square$

We have used the following result.

Theorem 2.3.4. [32, Theorem 1.1] *Let the notations be as above. The canonical map*

$$\psi : V \to V /\!\!/ G \times \mathfrak{u} : v \mapsto (\bar{v}, \mu_{\mathbb{R}}(v))$$

is proper.

2.4. **End of proof.** Let $k = \overline{\mathbb{F}}_p$. We choose λ generic with respect to α. Now recall that Kac has shown [20] that $a_{\alpha}(q)$ is a polynomial. We first show that X_k is pure. By Corollary 2.3.2 we may as well show that $X_{s,k}$ is pure. *Since we will now work exclusively over k we drop the corresponding subscript.*

Define $X_s^0 = \operatorname{Rep}(\Pi^0, \alpha) /\!\!/ G(\alpha)$. Then the canonical map $u : X_s \to X_s^0$ is projective [23]. Let $\nu : \mathbb{G}_m \times \operatorname{Rep}(\bar{Q}, \alpha) \to \operatorname{Rep}(\bar{Q}, \alpha)$ be the action which has the property that $\eta \in \mathbb{G}_m$ multiplies all arrows by η. This action induces $\mathbb{G}_m$-actions on X_s and X_s^0 and the map u commutes with these actions.

Now clearly $X_s^0 = \operatorname{Spec} R$ with $R = \mathcal{O}(\operatorname{Rep}(\Pi^0, \alpha))^{G(\alpha)}$. The ring R is graded via the $\mathbb{G}_m$-action we have defined in the previous paragraph and it is easy to see that the grading is of the form $R = k + R_1 + R_2 + \cdots$ with R_i finite dimensional.

Thus it follows that $(X_s^0)^{\mathbb{G}_m}$ consists of a single point o defined by the graded maximal ideal of R and furthermore $\lim_{t \to 0} \nu(t, x) = o$ for all $x \in X_s^0$. It also follows that $(X_s)^{\mathbb{G}_m} \subset u^{-1}(o)$. Since u is projective we deduce that $(X_s)^{\mathbb{G}_m}$ is projective.

By the valuative criterion for properness (applied to u) we deduce that $\lim_{t\to 0}\nu(t,x)$ exists for all $x\in X_s$ and is contained in $u^{-1}(o)$. Hence by Proposition A.2 X_s is pure.

By combining Proposition 2.2.1, Lemma A.1 with Corollary 2.3.2 and the fact that Kac has shown that $a_\alpha(q)$ is a polynomial in q [20], it follows

$$a_\alpha(q)=\sum_{i\geq 0}\dim H_c^{2d+2i}(X_{s,k},\mathbb{Q}_l)q^i$$

with $d=1-(\alpha,\alpha)$ and $k=\overline{\mathbb{F}}_p$ for $p\gg 0$. Since this is true for large characteristic we obtain

$$a_\alpha(q)=\sum_{i\geq 0}\dim H_c^{2d+2i}(X_{s,\mathbb{C}},\mathbb{C})q^i \tag{2.7}$$

where we have switched to a complex base field and complex coefficients (but we are still using sheaf cohomology).

Furthermore if $X_{s,\mathbb{C}}$ is non-empty then we compute

$$\dim X_{s,\mathbb{C}}=\dim\operatorname{Rep}(\Pi^0,\alpha)^\lambda-\dim G(\alpha)=\dim\operatorname{Rep}(\bar{Q},\alpha)^\lambda-2\dim G(\alpha)=2d$$

Thus the sum in (2.7) runs from $i=0$ to $i=d$. Applying Poincaré duality we obtain (1.1). We now switch to *ordinary singular cohomology*. See for example [38, Ch 5].

We now prove the equivalence of Conjecture B and Proposition 1.2. *In the rest of this section our base field will be* $\mathbb{C}$. Our starting point is the following commutative diagram

$$\begin{array}{ccc}\operatorname{Rep}(\Pi^0,\alpha)^\lambda & \xrightarrow{\text{open}} & \operatorname{Rep}(\Pi^0,\alpha)\\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q}\\ \operatorname{Rep}(\Pi^0,\alpha)^\lambda/G(\alpha) & \xrightarrow[u]{} & \operatorname{Rep}(\Pi^0,\alpha)/\!\!/G(\alpha)\end{array} \tag{2.8}$$

where all the maps are the obvious ones.

By (1.1) we have $a_\alpha(0)=\dim H^{2d}(X_s,\mathbb{C})$. With a similar argument as the one used in [37, Prop. 4.3.1] one shows that X_s is homotopy equivalent to $u^{-1}(0)$. Thus $H^{2d}(X_s,\mathbb{C})=H^{2d}(u^{-1}(0),\mathbb{C})$.

