A RELATION BETWEEN A CONJECTURE OF KAC AND THE STRUCTURE OF THE HALL ALGEBRA

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Abstract. In this paper we show that the Hall algebra of a quiver, as defined by Ringel, is the positive part of the quantived enveloping algebra of a generalized Kac-Moody Lie algebra. We give a potential application of this result to a conjecture of Kac which states that the constant coefficient of the polynomial counting the number of absolutely indecomposable representations of a quiver over a finite field is equal to the multiplicity of the corresponding root in the associated Kac-Moody Lie algebra.

1. Introduction

Throughout Q will be a fixed quiver without loops (although this is not very essential). We denote the vertices of Q by Q_0. We will consider representations over Q with coefficients in $\mathbb{F}_q$. If A is such a representation then we denote by $[A]$ its isomorphism class and by $\underline{\dim} A$ its dimension vector. The twisted Hall algebra (sometimes called Ringel algebra) [13] $H(Q)$ is the free $\mathbb{C}$-vector space with basis the isomorphism classes of Q representations. The product is given by

$$[A][B] = v^{-\langle B,A\rangle} \sum_{[C]} g^C_{AB}[C] \tag{1.1}$$

where $v = q^{-1/2}$, g^C_{AB} is the cardinality of the set

$$\{X \subset C \mid X \cong A, \quad C/X \cong B\}$$

and

$$\langle B, A\rangle = \dim \mathrm{Hom}(A, B) - \dim \mathrm{Ext}^1(A, B)$$

Note that we have taken Kapranov's definition of the product [10]. This yields an algebra which is the opposite of the one defined by Ringel.

The importance of the Hall algebra is given by the fact [13] that the subalgebra generated by the simple representations is the positive part of the quantized enveloping algebra of the Kac-Moody Lie algebra $\mathfrak{g}$ associated to the bilinear form on $\mathbb{C}^{Q_0}$ given by

$$(e_i, e_j) = \begin{cases} 2 & \text{if } i = j \\ -a_{ij} - a_{ji} & \text{otherwise} \end{cases}$$

where $(e_i)_j = \delta_{ij}$ and where a_{ij} is the number of edges going from i to j.

In this paper we will give a similar interpretation of the full Hall algebra. That is we show:

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Theorem 1.1. *The Hall algebra of a quiver over a finite field is the positive part of a quantized enveloping algebra of a **generalized** Kac-Moody Lie algebra.*

A generalized Kac-Moody Lie algebra differs from an ordinary Kac-Moody Lie algebra by the fact that it posesses so-called imaginary simple roots (see [1]). Theorem 1.1 gives us to some extent a complete description of the Hall algebra by generators and relations.

Quantized generalized Kac-Moody Lie algebras have been introduced by S.J. Kang in [9] in the case of a generic parameter. Unfortunately in the case of the Hall algebra the parameter is definitely not generic, and since Kang makes essential use of specialization arguments we cannot use his results directly. Luckily it turns out that as long as the defining parameter is not a root of unity one can simply generalize the arguments by Borcherds [1] and Kac [8] to the quantum case. Additionally the proof of Theorem 1.1 uses the positive definiteness of the Green bilinear form on the Hall algebra (see Proposition 3.2). This argument seems to be new.

As indicated in the previous paragraph, the methods in this paper are standard techniques in the theory of quantum groups. We refer the reader to the excellent books [5][6] or [11]. We have chosen to give our own presentation since the cited references sometimes use specialization arguments or arguments based on the theory of canonical bases which do not apply to our situation.

As a potential application of Theorem 1.1 we will now discuss a conjecture of Kac on the number of absolutely indecomposable representations of Q (recall than an indecomposable representation is said to be *absolutely indecomposable* if it remains indecomposable after extension of the base field). For a given $\alpha \in \mathbb{N}^{Q_0}$ let $a_\alpha(q)$ be the number of absolutely indecomposable representations of Q over $\mathbb{F}_q$ with dimension vector α .

Kac proves in [7] that $a_\alpha(q)$ is a polynomial in q with integral coefficients, and he conjectures that these coefficients are in fact non-negative. He makes furthermore a conjecture with regard to the constant term of $a_\alpha(q)$.

Conjecture 1.2. $a_\alpha(0)$ is equal to the multiplicity of α in $\mathfrak{g}$.

Using the known representation theory of finite and tame quivers [2][3], one can show that the conjecture is true in these cases. Otherwise there is very little evidence for the conjecture.

In this paper we give a reformulation (for many quivers) of Conjecture 1.2 in terms of the Hall algebra [13].

Theorem 1.3. *Assume that there is no subset of Q_0 which spans a tame subquiver. Then Conjecture 1.2 is true if and only if the multiplicities of the imaginary simple roots of the generalized Kac-Moody Lie algebra corresponding to the Hall algebra of Q are divisible by q.*

We don't really know how to use this result but it might become useful if one would be able to define new operations on Hall algebras which would act in some way on their imaginary simple roots. This could lead to information on their multiplicities.

