NORMALIZING EXTENSIONS OF THE VERONESE OF A THREE DIMENSIONAL ARTIN-SCHELTER REGULAR ALGEBRA ON TWO GENERATORS

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Abstract. We classify regular algebras of global dimension four that map surjectively onto the two-Veronese of a regular algebra of global dimension three on two generators. We also study the point modules.

Contents

1. Introduction

In the sequel k will always be an algebraically closed field. A graded algebra $A = k \oplus A_1 \oplus \cdots$ is (Artin-Schelter) regular if

- A has finite global dimension.
- A has finite Gelfand-Kirillov dimension, i.e. A has polynomial growth.
- A is Gorenstein meaning that there exists a δ such that

$$\operatorname{Ext}^n_A(k, A) = \begin{cases} k & \text{if } n = \delta \\ 0 & \text{otherwise} \end{cases}$$

Roughly speaking a regular algebra is a non-commutative analog of a polynomial algebra.

Three-dimensional Artin-Schelter regular algebras generated in degree one were classified in [3][4][5]. It turns out that two possibilities can occur

- A has three generators in degree one and three relations in degree two.
- A has two generators in degree one and two relations in degree three.

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In [20][21] Stephenson succeeded in classifying all three-dimensional regular algebras, (that is, not only those generated in degree one). By contrast it seems rather unlikely that such a classification in dimension four or higher will be given any time soon. Nevertheless in recent years some interesting new examples have been constructed that could be studied with methods introduced in the three dimensional case. Perhaps the most typical of these examples are the four dimensional Sklyanin algebras [11][12][13][14][15][16].

The current paper is to some extent inspired by [8] where a classification is given of four-dimensional regular algebras having a central element of degree one such that the corresponding quotient is a three-dimensional regular algebra on three generators. In this article we study Koszul algebras D that have a normalizing element of degree two C, such that the quotient $D/(C) = A_e = \oplus_i A_{2i}$ is the two-Veronese of a three-dimensional regular algebra A on *two* generators. We say that D is a *normalizing extension* of the two-Veronese of A. Standard results yield that such a normalizing extension is automatically regular and has other desirable properties (Proposition 3.1).

If D, C are as in the previous paragraph then conjugation by C defines an automorphism of A_e. Conversely given A and an automorphism α of A_e we might try to construct D such that the automorphism defined by C is precisely α. In this paper we give the exact conditions under which this is possible provided α is the restriction of an automorphism of A. It turns out that in this case D is even unique (Proposition 3.2).

We prove this result by applying an idea of [19]. From that paper it follows that normalizing extensions of A_e are in principle classified by the normalizing elements of degree two in the Koszul dual $A_e^!$ of A_e. Therefore we start in §2 by giving a detailed description of $A_e^!$ (Proposition 2.2) which may have some independent interest. Once we know the structure of $A_e^!$ we can classify the normalizing elements in degree two corresponding to an automorphism coming from A (Proposition 2.7).

Combining our results with those of [17] we see that among the algebras we find are the four-dimensional Sklyanin algebras as well as examples of related algebras which are very likely of the kind that are studied in [19]. If the underlying three-dimensional algebra is linear (see [5]) we get examples of the algebras studied in [22]. These surject onto the twisted homogeneous coordinate ring of a quadric in $\mathbb{P}^3$.

To distinguish our algebras from those already in the literature we study there so called "point-variety" which is a certain geometric invariant introduced in [4]. In particular we find new classes of four-dimensional regular algebras that have no central element in degree one (see §5.4).

2. The two-Veronese of a three-dimensional regular algebra

Throughout this paper A will be a three dimensional two generator regular algebra (see the introduction). With A_e we denote the 2-Veronese $\oplus_i A_{2i}$ of A. Similarly $A_o = \oplus_i A_{2i+1}$. In order to avoid a proliferation of different gradings we will usually consider A_e, A_o to be $\mathbb{N}$-graded, with support respectively in $2\mathbb{N}$ and $2\mathbb{N}+1$.

Let $V = A_1$ and let $R \subset V^{\otimes 3}$ be the vector space of relations of A. Then the generators of A_e are given by $W = V \otimes V$ and the relations are given by $S = V \otimes R + R \otimes V$. In particular A_e is quadratic. In [3] it is proved that $T = V \otimes R \cap R \otimes V$ is one-dimensional so $\dim S = 7$.

We start with the following result :

Proposition 2.1. *A_e is Koszul.*

Proof. According to [3] we have the following resolution

$$0 \to A \otimes T \to A \otimes R \to A \otimes V \to A \to k \to 0 \tag{2.1}$$

where the maps are the natural ones.

We consider V, R, T as graded vector spaces, living purely in degrees $1, 3, 4$ respectively. In this way (2.1) becomes a graded exact sequence. It splits in an even and odd part :

$$0 \to A_e \otimes T \to A_o \otimes R \to A_o \otimes V \to A_e \to k \to 0 \tag{2.2}$$

$$0 \to A_o \otimes T \to A_e \otimes R \to A_e \otimes V \to A_o \to 0 \tag{2.3}$$

From (2.3) it is clear that A_o has a free A_e-resolution of the form

$$\cdots \to A_e \otimes R \otimes T \to A_e \otimes V \otimes T \to A_e \otimes R \to A_e \otimes V \to A_o \to 0 \tag{2.4}$$

So in particular $\operatorname{Tor}_i^{A_e}(k, A_o)$ lives purely in degree $2i+1$.

Let m_e be the graded maximal ideal of A_e. Then (2.2) splits into two short exact sequences :

$$0 \to A_e \otimes T \to A_o \otimes R \to Z \to 0 \tag{2.5}$$

$$0 \to Z \to A_o \otimes V \to m_e \to 0 \tag{2.6}$$

Applying the long exact sequence for $k \otimes_{A_e} -$ to (2.5) yields that $\operatorname{Tor}_i^{A_e}(k, Z)$ lives in degree $2i+4$, provided $k \otimes_{A_e} (A_e \otimes T) \to k \otimes_{A_e} (A_o \otimes R)$ is injective. Inspection reveals that this is just the canonical map $T = V \otimes R \cap R \otimes V \to V \otimes R$ which is by definition injective.

Looking at $k \otimes_{A_e} -$ for (2.6) reveals that $\operatorname{Tor}_i^{A_e}(k, m_e)$ lives in in degree $2i+2$ and hence $\operatorname{Tor}_i^{A_e}(k, A_e)$ lives in degree $2i$. So A_e is Koszul. $\square$

Now we give an explicit description of the Koszul dual of A_e. Consider the following vector spaces $B_n \subset V^{\otimes 2n}$

$$\begin{aligned}
B_0 &= k \\
B_1 &= V \otimes V \\
B_2 &= V \otimes R + R \otimes V \\
B_{2i+3} &= R \otimes T^{\otimes i} \otimes R + V \otimes T^{\otimes (i+1)} \otimes V; \quad i \geq 0 \\
B_{2i+4} &= R \otimes T^{\otimes (i+1)} \otimes V + V \otimes T^{\otimes (i+1)} \otimes R; \quad i \geq 0
\end{aligned} \tag{2.7}$$

A straightforward verification shows that

$$B_{n+m} \subset B_n \otimes B_m \tag{2.8}$$

Dualizing (2.8) defines an algebra structure on $B^* = \oplus_n B_n^*$ and one has the following result :

Proposition 2.2. *The identity map $W^* = (A_e^!)_2 \to B_1^* = W^*$ induces an isomorphism $A_e^! \to B^*$.*

Proof. By definition $A_e^! = k\langle W^*\rangle/(S^\perp)$. This yields a map $A_e^! \to B^*$ since $S^\perp = (V\otimes R + R\otimes V)^\perp$ is sent to zero in $B_2^* = (V\otimes R + R\otimes V)^*$. Furthermore since (2.8) is injective, B^* is generated by B_1^*. So B^* is a quotient of $A_e^!$. To show that they are isomorphic it suffices to show that they have the same Hilbert series.