Let $(-)^{\text{nil}}$ denote the nilpotent representations in $\operatorname{Rep}(\Pi^0,\alpha)$ and $\operatorname{Rep}(\Pi^0,\alpha)^\lambda$. We have $\operatorname{Rep}(\Pi^0,\alpha)^{\text{nil}}=q^{-1}(0)$. So by the commutativity of (2.8) we also have $\operatorname{Rep}(\Pi^0,\alpha)^{\lambda,\text{nil}}=q^{-1}(u^{-1}(0))$.

Since the leftmost map in (2.8) is a principal $G(\alpha)$-bundle and the top map is an open immersion we find that if $X_s\neq\emptyset$ then $\dim u^{-1}(0)=\dim\operatorname{Rep}(\Pi^0,\alpha)^{\text{nil}}-\dim G(\alpha)$. Since $\operatorname{Rep}(\Pi^0,\alpha)^{\text{nil}}$ [27, 12.9] is a Lagrangian subvariety of $\operatorname{Rep}(\bar{Q},\alpha)$ it follows that $u^{-1}(0)$ is equidimensional and furthermore $\dim u^{-1}(0)=(1/2)\dim\operatorname{Rep}(\bar{Q},\alpha)-\dim G(\alpha)=d$. Hence (even if $X_s=\emptyset$), $\dim H^{2d}(X_s,\mathbb{C})$ is equal to the number of irreducible components of $u^{-1}(0)$. Using again that the leftmost map is a principal $G(\alpha)$-bundle this is equal to the number of irreducible components of $\operatorname{Rep}(\Pi^0,\alpha)^{\lambda,\text{nil}}$. This finishes the proof.

3. Proof of Conjecture B for indivisible roots

In this section our ground field is $\mathbb{C}$.

At the end of the previous section it was shown that Proposition 1.2 and Conjecture B are equivalent. So we only prove Proposition 1.2. The idea for the proof of Proposition 1.2 came partially from a talk by Reineke [33].

In the previous section we have used the notion of λ-stability introduced by King [23] which is derived from geometric invariant theory. A technical inconvenience of this notion is that if we work in $\operatorname{Rep}(\bar{Q}, \alpha)$ then $\lambda \cdot \alpha$ must be zero. Hence we cannot use the same λ for all α. Following Reineke [33] we use therefore an alternative notion of stability we will call slope stability (for a general discussion on stability notions in arbitrary abelian categories see [34]).

We fix an element $\Theta \in \mathbb{Z}^I$ and we define the corresponding "slope function" $s(\alpha) = (\Theta \cdot \alpha)/\dim \alpha$ where $\dim \alpha = \sum \alpha_i$. If V is a finite dimensional representation of $\bar{Q}$ then we put $s(V) = s(\underline{\dim} V)$. If $X \subset \operatorname{Rep}(\bar{Q}, \alpha)$ is irreducible then we write $s(X) = s(\alpha)$.

A representation V of $\bar{Q}$ is (Θ-slope) stable (resp. semistable) if for all proper subrepresentations W of V we have $s(W) < s(V)$ (resp. $s(W) \le s(V)$). It is easy to see that for a fixed dimension vector α, King (semi)stability and slope (semi)stability are equivalent for suitable λ and Θ. Below the notion of (semi)stability will refer to Θ-slope (semi)stability for an arbitrary but fixed Θ.

The following lemma is standard.

Lemma 3.1. *Assume that V, W are semistable representations such that $s(V) > s(W)$. Then* $\operatorname{Hom}(V, W) = 0$.

The following result is proved in [16, 33].

Theorem 3.2. *Let V be a representation of $\bar{Q}$. Then there exists a unique filtration*

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = V$$

such that all V_{i+1}/V_i are semistable and such that $s(V_{i+1}/V_i)$ is a strictly decreasing function of i.

The filtration introduced in the last theorem is called the Harder-Narasimhan filtration. Let us write

$$t(V) = (\underline{\dim}(V_1/V_0), \ldots, \underline{\dim}(V_n/V_{n-1}))$$

We call $t(V)$ the HN-type of V.

If X is a variety then we write $\operatorname{Irr} X$ for the set of irreducible components of X. If $\alpha \in \mathbb{N}^I$ then we write Λ_α for $\operatorname{Rep}(\Pi^0, \alpha)^{\mathrm{nil}}$. According to [27] this is a Lagrangian subvariety of $\operatorname{Rep}(\bar{Q}, \alpha)$ and furthermore $\operatorname{Irr} \Lambda_\alpha$ indexes a basis for $U(\mathfrak{g}^+)_\alpha$[21, 28].