2. Quantized Generalized Kac-Moody Lie algebras

Below k is an algebraically closed ground field. Y will be a $\mathbb{Q}$-vector space containing a countable linearly independent subset I. We assume that Y is equipped with a non-degenerate $\mathbb{Q}$-valued generalized Kac-Moody bilinear form $(-,-)$ [1]. That is $(-,-)$: is symmetric and for $i,j \in I$ one has $(i,j) \le 0$ if $i \ne j$ and $2(i,j)/(i,i)$ is an integer if (i,i) is positive. We will refer to the elements of I as simple roots. We introduce a decomposition $I = I^{\mathrm{re}} \cup I^{\mathrm{im}}$ where I^{re}, I^{im} contain respectively the elements $i \in I$ such that $(i,i) > 0$ and $(i,i) \le 0$. Note in passing that we do not necessarily assume that the elements of I are a basis of Y. This allows us to make the assumption that $(-,-)$ is non-degenerate (by extending Y if necessary). In fact the reader will notice that the definitions below do not depend on Y but only on the numbers (i,j) for $i,j \in I$.

Convention. Let $(T_a)_{a>0}$ be the inverse system with $T_a = k^*$ for all a and with transition maps $T_a \to T_{ab} : x \mapsto x^b$. Let $\mathbf{k}^*$ be the inverse limit of this inverse system. Fix $v \in k^*$ and let $\mathbf{v} = (v_a)_a$ be a fixed element of $\mathbf{k}^*$ such that $v_1 = v$. If $c/d \in \mathbb{Q}$ then we define v^d as v_d^c. This is clearly well defined.

We assume throughout that v is not a root of unity. This implies that if $v^\alpha = v^\beta$ for $\alpha, \beta \in \mathbb{Q}$ then $\alpha = \beta$.

We now define the algebras $\tilde{U}$ and U. $\tilde{U}$ has generators E_i, F_i, K_i, K_i^{-1} for $i \in I$ satisfying the following relations.

$$\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1 \\
K_i K_j &= K_i K_j \\
E_i F_j - F_j E_i &= (K_i - K_i^{-1})\delta_{i,j} \\
K_i E_j K_i^{-1} &= v^{(i,j)} E_j \\
K_i F_j K_i^{-1} &= v^{-(i,j)} F_j
\end{aligned}$$

The algebra $\tilde{U}$ is Y-graded by $\deg E_i = -\deg F_i = i$, $\deg K_i = 0$. To save space we will denote the degree of a homogeneous element a by $|a|$. With a method similar to [11, §3.2] one shows that $\tilde{U}$ has a triangular decomposition $\tilde{U}_- \otimes \tilde{U}_0 \otimes \tilde{U}_+$ where $\tilde{U}_-$, $\tilde{U}_+$ are respectively the free algebras generated by $(E_i)_i$ and $(F_i)_i$ and $\tilde{U}_0$ is the Laurent polynomial algebra generated by the K_i. If $\alpha \in \mathbb{Z}I$ then it will be convenient to write K_α for the product $\prod_i K_i^{\alpha_i}$.

It is easy to see that $\tilde{U}$ is a Hopf algebra with counit $\epsilon(E_i) = \epsilon(F_i) = 0$, $\epsilon(K_i) = 1$, comultiplication

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i \\
\Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i \\
\Delta(F_i) &= K_i^{-1} \otimes F_i + F_i \otimes 1
\end{aligned}$$

and antipode

$$\begin{aligned}
S(K_i) &= K_i^{-1} \\
S(E_i) &= -E_i K_i^{-1} \\
S(F_i) &= -K_i F_i
\end{aligned}$$

There is a unique automorphism ω of $\tilde{U}$ satisfying $\omega(E_i) = F_i$, $\omega(F_i) = E_i$, $\omega(K_i) = K_i^{-1}$. It is clear that ω is an anti-automorphism on the level of coalgebras.

In the theory of quantum groups it is customary to define v-binomial coefficients differently from combinatorics [12]. In order to avoid confusion we use a different notation.

$$\{n\} = \frac{v^n - v^{-n}}{v - v^{-1}}, \quad \{n\}! = \prod_{t=1}^{n} \{t\}, \quad \text{and} \quad \begin{Bmatrix} n \\ t \end{Bmatrix} = \frac{\{n\}!}{\{t\}!\{n-t\}!}$$

These are Laurent polynomials in v with integer coefficients. Given a Laurent polynomial $\phi(v)$ and a rational number d we use the notation ϕ_d for the Laurent polynomial $\phi(v^d)$.

For a real simple root i put

$$a_{ij} = -2\frac{(i,j)}{(i,i)}$$
$$d_i = \frac{(i,i)}{2}$$

We define the *quantized generalized Kac-Moody Lie algebra U* to be the quotient of $\tilde{U}$ by the quantum Serre relations.

$$\begin{aligned} \sum_{p=0}^{a_{ij}+1} (-)^p \begin{Bmatrix} a_{ij}+1 \\ p \end{Bmatrix}_{d_i} E_i^p E_j E_i^{a_{ij}+1-p} &= 0 \qquad i \text{ real} \\ \sum_{p=0}^{a_{ij}+1} (-)^p \begin{Bmatrix} a_{ij}+1 \\ p \end{Bmatrix}_{d_i} F_i^p F_j F_i^{a_{ij}+1-p} &= 0 \qquad i \text{ real} \\ E_iE_j - E_jE_i &= 0 \qquad \text{if } (i,j) = 0 \\ F_iF_j - F_jF_i &= 0 \qquad \text{if } (i,j) = 0 \end{aligned} \tag{2.1}$$

We denote images of E_i, F_i, K_i in U by the same symbol. The triangular decomposition of $\tilde{U}$ induces a triangular decomposition of U (this can be proved as in [11, Cor. 3.2.5]).