We have $\dim A_{2n} = (n+1)^2$, so

$$H(A_e, t) = \sum_n (n+1)^2 t^n = \frac{1+t}{(1-t)^3}$$

Since A_e is Koszul $H_{A_e}(t)H_{A_e^!}(-t) = 1$, so

$$H(A_e^!, t) = \frac{(1+t)^3}{1-t} = 1 + 4t + 7t^2 + 8t^3 + 8t^4 + \cdots$$

On the other hand, according to Lemma 2.4 below the sums in the last two equations of (2.7) are actually direct sums. So this yields that the Hilbert function of B^* is also equal to $1, 4, 7, 8, 8, \ldots$. This finishes the proof. $\square$

We have used Lemma 2.4 below. To prove it we need the following preliminary result.

Lemma 2.3. $R\otimes R \cap V^{\otimes 2}\otimes R\otimes V = 0$

Proof. Consider the standard exact sequence

$$0 \to A\otimes T \to A\otimes R \to A\otimes V \to A \to k \to 0$$

Taking the degree 6 part yields that the natural map

$$V^{\otimes 2}\otimes (V\otimes R \cap R\otimes V) \to (V^{\otimes 3}/R)\otimes R$$

is injective. Therefore

$$R\otimes R\cap (V^{\otimes 2}\otimes R\otimes V) = R\otimes R\cap (V^{\otimes 2}\otimes R\otimes V\cap V^{\otimes 3}\otimes R) = 0 \quad \square$$

Lemma 2.4. *For $a\ge 0$ one has*

$$V\otimes T^{\otimes(a+1)}\otimes R\cap R\otimes T^{\otimes(a+1)}\otimes V = 0$$
$$R\otimes T^{\otimes a}\otimes R\cap V\otimes T^{\otimes(a+1)}\otimes V = 0$$

Proof. This follows from the foregoing lemma and the following facts

$$V\otimes T^{\otimes(a+1)}\otimes R \subset V^{\otimes b}\otimes R\otimes R$$
$$R\otimes T^{\otimes(a+1)}\otimes V \subset V^{\otimes b}\otimes V^{\otimes 2}\otimes R\otimes V$$
$$R\otimes T^{\otimes a}\otimes R \subset V^{\otimes c}\otimes R\otimes R$$
$$V\otimes T^{\otimes(a+1)}\otimes V \subset V^{\otimes c}\otimes V^{\otimes 2}\otimes R\otimes V$$

for appropriate $b, c\in\mathbb{N}$. $\square$

Remark 2.5. There is a different strategy for proving that A_e is Koszul. Assume for simplicity that A is elliptic (see [4]). Then there is a normalizing element $N\in A_4$ such that $B = A_e/(N)$ is a twisted homogeneous coordinate ring [6] associated to a triple $(X,\sigma,\mathcal{L})$ where X is a connected proper one-dimensional Cohen-Macaulay scheme with $\mathcal{O}_X = \omega_X$ and $\mathcal{L}$ has degree 4. If X is smooth then it is shown in [19] that B is Koszul. If this result could be extended to more general X (which is very likely possible) then we would obtain that A_e is Koszul. This is however not sufficient for our purposes since in the proof of Proposition 2.7 we need the explicit structure of $A_e^!$ as well.

The following proposition shows that A_e has some expected properties.

Proposition 2.6. *A_e is Auslander-Gorenstein and satisfies the Cohen-Macaulay property with respect to GK-dimension (for the definitions of these notions see* [10]*).*

Proof. The proof follows easily by combining some results of Yekutiely. We distinguish two cases (see [4] for definitions).

***A* elliptic** In this case there is a normalizing element in A_4 such that $B = A_e/(N)$ is a twisted homegenous coordinate ring associated to a triple $(X, \sigma, \mathcal{L})$ as in Remark 2.5 above. According to [25, lemma 2.6, Prop. 2.8] B satisfies the "generalized" A-G property with respect to the dualizing complex of B and the generalized Cohen-Macaulay property with respect to Krull dimension. Now from the formula [24, Def. 7.2] it follows that the dualizing complex of B is given by $\Gamma_*(\omega_X)$ and hence (since $\omega_X = \mathcal{O}_X$) is isomorphic to B itself as left and as right module (but not necessarily as bimodule). Thus in this case the generalized A-G property is the ordinary A-G property and similarly for the Cohen-Macaulay property.

Furthermore from the fact that X has dimension one and the category equivalence $B\text{-gr}/\text{fin} \cong X\text{-coh}$ [6] we easily obtain that $\text{Kdim} = \text{GKdim}$. So B is Auslander Gorenstein and Cohen-Macaulay. Using [10] this can be lifted to A_e. This finishes the proof in the elliptic case.

***A* linear** Now A_e is itself a twisted homegeneous coordinate ring associated to a triple $(X = \mathbb{P}^1 \times \mathbb{P}^1, \sigma, \mathcal{L})$ where $\mathcal{L}$ is of bidegree $(1,1)$. One has again that the dualizing complex of A is isomorphic to $\Gamma_*(\omega_X)$ and since $\omega_X = \mathcal{O}(-2,-2)$ we deduce that this is equal to $A_e(-4)$ as left and right module. Furthermore one verifies again that $\text{GKdim} = \text{Kdim}$. So the reasoning we applied above for B now yields directly that A_e has the required properties. □

Now we give a description of a certain subset of the normalizing elements of $A_e^!$. Let α be an automorphism of A_e and let $\alpha^!$ be corresponding automorphism of $A_e^!$. Take $\phi \in (A_e^!)_4^* = S^*$. Then ϕ is a normalizing element in $A_e^!$ corresponding to $\alpha^!$ if

$$\phi\theta = \alpha^!(\theta)\phi \tag{2.9}$$

for all $\theta \in W^*$.

We are only able to describe in full generality the normalizing elements in $(A_e^!)_4$ corresponding to automorphisms of A_e obtained by restriction from automorphisms of A. Somewhat surprisingly, if α is of this form and if ϕ exists then ϕ is uniquely determined up to scalar multiples.

To state the following proposition and its proof we recall some notations and results from [3]. We can choose generators $x_1, x_2 \in A_1$ and defining relations f_1, f_2 of degree 3

$$f = Mx$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

such that

$$x^t M = (Qf)^t$$

for some $Q \in \text{GL}(2,k)$. The one-dimensional vector space T is generated by the element $w = x^t f = x^t M x = (Qf)^t x \in A_1^{\otimes 4}$.