If $X \in \operatorname{Irr} \Lambda_\alpha$ then we say that X is semistable if it contains a semistable representation.

Let S_α be the set of tuples $Z^* = (Z_1, \ldots, Z_n)$ with Z_i semistable elements of certain $\operatorname{Irr} \Lambda_{\alpha_i}$ such that $\alpha = \sum \alpha_i$ and such that $s(Z_i)$ is strictly decreasing.

For $Z^* \in S_\alpha$ we define $m'(Z^*)$ as the set of all $V \in \Lambda_\alpha$ such that if $(V_i)_i$ is the HN-filtration on V then $V_i/V_{i-1} \in Z_i$.

The following is our main theorem.

Theorem 3.3. (1) *If $Z^* \in S_\alpha$ then $m'(Z^*)$ has a dense intersection with unique $Z \in \operatorname{Irr} \Lambda_\alpha$. Put $m(Z^*) = Z$.*
(2) *The map m defines a bijection between S_α and* $\operatorname{Irr} \Lambda_\alpha$.

Proof. Let us call a subset Z of Λ_α *good* if it has the following properties.

(1) The elements of Z have constant HN-type denoted by $t(Z)$.
(2) Z is constructible.
(3) Z has a dense intersection with a unique irreducible component of Λ_α.

We prove a claim which is used in the proof of 1. and 2.

Claim. Let Z_1 be an open subset of a semistable irreducible component of Λ_β and let $Z_2 \subset \Lambda_\gamma$ be good. Assume that $s(Z_1) > t(Z_2)_1$. Define $Z \subset \Lambda_{\beta+\delta}$ as the set of all $V \in \Lambda_{\beta+\delta}$ which contain a semistable subrepresentation $U \subset V$ such that $U \in Z_1$, $V/U \in Z_2$. Then Z is good.

The only non-obvious property to prove is that Z has a dense intersection with a unique irreducible component of $\Lambda_{\beta+\gamma}$. So this is what we do below.

Let Z_1° be the semistable locus of Z_1 and let E be the set of 5-tuples (U, V, W, u, w) with $U \in Z_1^\circ$, $V \in \Lambda_{\beta+\gamma}$, $W \in Z_2$, $u \in \operatorname{Hom}(U, V)$, $w \in \operatorname{Hom}(V, W)$ such that

$$0 \to U \xrightarrow{u} V \xrightarrow{w} W \to 0$$

is exact. It is easy to see that E is a constructible subset of $\operatorname{Rep}(Q, \beta) \times \operatorname{Rep}(Q, \beta + \gamma) \times \operatorname{Rep}(Q, \gamma) \times M_{\beta\times(\beta+\gamma)}(k) \times M_{(\beta+\gamma)\times\gamma}(k)$.

Due to the uniqueness of the HN-filtration the non-empty fibers of the projection map $p : E \to \Lambda_{\beta+\gamma} : (U, V, W, u, w) \mapsto V$ are isomorphic to $\operatorname{Gl}(\beta) \times \operatorname{Gl}(\gamma)$ and hence they have dimension $\beta \cdot \beta + \gamma \cdot \gamma$.

There is another projection map $q : E \to Z_1^\circ \times Z_2 : (U, V, W, u, w) \mapsto (U, W)$. Its fibers are non-empty since we can take $V = U \oplus W$. According to [8, Lemma 5.1] its fibers have dimension

$$(\beta + \gamma) \cdot (\beta + \gamma) + \dim \operatorname{Ext}^1(W, U) - \dim \operatorname{Hom}(W, U)$$

and the proof also shows that these fibers are irreducible and locally closed.

According to [6, Lemma 1] we also have

$$\dim \operatorname{Hom}(U, W) - \operatorname{Ext}^1(W, U) + \dim \operatorname{Hom}(W, U) = 2(\beta, \gamma)$$

and furthermore according to lemma 3.1 we have $\operatorname{Hom}(U, W) = 0$. Substituting we find that the fibers of q have dimension:

$$(\beta + \gamma) \cdot (\beta + \gamma) - 2(\beta, \gamma)$$

According to lemma 3.4 below we find that E contains a dense irreducible locally closed subset E' such that $\dim(E - E') < \dim E$. Furthermore the dimension of E is:

$$\dim \Lambda_\beta + \dim \Lambda_\gamma + (\beta + \gamma) \cdot (\beta + \gamma) - 2(\beta, \gamma) \tag{3.1}$$