Let us define $\tilde{U}_{\geq 0} = \tilde{U}_0 \otimes \tilde{U}_+$, $\tilde{U}_{\leq 0} = \tilde{U}_- \otimes \tilde{U}_0$. It follows from [11] that there is a unique symmetric Hopf pairing $[-,-]$ of $\tilde{U}_{\geq 0}$ with itself satisfying $[K_i, K_j] = v^{-(i,j)}$, $[E_i, E_j] = \delta_{i,j}$, $[K_i, E_j] = 0$. We put $[-,-]_0 = [-, \omega(-)]$. This defines a skew Hopf pairing between $\tilde{U}_{\geq 0}$ and $\tilde{U}_{\leq 0}$.

The following lemma gives the connection between $[-,-]_0$ and the Drinfeld double.

Lemma 2.1. *Let $a \in \tilde{U}_{\geq 0}$, $b \in \tilde{U}_{\leq 0}$. Then*

$$\sum_{a,b} b_{(2)} a_{(2)} [a_{(1)}, b_{(1)}]_0 = \sum_{a,b} a_{(1)} b_{(1)} [a_{(2)}, b_{(2)}]_0 \tag{2.2}$$

where $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ (Sweedler convention).

Proof. With a reasoning similar to [15, Lemma 3.1] it suffices to establish this for $a \in \{(K_\alpha)_\alpha, E_i\}$, $b \in \{(K_\alpha)_\alpha, F_i\}$. But then it follows immediately from the relations. □

It is shown in [11] (and also in [14]) that the relations (2.1) are in the radical of $[-,-]$ and $[-,-]_0$ (for their respective domains of definitions). Hence $[-,-]$ and $[-,-]_0$ factor through U. We use the same notations for these induced bilinear forms.

Finally we denote by $\bar{U}$ the Hopf algebra obtained from $\tilde{U}$ (or U) by factoring out the left and right radical of $[-,-]_0$, restricted to $U_- \times U_+$. Again we have a triangular decomposition etc...

We fix homogeneous bases $(a_u)_u$, $(b_u)_u$ for $\bar{U}_+$, $\bar{U}_-$ which are dual for the form $[-,-]_0$ defined above. Put $\bar{u} = |b_u| = -|a_u|$. For $\alpha = \sum_i \alpha_i i \in \mathbb{N}I$ write $\operatorname{ht}\alpha = \sum \alpha_i$. Let M be a $\bar{U}$-module such that for all $m \in M$ we have that $(\bar{U}_-)_{-\alpha} m = 0$ for $\operatorname{ht}\alpha \gg 0$.

Then the action of the operator

$$\Omega = \sum_u S^{-1}(a_u) K_{\bar{u}} b_u K_{\bar{u}}$$

is well defined on M. Using (2.2), one verifies that it satisfies [11]

$$\begin{aligned} \Omega E_i &= E_i \Omega K_i^2 \\ \Omega K_i &= K_i \Omega \\ \Omega F_i &= K_i^{-2} F_i \Omega \end{aligned} \tag{2.3}$$

We denote by $\mathcal{O}$ the category of Y-graded $\bar{U}$-modules M satisfying

- K_i acts on M_α by multiplication with $v^{(\alpha,i)}$.
- All M_α are finite dimensional.
- The set of α such that $M_\alpha \neq 0$ is contained in the union of a finite number of sets of the form $\mu + \mathbb{N}I$.

One can use the operator Ω to construct a true Casimir operator on objects in $\mathcal{O}$. This is done as follows (cfr [11]). Enlarging Y if necessary we may assume that there exists $\rho \in Y$ such that $(\rho, i) = \frac{1}{2}(i,i)$ for all $i \in I$. Let $M \in \mathcal{O}$ and let C be the operator which acts on M_α by $v^{-(\alpha,\alpha-2\rho)}\Omega$. Then C commutes with the U-action on M. This follows from the formulas (2.3).

Particular objects in $\mathcal{O}$ are the so-called (lowest weight) Verma modules defined by

$$M(\mu) = \bar{U}/(\sum_i \bar{U} F_i + \sum_i \bar{U}(K_i - v^{(\mu,i)}))$$

As usual we denote by $L(\mu)$ the unique simple quotient of $M(\mu)$ in the category $\mathcal{O}$. It is clear that C acts on $M(\mu)$ and $L(\mu)$ by multiplication with $v^{-(\mu,\mu-2\rho)}$.

If $i \in I^{\mathrm{re}}$ then we write $s_i(\chi) = \chi - \frac{2(\chi,i)}{(i,i)} i$. Clearly s_i is a reflection on Y. We denote the group generated by the s_i by W. This is the Weyl group of the root datum $(Y, I, (-,-))$. We define the character $\epsilon : W \to \{-1,+1\}$ by $\epsilon(s_i) = -1$.