Proposition 2.7. *Let α be an automorphism of A, coming from an automorphism of A induced by $x \mapsto \Lambda x$ with $\Lambda \in \mathrm{GL}(2,k)$. Then there exists a $\phi \in S^*$ which is normalizing in $A_e^!$ with automorphism $\alpha^!$ if and only if the following two conditions are satisfied*

$$\begin{cases} w \text{ is invariant under } \Lambda \\ \det Q = \det \Lambda \text{ or } \operatorname{tr}(\Lambda Q^{-t}) = 0 \end{cases} \tag{2.10}$$

In this case ϕ is unique, up to a scalar multiple, and is given by

$$\begin{aligned} \phi(f_i x_j) &= \eta(\Lambda^{-1})_{ji} \\ \phi(x_i f_j) &= \eta(\Lambda Q^{-t})_{ij} \end{aligned} \tag{2.11}$$

where $\eta \in k^*$ is arbitrary. Such a ϕ is automatically a non-zero divisor in $A_e^!$.

Proof. One has $\alpha^!(\theta) = \theta \circ \alpha$ so (2.9) is equivalent to $\phi\theta - (\theta \circ \alpha)\phi = 0$. Now

$$(A_e^!)_6 = W^{\otimes 3}/(W \otimes S \cap S \otimes W)^\perp$$

So the condition for ϕ to be normalizing with automorphism $\alpha^!$ is

$$(\phi \otimes \theta - (\theta \circ \alpha) \otimes \phi) \mid W \otimes S \cap S \otimes W = 0$$

Since by Proposition 2.2 $B_3 = (A_e^!)_6^* = W \otimes S \cap S \otimes W$ we obtain from the definition of B_3 and Lemma 2.4

$$W \otimes S \cap S \otimes W = R \otimes R \oplus V \otimes T \otimes V$$

It follows that $\phi \in S^*$ is normalizing with corresponding automorphism $\alpha^!$ if for all $\theta \in W^*$

$$(\phi \otimes \theta - (\theta \circ \alpha) \otimes \phi) \mid R \otimes R = 0 \tag{2.12}$$

$$(\phi \otimes \theta - (\theta \circ \alpha) \otimes \phi) \mid V \otimes T \otimes V = 0 \tag{2.13}$$

We have $f_i = \sum_j m_{ij} x_j$ and also $f_k = \sum_{j,l} Q_{kl}^{-1} x_j m_{jl}$, as $x^t M = (Qf)^t$. So the elements of $R \otimes R$ can be written as linear combinations of

$$f_i f_k = \sum_j m_{ij} x_j f_k = \sum_{j,l} Q_{kl}^{-1} f_i x_j m_{jl}$$

Therefore,

$$(\phi \otimes \theta)(f_i f_k) = \sum_{j,l} Q_{kl}^{-1} \phi(f_i x_j) \theta(m_{jl})$$

which is the (ik)'th entry of the matrix $BY(Q^t)^{-1}$ with

$$Y = \begin{pmatrix} \theta(m_{11}) & \theta(m_{12}) \\ \theta(m_{21}) & \theta(m_{22}) \end{pmatrix}, \quad B = \begin{pmatrix} \phi(f_1 x_1) & \phi(f_1 x_2) \\ \phi(f_2 x_1) & \phi(f_2 x_2) \end{pmatrix}$$

On the other hand,

$$((\theta \circ \alpha) \otimes \phi)(f_i f_k) = \sum_j (\theta \circ \alpha)(m_{ij}) \phi(x_j f_k)$$

which is the (ik)'th entry of the matrix $Y'B'$ where

$$Y' = \begin{pmatrix} (\theta \circ \alpha)(m_{11}) & (\theta \circ \alpha)(m_{12}) \\ (\theta \circ \alpha)(m_{21}) & (\theta \circ \alpha)(m_{22}) \end{pmatrix}, \quad B' = \begin{pmatrix} \phi(x_1 f_1) & \phi(x_1 f_2) \\ \phi(x_2 f_1) & \phi(x_2 f_2) \end{pmatrix}$$

So (2.12) translates into (2.14) below.

To analyze (2.13) we note that $V \otimes T \otimes V$ has generators

$$x_i w x_k = \sum_j x_i x_j f_j x_k = \sum_{j,l} Q_{jl} x_i f_l x_j x_k$$

This gives

$$(\phi \otimes \theta)(x_i w x_k) = \sum_{j,l} Q_{jl} \phi(x_i f_l) \theta(x_j x_k)$$

which is the (ik)'th entry of the matrix $B' Q^t X$ with

$$X = \begin{pmatrix} \theta(x_1^2) & \theta(x_1 x_2) \\ \theta(x_2 x_1) & \theta(x_2^2) \end{pmatrix}$$

Similarly

$$((\theta \circ \alpha) \otimes \phi)(x_i w x_k) = \sum_j (\theta \circ \alpha)(x_i x_j) \phi(f_j x_k)$$

is the (ik)'th entry of the matrix $X' B$, where

$$X' = \begin{pmatrix} (\theta \circ \alpha)(x_1^2) & (\theta \circ \alpha)(x_1 x_2) \\ (\theta \circ \alpha)(x_2 x_1) & (\theta \circ \alpha)(x_2^2) \end{pmatrix}$$

This shows that (2.12) and (2.13) can be written as

$$BY = Y' B' Q^t \tag{2.14}$$

$$X' B = B' Q^t X \tag{2.15}$$

for all $\theta \in W^*$.

Now obviously under α

$$x x^t \mapsto \Lambda x x^t \Lambda^t$$

On the other hand, we know from Proposition (2.4) of [3] that A determines w up to scalar multiple. As α is an automorphism of A, this means that $w \mapsto \mu w$ for some $\mu \in k^*$. Therefore

$$M \mapsto \mu \Lambda^{-t} M \Lambda^{-1}$$

This implies $X' = \Lambda X \Lambda^t$ and $Y' = \mu \Lambda^{-t} Y \Lambda^{-1}$ for some $\mu \in k^*$, and (2.12) and (2.13) are thus equivalent to

$$BY = \mu \Lambda^{-t} Y \Lambda^{-1} B' Q^t \tag{2.16}$$

$$B' Q^t X = \Lambda X \Lambda^t B \tag{2.17}$$

Taking X equal to the elementary matrices E_{ij}, (2.17) reduces to

$$\Lambda^t B = \Lambda^{-1} B' Q^t = \eta I$$

for some $\eta \in k^*$ or,

$$B = \eta \Lambda^{-t} \tag{2.18}$$

$$B' = \eta \Lambda Q^{-t} \tag{2.19}$$

This reduces condition (2.16) to $\mu = 1$, which is equivalent to requesting that w should be invariant under Λ.

Now, $\phi \in S^*$ has to be compatible with $x^t M x = (Qf)^t x$, that is $\sum_j \phi(x_j f_j) = \sum_{j,l} Q_{jl}\phi(f_l x_j)$. This yields the extra condition $\operatorname{tr}(B') = \operatorname{tr}(BQ)$, or $\operatorname{tr}(\Lambda Q^{-t}) = \operatorname{tr}(\Lambda^{-t} Q)$. A trite computation shows that this is equivalent to

$$\det Q = \det \Lambda \qquad \text{or} \qquad \operatorname{tr}(\Lambda Q^{-t}) = 0$$

On the other hand, if Λ satisfies the two stated requirements, $\phi \in S^*$ exists and is uniquely determined (up to scalar multiple) by (2.18) and (2.19). Substituting the definitions of B and B' one obtains (2.11).