Now we have for $\alpha \in \mathbb{Z}^I$:

$$\dim \Lambda_\alpha = \frac{1}{2} \dim \operatorname{Rep}(\bar{Q}, \alpha) = \alpha \cdot \alpha - (\alpha, \alpha)$$

A trite computation shows that $Z = p(E)$ has dimension

$$(\beta + \gamma) \cdot (\beta + \gamma) - (\beta + \gamma, \beta + \gamma) = \dim \Lambda_{\beta+\gamma}$$

and $p(E - E')$ has smaller dimension. Hence $\dim p(E') = \dim \Lambda_{\beta+\gamma}$. Since E' is irreducible it follows that $p(E')$ is dense in some irreducible component Z of $\Lambda_{\beta+\gamma}$. This finishes the proof of the claim.

It is clear that the claim implies 1. by induction (in this case we take Z_1 to be a semistable irreducible component of Λ_α and not just an open subset).

Assume that 1. is proved. By the existence of the HN-filtration we have

$$\Lambda_\alpha = \bigcup_{Z^* \in S_\alpha} m'(Z^*)$$

Thus if X is an irreducible component of Λ_α.

$$X = \bigcup_{Z^* \in S_\alpha} m'(Z^*) \cap X$$

Hence some $m'(Z^*) \cap X$ must be dense in X. This implies the surjectivity of m.

Now for every $Y \in \operatorname{Irr} \Lambda_\alpha$ select an open subset Y° such that $Y^\circ \cap Z^\circ = \emptyset$ for $Y \neq Z$. We define $m'(Z^*)^\circ \subset m'(Z^*)$ as $m'(Z^*)^\circ$ but with Z_i replaced by Z_i°. The claim still applies and we find that there is a unique irreducible component of Λ_α intersected densely by $m'(Z^*)^\circ$. This component must be $m(Z^*)$.

Now we may prove injectivity of m. Assume that $Z^* \neq Z'^*$ and that $m'(Z^*) \cap X$ and $m'(Z'^*) \cap X$ are both dense in X. Then $m'(Z^*)^\circ \cap X$ and $m'(Z'^*)^\circ \cap X$ are dense as well. But $m'(Z^*)^\circ \cap m'(Z'^*)^\circ = \emptyset$ yielding a contradiction which completes the proof. □

If X is an algebraic variety and $S \subset X$ is a constructible set then let us say that S is weakly irreducible if S contains a dense subset S' which is irreducible locally closed in X and has the property that $\dim(S - S') < \dim S$.

Lemma 3.4. *Let $q : X \to Y$ be a morphism between (reduced) algebraic varieties. Let $S \subset X$, $T \subset Y$ be constructible subsets with $T = q(S)$ such that the fibers of $q : S \to T$ are locally closed in X, irreducible and of constant dimension. If T is weakly irreducible then so is S.*

Proof. Left to the reader. □

For $\alpha \in \mathbb{N}^I$ let us put n_α for the number of components of Λ_α and m_α for the number of semistable components. By [28] we have $n_\alpha = \dim U(\mathfrak{g}^+)_\alpha$. Theorem 3.2 yields the formula

$$n_\alpha = \sum_{\substack{\alpha_1,\dots,\alpha_n \\ s(\alpha_1) > \cdots > s(\alpha_n) \\ \sum \alpha_i = \alpha,}} \prod_i m_{\alpha_i}$$

and this formula allows us to determine the m_α recursively from the n_α.

Put $r_\alpha = \dim \mathfrak{g}_\alpha$. It turns out that we can give an explicit expression for m_α in terms of the r_α. Put an arbitrary total ordering on $\mathbb{N}^I$ with the property $s(\beta) > s(\gamma) \Rightarrow \beta > \gamma$ and $\beta > \gamma \Rightarrow s(\beta) \geq s(\gamma)$.

Lemma 3.5. *The following formula holds.*

$$m_\alpha = \sum_{\substack{(u_1,\beta_1),\dots,(u_n,\beta_n) \\ \beta_1 > \dots > \beta_n \\ s(\beta_1) = \cdots = s(\beta_n) = s(\alpha) \\ \sum u_i \beta_i = \alpha,}} \prod_i \binom{r_{\beta_i} + u_i - 1}{u_i} \tag{3.2}$$

Proof. By the PBW-theorem we have

$$n_\alpha = \sum_{\substack{(u_1,\beta_1),\dots,(u_n,\beta_n) \\ \beta_1 > \dots > \beta_n \\ \sum u_i \beta_i = \alpha,}} \prod_i \binom{r_{\beta_i} + u_i - 1}{u_i}$$