For $\alpha \in Y$ we introduce a symbol $e(\alpha)$, in such a way that $e(\alpha+\beta) = e(\alpha)e(\beta)$. We also put $w \cdot e(\alpha) = e(w\alpha)$ for $w \in W$. For a Y-graded vector space V we put

$$\operatorname{Ch}(V) = \sum_{\alpha \in Y} \dim V_\alpha e(\alpha)$$

We will now recall how the Weyl group acts on $\operatorname{Ch}(U_+)$.

For $i \in I^{\mathrm{re}}$ let U_i be the subalgebra of U generated by E_i, F_i, K_i. Since $U_v(\mathfrak{sl}_2)$ is simple as a Hopf algebra it follows that U_i is isomorphic to the quantum enveloping algebra of $\mathfrak{sl}_2$. As usual we define a twisted adjoint action of U_i on $\tilde{U}$ and U by

$$\begin{aligned} \operatorname{Ad}^\circ(E_i)(a) &= E_i a - v^{(i,|a|)} a E_i \\ \operatorname{Ad}^\circ(F_i)(a) &= (F_i a - a F_i) K_i \\ \operatorname{Ad}^\circ(K_i)(a) &= v^{(i,|a|)} a \end{aligned}$$

For $i,j\in I$, $p\in\mathbb{N}$ put $\tilde{T}_{i,j,p}=\mathrm{Ad}^\circ(E_i)^p(E_j)\in\tilde{U}_+$ and denote by $T_{i,j,p}$ the corresponding images in U. Let $V_i,\tilde{V}_i$ be respectively the subalgebras of $U,\tilde{U}$ generated by all $T_{i,j,p},\tilde{T}_{i,j,p}$, $j\neq i$.

The following properties are easily established.

1. V_i is $\mathrm{Ad}^\circ U_i$-stable. It is is easily seen to be sufficient to check this on the $T_{i,j,p}$. Stability under $\mathrm{Ad}^\circ E_i$ and $\mathrm{Ad}^\circ K_i$ is obvious. Stability under $\mathrm{Ad}^\circ F_i$ follows from the relations in U_i.
2. The action of U_i on V_i is locally finite. This follows from the fact that the Serre relation between E_i and E_j implies that $T_{i,j,a_{ij}+1}=0$ and hence V_i is generated by the finite dimensional U_i-representations given by $\sum_p T_{i,j,p}\cdot k$.
3. Every element of U_+ can be written as a sum $\sum_n E_i^n b_n$ with $b_n\in V_i$.

Let us now temporarily define $\tilde{U}_+ * \tilde{U}_+$ as $\tilde{U}_+\otimes\tilde{U}_+$ with the new multiplication

$$(a*b)(c*d)=v^{-(|b|,|c|)}(ac*bd)$$

(for homogeneous a,b,c,d).

It is easy to see that there is an algebra homomorphism $\delta:\tilde{U}_+\to\tilde{U}_+*\tilde{U}_+$ given by $\delta(E_i)=1\otimes E_i+E_i\otimes 1$. By induction on degree one shows

$$[a,bc]=\sum_i [a_i,b][a_i',c] \tag{2.4}$$

if $\delta(a)=\sum_i a_i\otimes a_i'$. We now define linear maps $\delta_i:\tilde{U}_+\to\tilde{U}_+$ by the properties

$$\delta_i(E_j)=\delta_{ij}$$
$$\delta_i(bc)=\delta_i(b)c+v^{-(i,|b|)}b\delta_i(c)$$

(for homogeneous b). One easily shows by induction that $[E_ia,b]=[a,\delta_i b]$. Furthermore it is also easy to see that if $a\in\tilde{U}_+$ is such that $\delta_i(a)=0$ then one also has $\delta_i(\mathrm{Ad}^\circ(E_i)(a))=0$. This allows us to prove the following fact

4. If $a,b\in\tilde{V}_i$ then $[E_i^m a,E_i^n b]=0$ unless $m=n$. In that case it is a non-zero multiple of $[a,b]$.

Proof. By the remark before the lemma it follows that $\delta_i\mid\tilde{V}_i=0$. Thus

$$[E_i^m a,E_i^n b]=[E_i^{m-1}a,\delta_i(E_i^n b)]=[E_i^{m-1}a,\delta_i(E_i^n)b]=\lambda[E_i^{m-1}a,E_i^{n-1}b]$$

where λ is some non-zero scalar (using the fact that v is not a root of unity). The lemma follows easily from this. ☐

By taking images a similar statement is of course true for U_+.

Let $\bar{V}_i$ be the image of V_i in $\bar{U}_+$. The corresponding properties hold also for $\bar{V}_i$. Note that it follows from 4. that in $\bar{U}$ the decomposition given by 3. is unique, since $[-,-]$ is non-degenerate. From this one deduces the following result.