The only thing that remains to be shown is that ϕ is a non-zero divisor. This follows easily from the explicit description of both $A_e^!$ and ϕ, as we show below.

The normalizing element ϕ will be a (left) non-zero divisor in $A_e^!$ if $(A_e^!)_{2i} \to (A_e^!)_{2i+4} : a \mapsto \phi a$ is injective or, dually, if

$$c : B_{i+2} \to B_2 \otimes B_i \overset{\phi\otimes 1}{\to} B_i$$

is surjective. The surjectivity of c follows from the fact that c is build up from maps of the following nature

$$\begin{array}{llll} c: & R\otimes T^{\otimes(t+1)}\otimes V \to R\otimes T^{\otimes t}\otimes V: & f_i w^{t+1} x_k \mapsto \sum_j \phi(f_i x_j) f_j w^t x_k \\ c: & V\otimes T^{\otimes(t+1)}\otimes R \to V\otimes T^{\otimes t}\otimes R: & x_i w^{t+1} f_k \mapsto \sum_{j,l} Q_{jl}\phi(x_i f_l) x_j w^t f_k \\ c: & R\otimes T^{\otimes(t+1)}\otimes R \to R\otimes T^{\otimes t}\otimes R: & f_i w^{t+1} f_k \mapsto \sum_j \phi(f_i x_j) f_j w^t f_k \\ c: & V\otimes T^{\otimes(t+1)}\otimes V \to V\otimes T^{\otimes t}\otimes V: & x_i w^{t+1} x_k \mapsto \sum_{j,l} Q_{jl}\phi(x_i f_l) x_j w^t x_k \end{array}$$

The explicit form of ϕ given by (2.11) shows that all these maps are bijections, so in particular surjective. □

3. Normalizing extensions

In this section we look for D which are Koszul and have a regular normalizing element $C \in D_2$ such that $A_e = D/(C)$. Such a D will be called a *normalizing* extension of A_e. We recall the following.

Proposition 3.1. *A normalizing extension D of A_e is Auslander regular of global dimension 4 and Cohen-Macaulay with respect to GK-dimension. In particular D satisfies the following properties.*

1. *A is a domain.*
2. *A is a maximal order.*
3. *A is Artin-Schelter regular.*

Proof. By Proposition 2.6 A_e is Auslander Gorenstein and Cohen-Macaulay. These conditions lift to D by [10]. Since D is Koszul and has Hilbert series $1/(1-t)^4$ we deduce that D has global dimension four. (1)(3) follow from [10] and (2) follows from [18]. □

Our main result is the following

Proposition 3.2. *Assume that α is an automorphism of A. Then there exists a normalizing extension D of A_e such that the automorphism of A_e induced by conjugation with the normalizing element $C \in D_2$ is given by α^{-1} if and only if α satisfies the condition* (2.10). *If this condition holds then D is unique.*

Proof. We have $A_e = k\langle W\rangle/(S)$, $D = k\langle W\rangle/(S')$ where $S' = \ker\phi$ for some $\phi \in S^*$. Dualizing yields that $D^! = A_e^!/(\phi)$. By [19, lemma 2.5] is ϕ is again a normalizing element and inspecting Hilbert series reveals that ϕ is regular.

Conversely let ϕ be a regular normalizing element in $(A_e^!)_2$. Then by a generalization of [7] and Proposition 2.1 $P = A_e^!/(\phi)$ is Koszul. Put $D = P^!$. There is a surjection $D \to A_e$ and applying again [19, lemma 2.5] together with Hilbert series yields that there is a regular normalizing element $C \in D_2$ such that $A_e = D/(C)$.

We can now use the classification of regular normalizing elements given by Proposition 2.7. To conclude is suffices to observe that if (for α arbitrary) ϕ induces an automorphism $\alpha^!$ then C induces α^{-1}.

To show this we slightly adapt the proof of [17]. Let $v \in W$. As C is normalizing in D, $Cv = v'C$ for some $v' \in D_1$. Therefore, $C \otimes v - v' \otimes C \in S' \otimes W + W \otimes S'$, or

$$C \otimes v - v' \otimes C = \lambda + \rho$$

with $\lambda \in S' \otimes W$ and $\rho \in W \otimes S'$. As $C \in S$, $C \otimes v \in S \otimes W$ and $v' \otimes C \in W \otimes S$. So we have $C \otimes v - \lambda \in S \otimes W$ and $v' \otimes C + \rho \in W \otimes S$. And as a result

$$C \otimes v - \lambda = v' \otimes C + \rho \in S \otimes W \cap W \otimes S$$

Since $\phi \in S^*$ is normalizing in $A_e^!$ with automorphism $\alpha^!$,

$$\begin{aligned} 0 &= (\phi \otimes \theta - (\theta \circ \alpha) \otimes \phi)(C \otimes v - \lambda) \\ &= (\phi \otimes \theta)(C \otimes v) - (\phi \otimes \theta)(\lambda) - ((\theta \circ \alpha) \otimes \phi)(C \otimes v - \lambda) \\ &= \phi(C) \otimes \theta(v) - ((\theta \circ \alpha) \otimes \phi)(v' \otimes C + \rho) \\ &= \phi(C) \otimes \theta(v) - ((\theta \circ \alpha) \otimes \phi)(v' \otimes C) + ((\theta \circ \alpha) \otimes \phi)(\rho) \\ &= \phi(C) \otimes \theta(v) - (\theta \circ \alpha)(v') \otimes \phi(C) \end{aligned}$$

Since $\phi(C) \neq 0$, $\theta(v) = (\theta \circ \alpha)(v')$, and this for all $\theta \in W^*$. Therefore, $v' = \alpha^{-1}(v)$. So $Cv - \alpha^{-1}(v)C$ for all $v \in D_1$, that is C is normalizing with automorphism α^{-1}. $\square$

Remark 3.3. It follows from [3, (2.12)] that Q^t itself defines an automorphism of A, preserving w. So putting $\Lambda = Q^t$ yields a canonical solution to (2.10). Hence there is always at least one normalizing extension of A_e.

If D is a normalizing extension of A_e by some Λ satisfying (2.10) we write $D = [A, \Lambda]$. For completeness we indicate here how one finds the relations of D. The relations of A_e are given by $(f_i x_j)_{ij}$, $(x_i f_j)_{ij}$. Put $X_1 = x_1^2$, $X_2 = x_1 x_2$, $X_3 = x_2 x_1$, $X_4 = x_2^2$ and denote by X, M the matrices obtained from xx^t and $(m_{ij})_{ij}$ by replacing $x_1^2, \ldots, x_2^2$ by $X_1, \ldots, X_4$. Since $fx^t = Mxx^t$, $xf^t = xx^t MQ^{-t}$ we find that the relations of A_e are given by the entries of MX, XM.

The relations of D are given by $\ker \phi$ were is ϕ is given by (2.11). This yields

$$\begin{aligned} \phi(MX) &= \Lambda^{-t} \\ \phi(XM) &= \Lambda \end{aligned} \tag{3.1}$$

(we have dropped the scalar η). The equations (3.1) make it easy to write down explicitly the relations of D.

4. Classification results

In this section we assume that k has characteristic zero.