In this formula we may collect the β_i's with equal slope. Let m'_α be given by the righthand side of (3.2). Then we have

$$n_\alpha = \sum_{\substack{\alpha_1,\dots,\alpha_n \\ s(\alpha_1)>\cdots>s(\alpha_n) \\ \sum \alpha_i=\alpha,}} \prod_i m'_{\alpha_i}$$

and by induction it follows $m'_{\alpha_i} = m_\alpha$. This finishes the proof of (3.2). □

Proof of Proposition 1.2. Recall that $\lambda \in \mathbb{Z}^I$ is such that $\lambda \cdot \alpha = 0$ and $\lambda \cdot \beta \neq 0$ for all $0 < \beta < \alpha$.

Now it is clear that King semistability for λ is equivalent to slope semistability for $\Theta = -\lambda$. Hence for this particular Θ we need to show that $m_\alpha = r_\alpha$. This follows immediately from (3.2). □

4. Closing comments

The authors are often posed the following natural questions:

Question 4.1. How essential is the indivisibility of α in the proof of Conjecture A and B?

Question 4.2. Lusztig has shown that the irreducible components of Λ_α index a basis of $U(\mathfrak{g}^+)_\alpha$. Can Proposition 1.2 somehow be strenghtened by establishing an explicit bijection between the stable irreducible components in Λ_α and a basis for $\mathfrak{g}_\alpha$?

We think that a positive answer to Question 4.2 is rather unlikely. For example the set of stable components of Λ_α depends on $\lambda \in \mathbb{Z}^I$ (or $\Theta \in \mathbb{Z}^I$) and there does not seem to be a natural choice. A preliminary difficulty is that we don't actually know if the desired bijection can exist since we have only been able to count the *semistable* components of Λ_α (3.2).

If α is indivisible then the stable and semistable components of Λ_α coincide. The indivisibility hypotheses is also used in a very essential way in the proof of Proposition 2.2.1 which establishes a connection between the absolutely indecomposable representations of Q and the simple representations of Π^λ. Finally the indivisibility hypotheses is used to establish the smoothness of $X_{s,\mathbb{C}}$ which allows us to go from cohomology with compact support (2.7) to ordinary singular cohomology using Poincare duality (1.1). This is necessary since we need a cohomology theory which is homotopy invariant.

Appendix A. Purity

For the benefit of the reader we recollect some basics. As usual q is a power of a prime number p and $l \neq p$ is another prime number. We put $k = \bar{\mathbb{F}}_p$.

Assume that Z/k is a variety defined over $\mathbb{F}_q$, i.e. there is some $Z_0/\mathbb{F}_q$ such that $Z = (Z_0)_k$. Let $F : Z \to Z$ be the corresponding Frobenius morphism. The key method for counting rational points on Z_0 is given by the trace formula [10, Thm 3.2]

$$|Z_0(\mathbb{F}_{q^r})| = \sum_{i=0}^{2\dim Z} (-1)^i \operatorname{Tr}(F^r; H^i_c(Z, \mathbb{Q}_l))$$

For this formula to be effective one needs information on the eigenvalues of F. Let us say that Z is *(cohomologically) pure* if the eigenvalues of F acting on $H^i_c(Z,\mathbb{Q}_l)$ have absolute value $q^{i/2}$. This definition only depends on Z and not on the particular choice of $\mathbb{F}_q$ and Z_0. The Weil conjectures [12] imply that if Z is smooth proper over k then Z is pure.

We have used the notion of purity in the following context:

Lemma A.1. *Assume that Z is pure and that there is a polynomial $p(t) \in \mathbb{Z}[t]$ such that $|Z_0(\mathbb{F}_{q^r})| = p(q^r)$. Then $p(q^r) = \sum_i \dim H^{2i}_c(Z,\mathbb{Q}_l)q^{ri}$ and in particular $p(t) \in \mathbb{N}[t]$.*

Proof. It is clearly sufficient to show that the action of F on $H^{2i}_c(Z,\mathbb{Q}_l)$ has a unique eigenvalue q^i and that in addition $H^{2i+1}_c(Z,\mathbb{Q}_l) = 0$.