Lemma 2.2. *$e(-\rho)\operatorname{Ch}(\bar{U}_+)^{-1}$ is W-skew invariant.*

Proof. Let $i\in I^{\mathrm{re}}$. Properties 3,4 imply that $\operatorname{Ch}(\bar{U}_+)=(\sum_n e(ni))\operatorname{Ch}(\bar{V}_i)$. Hence we find

$$e(-\rho)\operatorname{Ch}(\bar{U}_+)^{-1}=(e(-\rho)-e(-\rho+i))\operatorname{Ch}(\bar{V}_i)^{-1}$$

Properties 1,2 imply that $s_i\operatorname{Ch}(\bar{V}_i)=\operatorname{Ch}(\bar{V}_i)$. Since also $s_i(e(-\rho)-e(-\rho+i))=(e(-\rho+i)-e(-\rho))$, we obtain what we want. ☐

Using this lemma one can now generalize the Borcherds character formula [1]. Let P_- be the set of all $\mu \in Y$ such that

$$2(\mu,i)/(i,i) \in -\mathbb{N} \quad \text{if } i \in I^{\mathrm{re}}, \qquad (\mu,i) \le 0 \text{ for all } i \in I^{\mathrm{im}}$$

For $\mu \in P_-$ define $S_\mu = \sum \epsilon(s)e(s)$ where s runs over the sums of elements in I^{im} and $\epsilon(s)$ is $(-1)^n$ if such a sum consists of n distinct pairwise orthogonal terms, each orthogonal to μ. Otherwise $\epsilon(s) = 0$.

Proposition 2.3. *The character of $\bar{U}_+$ is given by*

$$\mathrm{Ch}(\bar{U}_+) = \Big(\sum_w \epsilon(w)e(\rho - w\rho)w(S_0)\Big)^{-1} \tag{2.5}$$

The character of $L(\mu)$, $\mu \in P_-$ is given by

$$\mathrm{Ch}\, L(\mu) = \Big(\sum_w \epsilon(w)e(\rho + w(\mu-\rho))w(S_\mu)\Big)\, \mathrm{Ch}(\bar{U}_+) \tag{2.6}$$

Proof. The proof is a copy of [1] or [8, Chapter 11]. One observes first that if μ is in P_- then $L(\mu)$ is integrable, in the sense that for $i \in I^{\mathrm{re}}$, E_i and F_i act locally nilpotently. Hence $\mathrm{Ch}(L(\mu))$ is W-invariant.

Secondly, as in the Lie algebra case [8, Prop 9.8]

$$\mathrm{Ch}(L(\mu)) = \sum_\lambda c_{\mu\lambda}\, \mathrm{Ch}(M(\lambda))$$

for $c_{\mu\lambda} \in \mathbb{Z}$, $c_{\mu\mu} = 1$ where the sum is over the λ such that $\lambda > \mu$ (that is $\lambda - \mu$ is a sum of simple roots) and $(\mu, \mu - 2\rho) = (\lambda, \lambda - 2\rho)$.

From $\mathrm{Ch}(M(\lambda)) = e(\lambda)\, \mathrm{Ch}(\bar{U}_+)$ one obtains

$$\frac{e(-\rho)L(\mu)}{\mathrm{Ch}(\bar{U}_+)} = \sum_\lambda c_{\mu\lambda} e(\lambda - \rho) \tag{2.7}$$

Using lemma 2.2 we find that the lefthand side of the (2.7) is skew invariant under W. Then as in [1] or [8, Chapter 11] it follows that the righthand side of (2.7) is equal to

$$\sum_{w \in W} \epsilon(w)e(w(\mu - \rho))w(S_\mu)$$

This yields (2.6). (2.5) follows from considering $\mu = 0$. □

The following result is also proved in the same way as in the ordinary Kac-Moody case [8]. See also [6, §4.1.17], where the notion of primitive vector is replaced by U_+-homology.

Proposition 2.4. *The bilinear form $[-,-]$ is non-degenerate on U_+. In particular $\bar{U} = U$.*

Proof. To start we need to know something about the minimal generators as a twosided ideal of $J = \ker(\tilde{U}_+ \to \bar{U}_+)$. We follows the method of [8]. It is easy to see that the degrees of the minimal generators of J are the same as the degrees of the minimal generators of the left $\bar{U}_+$-module $R = \ker(\oplus_{i\in I}\bar{U}_+(-i) \xrightarrow{(E_i)_i} \bar{U}_+)$, where $\bar{U}_+(-i)$ is the graded $\bar{U}_+$-module satisfying $(\bar{U}_+(-i))_\alpha = (\bar{U}_+)_{\alpha - i}$.

There is a canonical map of $\bar{U}$-modules $\oplus_{i\in I} M(i) \to M(0)$ with cokernel $L(0)$. It is clear that R is isomorphic as $\bar{U}_+$ module to $\ker(\oplus_{i\in I} M(i) \to M(0))$. In this way R acquires a $\bar{U}$-structure.

Since $R \subset \oplus_{i\in I} M(i)$ it follows in particular that R is in $\mathcal{O}$ and hence by [8, Remark 9.3] R is generated as $\bar{U}_+$ module by its primitive vectors (homogeneous elements which are annihilated by the F_i modulo a submodule). Let $v \in R$ be such a primitive vector of degree α. Looking at the action of C on v (modulo the submodule) yields that $(\alpha, \alpha - 2\rho) = (i, i - 2\rho)$, for at least one $i \in I$. This yields $(\alpha, \alpha) = 2(\rho, \alpha)$.

Assume that β has minimal height such that $(\mathrm{rad}[-,-] \cap U_+)_\beta \neq 0$. We claim that for all $i \in I$ we have $(\beta, i) \leq 0$. It is clearly sufficient to check this for $i \in I^{\mathrm{re}}$. By properties 3,4 and the minimality of β it follows that $(\mathrm{rad}[-,-] \cap V_i)_\beta \neq 0$.