As in the previous sections, A always denotes a three-dimensional Artin-Schelter regular algebra on two generators. Given A, we want to determine all the normalizing extensions of the Veronese A_e induced by some $\Lambda \in \mathrm{GL}(2, k)$ following Proposition 3.2. We determine them up to "equivalence". Suppose that $D_1 = [A, \Lambda_1]$ and

$D_2 = [A, \Lambda_2]$. Denote the corresponding normalizing elements of $A_e^!$ by respectively ϕ_1, ϕ_2, and the automorphisms of A by α_1, α_2. Then D_1 and D_2 are called *equivalent* if for some automorphism ψ of A, $\alpha_2 = \pm\psi^{-1}\alpha_1\psi$. Or equivalently (since $\pm\alpha_i$ and ϕ_i determine each other up to scalar multiple), $\phi_2 = \eta(\phi_1 \circ \psi)$, $\eta \in k^*$. We write $D_1 \sim D_2$. Clearly equivalence is stronger than isomorphism.

We start from the classification of the regular three-dimensional two-generator algebras of [3]. To every such regular algebra A we can associate a couple $(Q, w) \in \mathrm{GL}(2,k) \times V^{\otimes 4}$ where w is an element of a subspace W_Q of $V^{\otimes 4}$. (Table 4.1 recalls [3, Table (3.6)], listing the basis elements of W_Q for Q diagonal. The last column gives the condition on w in order to be in W_Q.) Change of the generators $\{x_1, x_2\} \subset R$ and relations $\{f_1, f_2\} \subset V$ induces an action of $\mathrm{GL}(2,k)$ on $\mathrm{GL}(2,k) \times V^{\otimes 4}$. The isomorphism class of A is then classified by the $\mathrm{GL}(2,k)$-orbit of (Q, w) under this action.

A regular algebra A is called *generic* if it corresponds to a generic point of the closed subset of $\mathrm{GL}(2,k) \times V^{\otimes 4}$ consisting of pairs (Q, w) where $w \in W_Q$. For such a generic point one can use the $\mathrm{GL}(2,k)$-action to put Q into diagonal form and then use the action of the stabilizer of Q on W_Q to put w into the standard form of [3, Table (3.9)] (which we recall in Table 4.2). Every generic algebra thus corresponds to some w listed in this table. The defining relations f can be deduced using $w = x^t M x = x^t f$. (Observe that ζ_n always denotes a primitive n^{th} root of unity.)

More generally, we will say that A is of *generic form* if the associated element (Q, w) can be put into this standard form (including the specializations for which this is possible).

w	Condition
$w_0 = x_2^4$	$\alpha_2 = 1$
$w_1 = x_2^3 x_1 + \alpha_2 x_2^2 x_1 x_2 + \alpha_2^2 x_2 x_1 x_2^2 + \alpha_2^3 x_1 x_2^3$	$\alpha_1 \alpha_2^3 = 1$
$w_2 = x_1^2 x_2^2 + \alpha_1 x_1 x_2^2 x_1 + \alpha_1^2 x_2^2 x_1^2 + \alpha_1^2 \alpha_2 x_2 x_1^2 x_2$	$\alpha_1^2 \alpha_2^2 = 1$
$w_2' = x_1 x_2 x_1 x_2 + \alpha_1 x_2 x_1 x_2 x_1$	$\alpha_1 \alpha_2 = 1$
$w_3 = x_1^3 x_2 + \alpha_1 x_1^2 x_2 x_1 + \alpha_1^2 x_1 x_2 x_1^2 + \alpha_1^3 x_2 x_1^3$	$\alpha_1^3 \alpha_2 = 1$
$w_4 = x_1^4$	$\alpha_1 = 1$

TABLE 4.1. Basis of W_Q, $Q = \mathrm{diag}(\alpha_1, \alpha_2)$

Type	Q	w
A	$(1,1)$	$w_0 + a w_2 + b w_2' + w_4$
E	$(1, \zeta_3)$	$w_1 + w_4$
H	$(\zeta_8, -\zeta_8)$	$w_1 + w_3$
S_1	(α, α^{-1})	$w_2 + a w_2'$
S_2	$(\alpha, -\alpha^{-1})$	w_2
S_2'	$(1, -1)$	$w_2 + w_4$

TABLE 4.2. Generic forms

For the rest of this section we assume that A is of generic form, i.e. A corresponds to some (Q, w) in Table 4.2.

Proposition 4.1. *If A is of generic form, the normalizing extensions of the Veronese A_e correspond to $\pm\Lambda$ for some $\Lambda \in \mathrm{GL}(2,k)$ listed in the Tables 4.3 and 4.4.*

Proof. The regular algebras of generic form are listed in Table 4.2. It now suffices to verify for each type which $\Lambda \in \mathrm{GL}(2,k)$ satisfy the two requirements (2.10). These are all straightforward computations. □

In the Tables 4.3 and 4.4, the column named "Λ" gives which $\pm\Lambda$ allow a normalizing extension (where Λ and $-\Lambda$ induce the same extension, as they determine the same automorphism of A_e). The column named "Dim" gives the dimension the family of normalizing extensions of A_e corresponding to Λ up to equivalence (see Proposition 4.2).

Type A	Λ	Dim
A	$(1,1)$	2
	$(1,-1)$	2
	$(i,-i)$	2
E	$(1,\zeta_3)$	0
H	$(\zeta_8,-\zeta_8)$	0
	$(1,1)$	0
	(ζ_8^2,ζ_8^2)	0
S_1	$(p,p^{-1}),\ p\in k^*$	3
	$(\alpha,-\alpha^{-1})$	2
S_2	$(p,-p^{-1}),\ p\in k^*$	2
	(α,α^{-1})	1
S_2'	$(1,-1)$	0
	$(1,1)$	0
	(i,i)	0

TABLE 4.3. Diagonal extensions of the generic forms

Type A	Condition on A	Λ	Condition on Λ	Dim
A		$\begin{pmatrix} 0 & q \\ \pm q & 0 \end{pmatrix}$	$q^4=1$	2
	$2a+b=\delta=\pm 1$	$\begin{pmatrix} p & q \\ -\delta q & p \end{pmatrix}$	$p^2+\delta q^2=1, p\neq 0$	2
		$\begin{pmatrix} p & q \\ \delta q & -p \end{pmatrix}$	$(p^2+\delta q^2)^2=1, p\neq 0$	2
S_1	$\alpha=1, a\neq -2$	$\begin{pmatrix} 0 & q \\ \pm q^{-1} & 0 \end{pmatrix}$	$q\in k^*$	1
	$\alpha^2=1, a=-1-\alpha$		$\det\Lambda=1$	1
			$\det\Lambda=-1, \operatorname{tr}\Lambda=0$	0
S_2	$\alpha^2=-1$	$\begin{pmatrix} 0 & q \\ \pm iq^{-1} & 0 \end{pmatrix}$	$q\in k^*$	0

TABLE 4.4. Non-diagonal extensions of the generic forms

If A is a specialization of generic form then extensions of A_e might be equivalent.