Write $p(t) = \sum_i b_{2i}t^i$ and $b_j = 0$ for j odd. Since Z is pure the eigenvalues of F acting on $H^i(Z,\mathbb{Q}_l)$ are given by $\epsilon_{ij}q^{i/2}$ where $j = 1\dots\beta_i$ and $|\epsilon_{ij}| = 1$. From the hypotheses and the trace formula we obtain

$$\sum_{i=0}^{2d}(-1)^i b_i q^{ri/2} = \sum_{i=0}^{2d}(-1)^i \sum_{j=1}^{\beta_i} \epsilon^r_{ij} q^{ri/2} \tag{A.1}$$

where $d = \dim Z$. Dividing by q^{rd} we find

$$b_{2d} = \lim_{r\to\infty} \sum_{j=1}^{\beta_{2d}} \epsilon^r_{2d,j}$$

Using a Van der Monde type argument we see that the limit on the righthand side only exists if $\epsilon_{2d,j} = 1$ for all j. Subtracting the leading term in q from (A.1) and repeating the same argument we ultimately find that $\epsilon_{ij} = 1$ for all i,j. Since $b_i = 0$ for odd i we find that $\beta_i = 0$ for odd i. This finishes the proof. $\square$

In this paper we use the following purity criterion:

Proposition A.2. *Assume that Z is smooth quasi-projective and that there is an action $\lambda : \mathbb{G}_m \times Z \to Z$ such that for every $x \in Z$ the limit $\lim_{t\to 0}\lambda(t,x)$ exists. Assume in addition that $Z^{\mathbb{G}_m}$ is projective. Then Z is pure.*

Proof. Let $Z^{G_m} = \bigcup_\alpha L_\alpha$ be the decomposition into connected components and for each α define

$$W_\alpha = \{x \in Z \mid \lim_{t\to 0}\lambda(t,x) \in L_\alpha\}$$

According to [4, Thm 4.1, proof of Thm 4.2] the L_α, W_α are smooth and the W_α are locally closed in Z. Furthermore the limit map $f_\alpha : W_\alpha \to L_\alpha$ is a Zariski locally trivial affine fibration. Furthermore in [5] it is shown that there is a filtration $\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z$ of Z by closed subsets such that for every i, $Z_{i+1} - Z_i$ is one of the W_α (this depends on Z being quasi-projective).

Looking at Zariski open sets we find

$$R^i f_{\alpha *}\mathbb{Q}_l = \begin{cases} \mathbb{Q}_l & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$H^i(W_\alpha,\mathbb{Q}_l) = H^i(L_\alpha,\mathbb{Q}_l) \tag{A.2}$$

By the Weil conjectures L_α is pure. Since L_α and W_α are smooth, (A.2) and lemma A.3 below imply that W_α is pure as well. Applying lemma A.4 finishes the proof. □

We have used the following lemmas

Lemma A.3. *If Z is smooth then Z is pure if and only if the eigenvalues of F acting on $H^i(Z, \mathbb{Q}_l)$ have absolute values $q^{i/2}$.*

Proof. This follows by Poincaré duality. □

Lemma A.4. *Assume that we have a decomposition $Z = Y \coprod U$ where Y is closed and Y, U are pure. Then Z is also pure and in addition we have short exact sequences*

$$0 \to H^i_c(Y, \mathbb{Q}_l) \to H^i_c(Z, \mathbb{Q}_l) \to H^i_c(U, \mathbb{Q}_l) \to 0 \tag{A.3}$$

Proof. This follows from the fact that in the long exact sequence

$$\to H^{i-1}_c(U, \mathbb{Q}_l) \to H^i_c(Y, \mathbb{Q}_l) \to H^i_c(Z, \mathbb{Q}_l) \to H^i_c(U, \mathbb{Q}_l) \to H^{i+1}_c(Y, \mathbb{Q}_l) \to$$

the connection maps must be zero by purity. □

Appendix B. Invariants over $\mathbb{Z}$

In this paper we have used lifting to characteristic zero. To do this rigorously we need that taking invariants commutes with base change over a Zariski open part of the base. This is of course well known but we have not found an explicit reference. For simplicity we will only consider the case where the base is $\mathbb{Z}_f$. Replacing $\operatorname{Spec} \mathbb{Z}_f$ by a Zariski open subset amounts to "increasing" f in the following sense:

Convention B.1. If $f \in \mathbb{Z}$ then *increasing* f means making f larger for the partial order given by divisibility.

Let G be reductive group defined over $\mathbb{Z}_f$ [35]. All G-actions below are rational. That is: they are obtained from a coaction of $\mathcal{O}(G)$.

First recall Seshadri's generalization of Geometric Invariant Theory to an arbitrary base ring.