Since U_i acts on $\mathrm{rad}[-,-] \cap V_i = \ker(V_i \to \bar{V}_i)$ we find that $\mathrm{Ch}(\mathrm{rad}[-,-] \cap V_i)$ is s_i-invariant. In particular $(\mathrm{rad}[-,-] \cap V_i)_{s_i\beta} \neq 0$. Now if $i \in I^{\mathrm{re}}$ were such that $(\beta, i) > 0$ then $\mathrm{ht}(s_i\beta) < \mathrm{ht}(\beta)$. This would yield a contradiction with the choice of β.

Choose $0 \neq r \in (\mathrm{rad}[-,-] \cap U_+)_\beta$. Clearly r is the image of a minimal generator of J. Hence β is the degree of a minimal generator of R. As was already said above R is generated by its primitive vectors, and so the degrees of the primitive vectors contain the degrees of the minimal generators. In particular β is also the degree of a primitive vector in R. We now have the following information on β.

1. $(\beta, i) \leq 0$ for all $i \in I$.
2. $(\beta, \beta) = 2(\rho, \beta)$.

As in [8, Lemma 11.13.2] this implies that β is the sum of pairwise orthogonal, not necessarily distinct elements of I^{im}. To finish the proof one shows directly (using the last two relations in (2.1)) that $(\mathrm{rad}[-,-] \cap U_+)_\beta$ is zero for such β. □

3. Quantized Generalized Kac-Moody Lie algebras and positive definite quadratic forms

For simplicity we will assume in this section that the base field is $\mathbb{R}$. If $v \in \mathbb{R}_+$ and $a \in \mathbb{R}$ then v^a denotes the usual a'th power of v. It is clear that this is compatible with the convention which was in force in the previous section.

Below let e_t the canonical t'th basis element of $\mathbb{Z}^n$. In this section we consider the following data.

1. A $\mathbb{N}^n$-graded $\mathbb{R}$-algebra with the following properties.
   (a) $A_0 = \mathbb{R}$.
   (b) $\dim A_\alpha < \infty$ for all $\alpha \in \mathbb{N}^n$.
   (c) $A_{e_t} \neq 0$ for all $t \in \{1, \ldots, n\}$.
2. A symmetric positive definite bilinear form $[-,-] : A \times A \to \mathbb{R}$, which is zero on $A_\alpha \times A_\beta$ if $\alpha \neq \beta$. We also assume $[1, 1] = 1$.
3. An element $v \in]0, 1[$.
4. A symmetric bilinear form $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $(e_i, e_i) > 0$ and such that
$$a_{ij} = 2(e_i, e_j)/(e_i, e_i)$$
   is a generalized Cartan matrix in the sense of [8, Chapter I].

We make $A \otimes A$ into an algebra by the rule

$$(a \otimes b)(c \otimes d) = v^{-(\deg b, \deg c)}(ac \otimes bd) \tag{3.1}$$

for homogeneous a, b, c, d.

Throughout we now make the following extra hypotheses.

Hypothesis (*). Let δ be the map $A \to A \otimes A$, which is adjoint under $[-,-]$ to the multiplication. Then δ is an algebra morphism for the algebra structure on $A \otimes A$, defined in (3.1).

Note that the adjointness property implies that δ is coassociative, and also that $\epsilon = [1,-]$ behaves as a counit.

For $\alpha \in \mathbb{N}^n - \{0\}$ define $H_\alpha = (\sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma\neq\alpha}} A_\beta A_\gamma)^\perp \subset A_\alpha$. Here $(-)^\perp$ is taken inside A_α with respect to $[-,-]$. Since $[-,-]$ is positive definite we can for each H_α choose an orthonormal basis. Let $(\theta_i)_{i\in I}$ be the union of these bases. We define a bilinear form on $\mathbb{Z}I$ by $(i,j) = (\deg\theta_i, \deg\theta_j)$ for $i,j \in I$.

Lemma 3.1. *We have*

$$\delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$$

Proof. Choose a homogeneous orthonormal basis $(f_j)_{j\in J}$ for A and assume that $(\theta_i)_{i\in I} \subset (f_j)_{j\in J}$. We have

$$\delta(\theta_i) = \sum_{j,k} c_{j,k} f_j \otimes f_k$$

for $c_{j,k} \in \mathbb{R}$ and hence we find

$$\begin{aligned} [\theta_i, f_l f_m] &= \sum_{j,k} c_{j,k} [f_j, f_l][f_k, f_m] \\ &= c_{l,m} \end{aligned}$$

The statement now follows from the definition of $(\theta_i)_{i\in I}$. $\square$

Proposition 3.2. 1. *The bilinear form $(-,-)$ on $\mathbb{Z}I$ satisfies the axioms of* [1]. *That is for $i,j \in I$ we have*
(a) $(i,j) \leq 0$ *if* $i \neq j$.
(b) *If* $(i,i) > 0$ *then* $2(i,j)/(i,i)$ *is an integer.*
In addition we have
(c) $(i,i) > 0$ *if and only if* $\theta_i \in \bigcup_t A_{e_t}$.
(d) $\dim A_{e_t} = 1$ *for all* t.