Proposition 4.2. *For generic A, two different extensions of A_e are never equivalent. For specializations of generic form, equivalences do occur, they are listed in Table 4.5.*

Proof. Given two extensions $D_1 = [A, \Lambda_1]$ and $D_2 = [A, \Lambda_2]$ then in order to be equivalent, Λ_1 and Λ_2 have to be in the same conjugacy class. If so, then D_1 and D_2 are actually equivalent if among the matrices $P \in \mathrm{GL}(2, k)$ for which $\Lambda_2 = P^{-1}\Lambda_1 P$, there is a P that defines an automorphism of A. $\square$

Type A	Condition on A	Equivalences
A	$2a+b=1$ or $a=b=\pm\frac{i}{\sqrt{3}}$	$\left[A, \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}\right] \sim \left[A, \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix}\right]$, $q^4 = 1$
	$2a+b=-1$ or $a=b=\pm\frac{i}{\sqrt{3}}$	$\left[A, \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}\right] \sim \left[A, \begin{pmatrix} iq & 0 \\ 0 & -iq \end{pmatrix}\right]$, $q^4 = 1$
	$2a+b=\delta=\pm 1$	$\left[A, \begin{pmatrix} p & q \\ \delta q & -p \end{pmatrix}\right] \sim \left[A, \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}\right]$, $\epsilon^2 = p^2 + \delta q^2 = 1,\ p \in k^*$
S_1	$\alpha = 1$ or $\alpha = -1, a = 0$	$[S_1, (p, p^{-1})] \sim [S_1, (p^{-1}, p)],\ p \in k^*$
	$\alpha = 1, a \neq -2$	$\left[S_1, \begin{pmatrix} 0 & q \\ \pm q^{-1} & 0 \end{pmatrix}\right] \sim \left[S_1, \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}\right]$, $q \in k^*$
	$\alpha^2 = 1, a = -1-\alpha$	$[S_1, \Lambda] \sim [S_1, J]$ if $\det\Lambda = 1$, then $J = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$, $p \in k^*$ or $J = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$, if $\det\Lambda = -1, \operatorname{tr}\Lambda = 0$, then $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
S_2	$\alpha^2 = -1$	$[S_2, (p, -p^{-1})] \sim [S_2, (-p^{-1}, p)],\ p \in k^*$
	$\alpha^2 = -1$	$\left[S_2, \begin{pmatrix} 0 & q \\ \pm iq^{-1} & 0 \end{pmatrix}\right] \sim \left[S_2(\alpha), \begin{pmatrix} 0 & 1 \\ \pm i & 0 \end{pmatrix}\right]$, $q \in k^*$

TABLE 4.5. Equivalences among $[A, \Lambda]$ with A of generic form

Example 4.3. The 3-dimensional algebra of type S_1 with $\alpha = 1$ and $a = -2$ is the Heisenberg algebra which has generators x, y subject to the relations

$$\begin{cases} yx^2 - 2xyx + x^2y = 0 \\ y^2x - 2yxy + xy^2 = 0 \end{cases} \tag{4.1}$$

So up to equivalence we obtain three isolated 4-dimensional regular algebras and one 1-parameter family surjecting onto the Veronese of this algebra.

Remark 4.4. Among the normalizing extensions that we obtain, we recover families of algebras that were already studied.

1. Some of them have a central (or normalizing) degree 1 element and can thus be treated as (twists of) central extensions in the sense of [8]. This is not always true. We come back to this issue in Section 5.4.
2. The normalizing extensions of type $[S_1(a, \alpha), (p, p^{-1})]$ form a 3-parameter family. This family has (up to isomorphism) a 2-parameter overlap with Le Bruyn's homogenized conformal $\mathfrak{sl}_2$-enveloping algebras $H_{ABC}(\mathfrak{sl}_2)$, which are the regular deformations of homogenized $\mathfrak{sl}_2$ [9].

3. It follows from [17] that the extensions of type $[A,(1,1)]$ are isomorphic to four-dimensional Sklyanin algebras with geometric data related to that of A. The other extensions of type A are very likely special cases of the generalized Sklyanin algebras that were studied by Stafford in [19]. (If Stafford's conjecture 2.4 in [19] is true then they are.)
4. If A is linear (see [5] and Section 5 of this paper), the extensions of A_e are examples of the algebras that were classified in [22].

5. The point variety

5.1. **Generalities.** Following Artin and Zhang [1][2], we associate to a graded algebra A its quantum space $\operatorname{Proj} A$, which is the quotient category of the category of all finitely generated graded left A-modules modulo the subcategory of finite length modules. Suppose that $M = \oplus_i M_i$ is a $\mathbb{Z}$-graded left A-module, then taking the degree 2 part (i.e. put $M_e = \oplus_i M_{2i}$) yields a graded left A_e-module. Conversely, for N a graded left A_e-module, $A \otimes_{A_e} N$ is a graded left A-module. Under suitable hypotheses this determines an equivalence between $\operatorname{Proj} A$ and $\operatorname{Proj} A_e$, see [23].

Important objects in $\operatorname{Proj} A$ are the linear modules, the simplest of which are the point modules. If M be a finitely generated graded left R-module then we say that M is a point module if $M = \oplus_{i\geq 0} M_i$ is cyclic and $\dim M_i = 1$ for all $i \geq 0$. Point modules of 3-dimensional regular algebras were completely classified in [4] and [5]. In this section we study the point modules of a normalizing extension $D = [A, \Lambda]$ (as defined in §3). Among other things this provides us with a geometric invariant which is useful to distinguish our algebras from various other four-dimensional algebras in the literature.

Firstly, a point module of A_e obviously defines a point module of D and conversely, point modules of D which are killed by the normalizing element C determine point modules of A_e. Secondly, the point modules of A_e are completely determined since they are the same as the point modules of the 3-dimensional regular algebra A (up to a few exceptions due to the possible existence of a so-called special point for A_e). So our problem reduces to examining what the "extra" point modules are.

Let us start with recalling some facts from [4] about the point modules of A. Let $\Gamma_A \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the locus of common zeros of $\tilde{f}_1$ and $\tilde{f}_2$ (the multilinearization of the defining relations of A). Write $\mathcal{P}_A = \operatorname{pr}_{12}(\Gamma_A)(= \operatorname{pr}_{23}(\Gamma_A)) \subset \mathbb{P}^1 \times \mathbb{P}^1$, which is the locus of zeros of $\det \tilde{M}$ with $M = (m_{ij})$. If $\det \tilde{M}$ is not identically zero, $\mathcal{P}_A$ is a curve of bidegree $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ (we say that A is elliptic). Otherwise it is all of $\mathbb{P}^1 \times \mathbb{P}^1$ (we say that A is linear). Γ_A defines an automorphism of $\mathcal{P}_A$ of the form $\sigma_A(p,q) = (p, f(q))$. The point modules of A are in one to one correspondence with the points of $\mathcal{P}_A$. We call $(\mathcal{P}_A, \sigma_A)$ the *point variety* of A.

The Plucker embedding identifies $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{V}(X_1X_4 - X_2X_3) \subset \mathbb{P}^3$. Hence

$$\mathcal{P}_A = \mathcal{V}(\det M) \cap \mathcal{V}(X_1X_4 - X_2X_3)$$

Let $\Gamma_{A_e} \subset \mathbb{P}^3 \times \mathbb{P}^3$ be the locus of common zeros of the multilinearizations of the quadratic relations of A_e.

By §3, the defining relations of Γ_{A_e} can be written as follows

$$M^{(1)}X^{(2)} = X^{(1)}M^{(2)} = 0 \tag{5.1}$$

By $\mathcal{P}_{A_e}$ we denote $\operatorname{pr}_1(\Gamma_{A_e})$.