Theorem B.2. [35, §II] *Let R be finitely generated $\mathbb{Z}_f$ algebra and let M be a finitely generated R-module. Assume that G acts rationally on R and M. Then R^G is a finitely generated $\mathbb{Z}_f$-algebra and M^G is a finitely generated R^G-module. In addition if $X = \operatorname{Spec} R$ and $X /\!/ G = \operatorname{Spec} R^G$ then $X /\!/ G$ has the usual behavior in the sense that if $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{Z}_f$ is a geometric point then the points in $(X /\!/ G)(k)$ correspond to the closed orbits in $X(k)$.*

It follows in particular that $\operatorname{Spec}(R \otimes k)^G \to \operatorname{Spec}(R^G \otimes k)$ is set-theoretically a bijection. We want it to be an isomorphism. The result we need is the following:

Theorem B.3. *Let R be finitely generated $\mathbb{Z}_f$ algebra and let M be a finitely generated R-module. Assume that G acts rationally on R and M. Then there exists a Zariski open subset U of $\operatorname{Spec} \mathbb{Z}_f$ such that for every geometric point $\operatorname{Spec} k \to U$ we have that the canonical map $M^G \otimes k \to (M \otimes k)^G$ is an isomorphism and in addition $H^i(G, M \otimes k) = 0$ for $i > 0$.*

We will informally say that the formation of M^G is compatible with base change for f large enough.

Proof. Recall that if H is a reductive algebraic group over an algebraically closed field k then an H-representation of countable dimension is said to have a good filtration if it has an ascending filtration by co-Weyl-modules $Y(\lambda)$, or equivalently if $H^i(H, Y(\lambda)\otimes U) = 0$ for all $i > 0$ and all λ [14]. In particular $(-)^H$ is exact on representations with a good filtration and the category of representations with good filtrations is stable under taking cokernels of surjective maps and extensions. It is a deep theorem [14, 29] that the category of representations with a good filtration is stable under tensor product.

Put $A = \mathbb{Z}_f$. If V is a G-module free of finite rank over A and if $V \otimes_A k$ (k as in the statement of the theorem) has a good filtration then it follows from exactness of $(-)^G$ that $\dim(V\otimes_A k)^G$ is the number of $Y(0)$'s in a good filtration of $V \otimes_A k$. This can be computed in terms of characters so we conclude

$$\dim(V\otimes_A k)^G = \dim(V\otimes_A \bar{\mathbb{Q}})^G = \operatorname{rk} V^G = \dim(V^G \otimes_A k) \tag{B.1}$$

By the universal coefficient theorem the canonical map

$$V^G \otimes_A k \to (V\otimes_A k)^G \tag{B.2}$$

is a monomorphism and hence by (B.1) it an isomorphism.

If V is not necessarily of finite rank but has a filtration $0 = V_0 \subset V_1 \subset V_2 \subset \cdots$ such that each V_{i+1}/V_i is free of finite rank and $(V_{i+1}/V_i)\otimes_A k$ has a good filtration then it is easy to see that (B.2) is still an isomorphism.

Since the action of G is locally finite there exist a finitely generated G module W such that R is a quotient of SW. By increasing f we may assume that W is free. If the characteristic of k is large with respect to λ (in a suitable sense) then $Y(\lambda)$ is simple [18, Ch. 6]. It follows that if $\operatorname{char} k$ is large then the finite dimensional G-representation $\Lambda(W\otimes_A k)$ has a good filtration. It then follows from [1, §4.3] that $SW\otimes_A k = S(W\otimes_A k)$ has a good filtration as well. From the proof it follows that this good filtration is compatible with the grading.

Now we filter SW by degree and we put the induced filtration on R. We choose a compatible filtration on M such that $\operatorname{gr} M$ is a finitely generated $\operatorname{gr} R$-module (confusingly such a filtration is also called a good filtration!) [31]. Since $\operatorname{gr} R$ and $\operatorname{gr} M$ are finite over the noetherian ring SV their $\mathbb{Z}$-torsion is supported on a finite set of primes. Hence by increasing f we may and we will assume that $\operatorname{gr} R$ and $\operatorname{gr} M$ are torsion free.

Since SW has finite global dimension it is easy to see that (at the cost of possibly increasing f) we may construct a graded resolution of $\operatorname{gr} M$ whose terms are of the form $U_i \otimes_A SW$ with U_i a free G-representation of finite rank. Increasing f again if necessary we may assume that all $U_i\otimes_A k$ have a good filtration. Thus it follows that $(\operatorname{gr} M)\otimes_A k$ will also have a good filtration compatible with the grading for all k. Thus $M\otimes_A k$ has vanishing cohomology. The rest of the theorem follows from the fact that (B.2) is an isomorphism with $V = M$. $\square$

From Theorem B.3 one easily deduces that all standard constructions are compatible with base change if we take f large enough. We give an example whose proof we leave to the reader.