2. *A is isomorphic to the positive part of a quantized generalized Kac-Moody Lie algebras with simple roots I and bilinear form $(-,-)$. $i \in I$ is a real simple root if and only if $\theta_i \in A_{e_t}$ for some t.*

Proof. Let us prove (1a) first. We have

$$\begin{aligned} \delta(\theta_i\theta_j) &= (\theta_i \otimes 1 + 1 \otimes \theta_i)(\theta_j \otimes 1 + 1 \otimes \theta_j) \\ &= \theta_i\theta_j \otimes 1 + \theta_i \otimes \theta_j + v^{-(i,j)}\theta_j \otimes \theta_i + 1 \otimes \theta_i\theta_j \end{aligned}$$

Assume $i \neq j$. We deduce

$$\begin{aligned} [\theta_i\theta_j, \theta_i\theta_j] &= 1 \\ [\theta_i\theta_j, \theta_j\theta_i] &= v^{-(i,j)} \end{aligned}$$

The positivity of $[-,-]$ implies for all $x, y \in \mathbb{R}$:

$$\begin{aligned} 0 &\leq [x\theta_i\theta_j + y\theta_j\theta_i, x\theta_i\theta_j + y\theta_j\theta_i] \\ &= x^2 + 2v^{-(i,j)}xy + y^2 \end{aligned}$$

Hence the determinant of the matrix

$$\begin{pmatrix} 1 & v^{-(i,j)} \\ v^{-(i,j)} & 1 \end{pmatrix}$$

must be positive. Since $v < 1$ this implies $(i,j) \leq 0$ which proves (1a). Since if $\theta_i, \theta_j \in A_{e_t}$ then $(i,j) = (\deg\theta_i, \deg\theta_j) = (e_t, e_t) > 0$, (1d) immediately follows.

Now we prove (1c). By hypotheses it is clear that if $\theta_i \in A_{e_t}$ then $(i,i) = (e_t, e_t) > 0$, hence let us prove the converse. Assume $\theta_i \notin \bigcup_t A_{e_t}$. Since $A_{e_t} \neq 0$ it follows that $(e_t, \deg\theta_i) \leq 0$ and hence $(\alpha, \deg\theta_i) \leq 0$ for every $\alpha \in \mathbb{N}^n$. Applying this with $\alpha = \deg\theta_i$ yields $(i,i) \leq 0$.

Let us finally prove (1b). Let $(i,i) > 0$. Then $\theta_i \in A_{e_t}$ for certain t. Hence $2(i,j)/(i,i)$ is equal to $2(e_t, \deg\theta_j)/(e_t, e_t)$. The latter is a linear combination of $a_{tu} = 2(e_t, e_u)/(e_t, e_t)$ and since by hypotheses the a_{tu} form a generalized Cartan-matrix, they are integers.

Now we prove 2. Let $\tilde{U}$ as in the previous section. Sending E_i to θ_i defines a surjective map of $\tilde{U}_+$ to A which is compatible with the bilinear forms (this follows from (2.4)). Since the bilinear form is non-degenerate on A this implies that A is equal to $\bar{U}_+$. By Proposition 2.4 it follows that A is isomorphic to U_+. □

If $S, T \subset \{1, \dots, n\}$ then we will say that S, T are connected to each other if there exists $s \in S$, $t \in T$ such that $a_{st} \neq 0$ (and hence $a_{ts} \neq 0$ by axiom (C3) of [8, §1.1]).

For $\alpha = \sum_t \alpha_t e_t \in \mathbb{N}^n$ write $\operatorname{Supp}\alpha$ for the set $\{1 \leq t \leq n \mid \alpha_t \neq 0\}$.

Lemma 3.3. *Assume that $\alpha_1, \alpha_2 \in \mathbb{N}^n$ are such that $(\alpha_1, \alpha_2) = 0$ and $(i, \alpha_1) \leq 0$, $(i, \alpha_2) \leq 0$ for all $i \in I^{re}$. Then there exist three sets $P, P_1, P_2 \in \mathbb{N}^n$ with the following properties.*

1. *P, P_1, P_2 are not connected to each other.*
2. $\operatorname{Supp}\alpha_1 = P_1 \cup P$, $\operatorname{Supp}\alpha_2 = P_2 \cup P$.
3. *P spans an affine submatrix of $(a_{ij})_{ij}$.*

Proof. Let $T_j = \operatorname{Supp}\alpha_j$ for $j = 1, 2$ and put $P_1 = T_1 \setminus T_2$, $P_2 = T_2 \setminus T_1$ and $P = T_1 \cap T_2$. The fact that $(\alpha_1, \alpha_2) = 0$ together with $(i, \alpha_1) \leq 0$ for $i \in I^{\mathrm{re}}$ implies that $(i, \alpha_1) = 0$ for $i \in T_2$. If $i \in P_2$ is such that $\{i\}$ is connected to T_1 then clearly $(i, \alpha_1) < 0$. We conclude that P_2 is not connected to T_1. Hence P_2 is not connected to P and also not to P_1. By symmetry the same statement holds with P_1 and P_2 interchanged. This proves 1. and 2. Let $\alpha_1' = \alpha_1 | P$. Then $\operatorname{Supp}\alpha_1' = P$ and for all $i \in P$: $(i, \alpha_1) = (i, \alpha_1') = 0$. This implies that P spans an affine submatrix of $(a_{ij})_{i,j}$. □

Corollary 3.4. *Let $\mathfrak{g}$ the Kac-Moody Lie algebra associated to the generalized Cartan matrix $(a_{ij})_{ij}$. Assume that there is no $P \subset \{1, \dots, n\}$ such that P spans an affine submatrix of $(a_{ij})_{ij}$. Let $q \in \mathbb{N}$.*

The following are equivalent.