Definition 5.1.1. A point $p \in \mathbb{P}^3$ is *special* if $M(p) = 0$.

Lemma 5.1.2. *There is at most one special point (even scheme theoretically). Such as special point does not lie on the quadric $\mathcal{V}(X_1X_4 - X_2X_3)$.*

Proof. As A is regular, $\operatorname{rk} M(p) \geq 1$ for $p \in \mathbb{P}^1 \times \mathbb{P}^1 = \mathcal{V}(X_1X_4 - X_2X_3)$. So if the (linear) variety of special points would be bigger than one point it would contain at least a line and this line would intersect $\mathcal{V}(X_1X_4 - X_2X_3)$ in a point which would also be special. This yields a contradiction. □

Lemma 5.1.3. *Denote by $\Gamma_{\mathcal{P}_A,\sigma_A^2} \subset \mathbb{P}^3 \times \mathbb{P}^3$ the graph of the automorphism σ_A^2 on $\mathcal{P}_A \subset \mathbb{P}^3$. Define*

$$\Gamma_{\mathrm{sp}} = \begin{cases} (p,p) & \text{if } A_e \text{ has a special point } p \\ \emptyset & \text{otherwise} \end{cases}$$

Then scheme theoretically

$$\Gamma_{A_e} = \Gamma_{\mathcal{P}_A,\sigma_A^2} \sqcup \Gamma_{\mathrm{sp}}$$

In particular $\mathcal{P}_{A_e}$ is the disjoint union of $\mathcal{P}_A$ (embedded in $\mathbb{P}^3$ through the Segre embedding) and possibly one extra special point.

Proof. The equations for Γ_{A_e} are given by (5.1). Since $\det X = 0$ implies $\operatorname{rk} M \geq 1$ and thus $M \neq 0$ (see [4]) we find that Γ_{A_e} has an open covering given by open sets $\{M^{(1)} \neq 0, M^{(2)} \neq 0\}^{(\mathrm{a})}$, $\{M^{(1)} \neq 0, \det X^{(2)} \neq 0\}^{(\mathrm{b})}$, $\{\det X^{(1)} \neq 0, M^{(2)} \neq 0\}^{(\mathrm{c})}$, $\{\det X^{(1)} \neq 0, \det X^{(2)} \neq 0\}^{(\mathrm{d})}$. We discuss these separately

(a) In this case $\det M^{(i)} = \det X^{(i)} = 0$ for $i = 1, 2$.
(b) It is clear from the equations of Γ_{A_e} that $\det X^{(2)} \neq 0$ implies $M^{(1)} = 0$. So this open set is empty.
(c) This is similar to (b).
(d) Now we obtain $M^{(i)} = 0$ for $i = 1, 2$.

So we obtain that scheme theoretically

$$\Gamma_{A_e} = \left(\Gamma_{A_e} \cap \{\det X^{(1)} = \det X^{(2)} = 0\}\right) \sqcup \Gamma_{\mathrm{sp}}$$

We now have to show that

$$\Gamma_{\mathcal{P}_A,\sigma_A^2} = \Gamma_{A_e} \cap \{\det X^{(1)} = \det X^{(2)} = 0\} \tag{5.2}$$

To analyze the righthand side of (5.2) we pull it back under the Segre embedding. This yields a closed subscheme of $(\mathbb{P}^1)^4$ with equations

$$\begin{cases} M^{(1,2)}x^{(3)}x^{(4)t} = 0 \\ x^{(1)}x^{(2)t}M^{(3,4)} = 0 \end{cases}$$

These equations are equivalent to

$$\begin{cases} M^{(1,2)}x^{(3)} = 0 \\ x^{(2)t}M^{(3,4)} = 0 \end{cases}$$

and hence they define $(\Gamma_A \times \mathbb{P}^1) \cap (\mathbb{P}^1 \times \Gamma_A) \cong \Gamma_{\mathcal{P}_A,\sigma_A^2}$ □

5.2. **The point variety of D.** We turn to the point variety of D. Write $\Gamma_D \subset \mathbb{P}^3 \times \mathbb{P}^3$ for the scheme defined the locus of common zeros of the multilinearizations of the quadratic relations of D. We will show that Γ_D is the graph of an automorphism σ_D of $\mathcal{P}_D$. As a result point modules of D correspond to points in $\mathcal{P}_D$ (see Corollary 5.2.4 below).

Lemma 5.2.1. *Suppose that $(p, p') \in \mathbb{P}^3 \times \mathbb{P}^3$. Then $(p, p') \in \Gamma_D$ if and only if for some $\epsilon \in k$,*

$$\begin{cases} M(p)X(p') = \epsilon\Lambda^{-t} \\ X(p)M(p') = \epsilon\Lambda \end{cases} \tag{5.3}$$

And $(p, p') \in \Gamma_{A_e}$ for $\epsilon = 0$.

Proof. Let S' be the space of quadratic relations of D. Evaluating in (p, p') defines (up to scalar multiple) an element of S^* which vanishes on S'. Since $S' = \ker \phi$ this implies that for $g \in S$ we have $g(p, p') = \epsilon\phi$, $\epsilon \in k$. (5.3) now follows from (3.1). We can invert this reasoning to show that if (p, p') satisfies (5.3) then (p, p') lies in Γ_{A_e}. □

Proposition 5.2.2. *Let P_* denote the reduced part of $\mathcal{P}_*$ for $* = A, A_e, D$.*

1. *$\mathcal{P}_D = \mathrm{pr}_1(\Gamma_D) = \mathrm{pr}_2(\Gamma_D)$ and Γ_D is the graph of an automorphism σ_D of $\mathcal{P}_D$.*
2. *$P_D \cap \mathcal{V}(\det M) = P_{A_e}$ and $P_D \cap \mathcal{V}(X_1X_4 - X_2X_3) = P_A$.*
3. *On $P_D \cap \mathcal{V}(X_1X_4 - X_2X_3)$, σ_D restricts to σ_A^2. On $P_D \cap \mathcal{V}(X_1X_4 - X_2X_3)^c$, σ_D is determined by $X \mapsto [\Lambda^t M]^{-1}$ except if there exists a special point $p_{\mathrm{sp}} \in \mathbb{P}^3$ then for this point $\sigma_D(p_{\mathrm{sp}}) = p_{\mathrm{sp}}$.*
4. *$(\mathcal{P}_D)_{\mathrm{red}} = (\mathcal{P}_A)_{\mathrm{red}} \cup \mathcal{V}(q_1, q_2, q_3)_{\mathrm{red}} \cup \{p_{\mathrm{sp}}\}$ with q_i quadratic in the X_j's.*

Proof. Suppose that $p \in P_D$ that is, $(p, p') \in \Gamma_D$ for some $p' \in \mathbb{P}^3$. If $p \in P_D \cap \mathcal{V}(\det M)$ then, using the previous lemma, we see that $\epsilon = 0$, since $\Lambda \in \mathrm{GL}(2, k)$. Therefore $(p, p') \in \Gamma_{A_e}$. By Lemma 5.1.3, this implies that either p and p' are in P_A or else $p = p'$ is special. In the first case it follows that $p' = \sigma_A^2(p)$ is the unique point in $\mathbb{P}^3$ for which $(p, p') \in \Gamma_D$. Suppose on the other hand that $p \notin \mathcal{V}(\det M)$ The previous lemma yields that,

$$X(p') = \epsilon[\Lambda^t M(p)]^{-1}, \quad \epsilon \neq 0$$

Substitution in the three remaining relations yields three quadratic relations, say q_1, q_2, q_3. So $p \in P_D$ if and only if $p \in \mathcal{V}(q_1, q_2, q_3)$ and (4) is done.