Lemma B.4. *Let X be of finite type over $\mathbb{Z}_f$ and assume that G acts rationally on X. Let L be a G-equivariant line bundle on X. Let X^{ss} be the L-semistable points on X* [35, §II]. *Then the formation of X^{ss} and $X^{ss}/\!\!/G$ is compatible with base change for f large enough.*

Appendix by Hiraku Nakajima

The following simple proof avoids the arguments in Section 2.3, showing directly that if λ is generic for α, then $\mathrm{Rep}(\Pi^\lambda,\alpha)^\lambda /\!\!/ G(\alpha)$ and $\mathrm{Rep}(\Pi^0,\alpha)^\lambda /\!\!/ G(\alpha)$ have the same number of points over sufficiently large finite fields $\mathbb{F}_q$.

Let $k = \overline{\mathbb{F}_p}$, the algebraic closure of a finite field.

Suppose that $\pi\colon \mathcal{X} \to \mathbb{A}^1$ is a smooth family of nonsingular quasi-projective varieties over the line $\mathbb{A}^1 = k$ with the following properties:

(1) there exists a $\mathbb{G}_m$-action on $\mathcal{X}$ such that π is equivariant with respect to a $\mathbb{G}_m$-action on $\mathbb{A}^1$ of weight one,
(2) for every $x \in \mathcal{X}$, the limit $\lim_{t\to 0} t \cdot x$ exists.

Such an action exists in the case of quiver varieties [30, §5].

Let $X_\lambda = \pi^{-1}(\lambda)$.

Theorem. *The number $\#X_\lambda(\mathbb{F}_q)$ of rational points is independent of λ (for $\mathbb{F}_q$ containing fields of definition of $\mathcal{X}$, π, λ and a finite number of auxilliary varieities).*

Proof. First note that X_λ is isomorphic to $X_{t\lambda}$ for $t \in k^*$. Therefore, it is enough to show that $\#X_0(\mathbb{F}_q)$ is equal to $\#X_1(\mathbb{F}_q)$.

Let $\bigsqcup \mathcal{F}_\alpha$ be the decomposition of the fixed point set $\mathcal{X}^{\mathbb{G}_m}$ into connected components. Each $\mathcal{F}_\alpha$ is a nonsingular projective variety. Moreover, $\mathcal{F}_\alpha$ is contained in X_0. (We have used the assumption (1).)

We consider the Bialynicki-Birula decomposition of $\mathcal{X}$ with respect to the $\mathbb{G}_m$-action:

$$\mathcal{X} = \bigsqcup_\alpha \mathcal{X}_\alpha,$$

where $\mathcal{X}_\alpha = \{x \in \mathcal{X} \mid \lim_{t\to 0} t\cdot x \in \mathcal{F}_\alpha\}$. By the assumption (2), the right hand side coincides with the whole space $\mathcal{X}$. It is known that the natural projection $\mathcal{X}_\alpha \to \mathcal{F}_\alpha$ is an affine fibration whose fiber is isomoprhic to the direct sum of positive weight space in the tangent space at $\mathcal{F}_\alpha$. Therefore, we have

$$\#\mathcal{X}(\mathbb{F}_q) = \sum_\alpha \#\mathcal{X}_\alpha(\mathbb{F}_q) = \sum_\alpha \#\mathcal{F}_\alpha(\mathbb{F}_q) q^{n_\alpha},$$

where n_α is the dimension of the fiber.

We also consider the Bialynicki-Birula decomposition of X_0:

$$X_0 = \bigsqcup_\alpha (X_0)_\alpha.$$

Then $(X_0)_\alpha$ is also an affine fibration over the *same* base $\mathcal{F}_\alpha$. The tangent space of $\mathcal{X}$ (at a point in $\mathcal{F}_\alpha$) decompose into the sum of the tangent space of X_0 (fiber direction) and $\mathbb{A}$ (base direction). Therefore, the dimension of the fiber is equal to $n_\alpha - 1$. Thus

$$\#X_0(\mathbb{F}_q) = \sum_\alpha \#(X_0)_\alpha(\mathbb{F}_q) = \sum_\alpha \#\mathcal{F}_\alpha(\mathbb{F}_q) q^{n_\alpha - 1} = \frac{1}{q}\#\mathcal{X}(\mathbb{F}_q).$$

On the other hand,

$$\#\mathcal{X}(\mathbb{F}_q) = \sum_{\lambda\in\mathbb{F}_q} \#X_\lambda(\mathbb{F}_q) = (q-1)\#X_1(\mathbb{F}_q) + \#X_0(\mathbb{F}_q).$$

Therefore the conclusion follows. $\square$

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