1. *For all $\alpha \in \mathbb{N}^n$ we have that* $\dim U_+(\mathfrak{g})_\alpha \cong \dim A_\alpha$ *modulo q.*
2. *For all $\alpha \in \mathbb{N}^n$ the cardinality of the set $\{i \in I^{\mathrm{im}} \mid \deg\theta_i = \alpha\}$ is divisible by q.*

Proof. This is a consequence of the Borcherds character formula for A, and the Kac-Weyl character formula for $U(\mathfrak{g})$.

Let us write

$$\mathrm{Ch}'(A) = \sum_{\alpha \in \mathbb{N}^n} \dim A_\alpha e(\alpha)$$

Then from Proposition 3.2.2 and (2.5) one deduces that

$$\mathrm{Ch}'(A) = (\sum_w \epsilon(w)e(\rho - w\rho)w(S_0'))^{-1} \tag{3.2}$$

In (3.2) we consider the Weyl group as acting on $\mathbb{C}^n \oplus \mathbb{C}\rho$ where ρ is a formal symbol satisfying $s_i(\rho) = \rho - i$ with s_i the reflection associated to the real simple root i. S_0' is $\sum_\alpha \epsilon(\alpha)e(\alpha)$ where α runs over the sums $\deg\theta_{i_1} + \cdots + \deg\theta_{i_t}$ where the i_p are pairwise orthogonal and distinct elements of I. Finally $\epsilon(\alpha) = (-1)^t$.

Note that we have changed the notations slightly compared to those in (2.5). This is because we are evaluating $\mathrm{Ch}'(A)$ and not $\mathrm{Ch}(A)$ (recall that $\mathrm{Ch}(A)$ would be the character for a much finer grading on A).

From the Weyl character formula it follows that (with the same notations)

$$\mathrm{Ch}(U_+(\mathfrak{g})) = (\sum_w \epsilon(w)e(\rho - w\rho))^{-1}$$

Comparing these two formulas, it follows that 1. is equivalent to

$$\sum_w \epsilon(w)w\big(e(-\rho)(S_0' - 1)\big) \cong 0 \qquad \text{modulo } q \tag{3.3}$$

Now $e(-\rho)(S_0' - 1)$ is a sum of terms which lie in the interior of the fundamental chamber of $\mathbb{C}^n \oplus \mathbb{C}\rho$. Since the stabilizer under the Weyl group of a point in the interior of the fundamental chamber is trivial [8, Prop. 3.12, beginning of proof] it follows from (3.3) that $S_0' \cong 1$ modulo q.

Now suppose we have a sum $\alpha = \deg\theta_{i_1} + \cdots + \deg\theta_{i_t}$ where the i_p are pairwise orthogonal and distinct. By the previous lemma and the hypotheses all $\deg\theta_{i_p}$ have disjoint pairwise not connected support.

An easy induction argument now shows that the number of such sums for all α is divisible by q if and only is the multiplicity of the imaginary simple roots is divisible by q. □

4. Proofs of Theorems 1.1 and 1.3

Let Q be a quiver without loops and let $H(Q)$ be the Hall algebra of Q as defined in the introduction. It is clear that $H(Q)$ is a $\mathbb{Z}^{Q_0}$ graded algebra by associating to a representation its dimension vector. We make $H(Q) \otimes H(Q)$ into an algebra by defining

$$([A] \otimes [B])([C] \otimes [D]) = v^{-(\underline{\dim} B, \underline{\dim} C)}[A][C] \otimes [B][D]$$

(the bilinear form $(-,-)$ was defined in the introduction). Following Green [4] we put

$$\delta[A] = \sum_{[B],[C]} v^{-\langle C,B\rangle} g_{BC}^A \frac{|\operatorname{Aut}(B)| \cdot |\operatorname{Aut}(C)|}{|\operatorname{Aut}(A)|}[B] \otimes [C]$$

and

$$\big[[A],[B]\big] = v^{-(\alpha,\beta)} \frac{\delta_{[A],[B]}}{|\operatorname{Aut}(A)|}$$

It was shown by Green that δ is an algebra map $H(Q) \to H(Q) \otimes H(Q)$ and that multiplication and δ are adjoint under $[-,-]$. In other words the pair $H(Q), v$

satisfies the hypotheses put on the pair A, v in the beginning of the previous section. Hence Theorem 1.1 follows from Proposition 3.2.

In [16, (2.7)] it is noted that Conjecture 1.2 is equivalent to $\dim U_+(\mathfrak{g}) = \dim H(Q)_\alpha$ for all $\alpha \in \mathbb{N}^{Q_0}$. Thus Theorem 1.3 follows from Corollary 3.4.

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