We showed as well that pr_1 determines isomorphism between $(\Gamma_D)_{\mathrm{red}}$ and $(\mathcal{P}_D)_{\mathrm{red}}$. Similarly, pr_2 determines an isomorphism. By Theorem 4.1.3 of [8] we may thus conclude that $\mathcal{P}_D = \mathrm{pr}_1(\Gamma_D) = \mathrm{pr}_2(\Gamma_D)$ and that Γ_D defines an automorphism σ_D of $\mathcal{P}_D$. This is statement (1). Also (2) (from which the second part goes similarly as the first) and (3) follow. □

Remark 5.2.3. It would be nicer if one could write down a scheme theoretic description of $\mathcal{P}_D$. However we have not succeeded in doing this.

Corollary 5.2.4. *The point modules of D are in one to one correspondence with the points of $\mathcal{P}_D$. For $M(p)$ the point module corresponding to a point $p \in \mathcal{P}_D$, $M(p)_{\geq 1}(1) \cong M(\sigma_D^{-1}(p))$* [4, Cor. 3.13].

5.3. **Explicit computations for the generic types.** In this section we assume that k has characteristic zero. As before write P_A and P_D for the reduced part of the point varieties of respectively $\mathcal{P}_A$ and $\mathcal{P}_D$ and write V for the reduced part of $\mathcal{V}(q_1, q_2, q_3)$. Let p_{sp} denote a special point. As an example, we apply the foregoing results to calculate how much P_D differs from P_A in the case where A is generic (see §4 and [3]).

Proposition 5.3.1. *Let A be generic and suppose that $D = [A, \Lambda]$ is a normalizing extension of A_e corresponding to $\Lambda \in \mathrm{GL}(2, k)$. Then $P_D = P_A \cup V \cup \{p_{\mathrm{sp}}\}$ where V and p_{sp} are given in Table 5.1. This yields the following cases :*

- *If $D = [A, \Lambda]$, then P_D is the union of a smooth elliptic curve on $\mathcal{V}(X_1X_4 - X_2X_3)$ together with four additional points.*
- *If $D = [E, (1, \zeta_3)]$, then P_D is the union of a smooth elliptic curve on $\mathcal{V}(X_1X_4 - X_2X_3)$ and 7 additional points (from which 1 special point).*
- *If $D = [H, (\zeta_8, -\zeta_8)]$, then P_D is the union of a smooth elliptic curve on $\mathcal{V}(X_1X_4 - X_2X_3)$ and 4 additional points. If $D = [H, (1, 1)]$ or $D = [H, (\zeta_8^2, \zeta_8^2)]$, then P_D is a non-singular quadric.*
- *If $D = [S_1, (p, p^{-1})]$ with p generic, then P_D is the union of two plane conics on $\mathcal{V}(X_1X_4 - X_2X_3)$, one line (being the intersection of the two planes) and two additional points. If p satisfies the quadratic relation $p^2 \pm (2\alpha + a^2)p + \alpha^2 = 0$, then P_D is the union of a plane, a plane conic on $\mathcal{V}(X_1X_4 - X_2X_3)$ and one point. If $D = [S_1, (\alpha, -\alpha^{-1})]$, then P_D is the union of two plane conics on $\mathcal{V}(X_1X_4 - X_2X_3)$ and two additional points.*
- *If $D = [S_2, (p, -p^{-1})]$ with p generic, then P_D is the union of 5 lines, from which 4 lying on the quadric $\mathcal{V}(X_1X_4 - X_2X_3)$. If $p^2 = \alpha^2$, then P_D is the union of 6 lines from which 4 on $\mathcal{V}(X_1X_4 - X_2X_3)$. If $p^2 = -\alpha^2$, then P_D is the union of two planes. Also for $D = [S_2, (\alpha, \alpha^{-1})]$, P_D is the union of two planes.*
- *If $D = [S_2', (1, 1)]$, then P_D is the union of two planes. If $D = [S_2', (1, -1)]$, then P_D is the union of a plane conic on $\mathcal{V}(X_1X_4 - X_2X_3)$ and three lines, with two of these lines lying on the quadric $\mathcal{V}(X_1X_4 - X_2X_3)$. For $D = [S_2', (i, i)]$, P_D is a singular quadric.*

Proof. Apply Proposition 5.2.2. □

5.4. **Central extensions or not?** As we pointed out already, a lot of the normalizing extensions are central extensions (or twists of) in the sense of [8]. That is, there exists a central (or more generally, normalizing) regular degree 1 element, say $z \in D_1$, such that $D/(z)$ is 3-dimensional A.S.-regular on three generators. If so, then $\mathcal{P}_D \cap \mathcal{V}(z)$ is the point variety of this underlying algebra, i.e. either a cubic curve in the hyperplane $\mathcal{V}(z) \subset \mathbb{P}^3$ or else all of $\mathcal{V}(z)$. Also, $\mathcal{P}_D \cap \mathcal{V}(z)$ is then fixed by σ_D, and outside $\mathcal{V}(z)$ it is the identity.

We know that $\mathcal{P}_A$ is a curve of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ or all of $\mathbb{P}^1 \times \mathbb{P}^1$. It is thus clear that the first statement is impossible in the situations where P_A is irreducible and $\mathcal{V}(q_1, q_2, q_3)$ is a finite number of points. Using this observation we conclude that the first three entries in Table 5.1 do not represent (twists of) central extensions.

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Type	Condition	$p_{\rm sp}$	V	Comment
$[A,\Lambda]$			8 pts	$V\cap P_A = 4$ pts
$[E,(1,\zeta_3)]$		1 pt	8 pts	$V\cap P_A = 2$ pts
$[H,(\zeta_8,-\zeta_8)]$			8 pts	$V\cap P_A = 4$ pts
$[H,(1,1)]$			non-sing. irred. quadric	$P_A\subset V$
$[H,(\zeta_8^2,\zeta_8^2)]$			non-sing. irred. quadric	$P_A\subset V$
$[S_1,(p,p^{-1})]$	p generic		a line and 2 pts	
	p satisfies*		a plane and 1 pt	$V\cap P_A = 1$ conic
$[S_1,(\alpha,-\alpha^{-1})]$			4 pts	$V\cap P_A = 2$ pts
$[S_2,(p,-p^{-1})]$	p generic		3 lines	$V\cap P_A = 2$ lines
	$p^2=\alpha^2$		4 lines	$V\cap P_A = 2$ lines
	$p^2=-\alpha^2$		2 planes	$P_A\subset V$
$[S_2,(\alpha,\alpha^{-1})]$			2 planes	$P_A\subset V$
$[S_2',(1,1)]$			2 planes	$P_A\subset V$
$[S_2',(1,-1)]$			2 lines	$V\cap P_A = 1$ line
$[S_2',(i,i)]$			sing. irred. quadric	$P_A\subset V$

(*) $p^2 \pm (a^2-2\alpha)p+\alpha^2=0$

TABLE 5.1. P_D for $D=[A,\Lambda]$ with A generic

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