SEMI–INVARIANTS OF QUIVERS FOR ARBITRARY DIMENSION VECTORS

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Abstract. The representations of dimension vector α of the quiver Q can be parametrised by a vector space $R(Q,\alpha)$ on which an algebraic group $\mathrm{Gl}(\alpha)$ acts so that the set of orbits is bijective with the set of isomorphism classes of representations of the quiver. We describe the semi–invariant polynomial functions on this vector space in terms of the category of representations. More precisely, we associate to a suitable map between projective representations a semi–invariant polynomial function that describes when this map is inverted on the representation and we show that these semi–invariant polynomial functions form a spanning set of all semi–invariant polynomial functions in characteristic 0. If the quiver has no oriented cycles, we may replace consideration of inverting maps between projective representations by consideration of representations that are left perpendicular to some representation of dimension vector α. These left perpendicular representations are just the cokernels of the maps between projective representations that we consider.

1. Notation and Introduction

In the sequel k will be an algebraically closed field. For our main result, Theorem 2.3 below, k will have characteristic zero.

Let Q be a quiver with finite vertex set V, finite arrow set A and two functions $i,t : A \to V$ where for an arrow a we shall usually write ia in place of $i(a)$, the initial vertex, and $ta = t(a)$, the terminal vertex. A representation, R, of the quiver Q associates a k-vector space $R(v)$ to each vertex v of the quiver and a linear map $R(a) : R(ia) \to R(ta)$ to each arrow a. A homomorphism ϕ of representations from R to S is given by a collection of linear maps for each vertex $\phi(v) : R(v) \to S(v)$ such that for each arrow a, $R(a)\phi(ta) = \phi(ia)S(v)$. The category of representations of the quiver $\mathrm{Rep}(Q)$ is an abelian category as is the full subcategory of finite dimensional representations. We shall usually be interested in finite dimensional representations in which case each representation has a dimension vector $\underline{\dim}\, R$, which is a function from the set of vertices V to the natural numbers $\mathbb{N}$ defined by $(\underline{\dim}\, R)(v) = \dim R(v)$. Now let α be a dimension vector for Q. The representations of dimension vector α are parametrised by the vector space

$$R(Q,\alpha) = \times_{a\in A}{}^{\alpha(ia)}k^{\alpha(ta)}$$

where ${}^mk^n$ is the vector space of m by n matrices over k (and ${}^mk$ and k^n are shorthand for ${}^mk^1$ and ${}^1k^n$ respectively). Given a point $p \in R(Q,\alpha)$, we denote the corresponding representation by R_p. The isomorphism classes of representations of

Date: July 27, 1999.

1991 *Mathematics Subject Classification.* Primary 13A50.

Key words and phrases. Invariant theory, quivers.

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dimension vector α are in 1 to 1 correspondence with the orbits of the algebraic group

$$\mathrm{Gl}(\alpha) = \times_{v\in V} \mathrm{Gl}_{\alpha(v)}(k)$$

We consider the action of $\mathrm{Gl}(\alpha)$ on the co–ordinate ring $S(Q,\alpha)$ of $R(Q,\alpha)$; $f \in S(Q,\alpha)$ is said to be semi–invariant of weight ψ where ψ is a character of $\mathrm{Gl}(\alpha)$ if $g(f) = \psi(g)f$, $\forall g \in \mathrm{Gl}(\alpha)$.

The invariants and semi–invariants for this action are of importance for the description of the moduli spaces of representations of the quiver for fixed dimension vector. In characteristic 0 we may apply Weyl's theory of invariants for $\mathrm{Sl}_n(k)$ to give an explicit description of all such semi–invariants. We shall find a set of semi–invariants that span all semi–invariants as a vector space in characteristic 0. It seems likely that the result we obtain here should hold in arbitrary characteristic and that this would follow from Donkin [2, 3]. However, we restrict ourselves in this paper to characteristic 0. In [6], the first author described all the semi–invariants when $\mathrm{Gl}(\alpha)$ has an open orbit on $R(Q,\alpha)$ in terms of certain polynomial functions naturally associated to the representation theory of the quiver. We shall begin by recalling the definition of these semi–invariants and some related theory.

Given the quiver Q, let $\mathrm{add}(Q)$ be the additive k–category generated by Q. To describe this more precisely, we define a path of length $n > 0$ from the vertex v to the vertex w to be a monomial in the arrows $p = a_1 \dots a_n$ such that $ta_i = ia_{i+1}$ for $0 < i < n$ and $ia_1 = v$, $ta_n = w$. We define $ip = v$ and $tp = w$. For each vertex v there is also the trivial path of length 0, e_v, from v to v.

For each vertex v, we have an object $O(v)$ in $\mathrm{add}(Q)$, and $\mathrm{Hom}(O(v), O(w)) = P(v,w)$ where $P(v,w)$ is the vector space with basis the paths from v to w including the trivial path if $v = w$; finally,

$$\mathrm{Hom}(\bigoplus_v O(v)^{a(v)}, \bigoplus_v O(v)^{b(v)})$$

is defined in the usual way for an additive category, where composition arises via matrix multiplication.

Any representation R of Q extends uniquely to a covariant functor from $\mathrm{add}(Q)$ to $\mathrm{Mod}(k)$ which we shall continue to denote by R; thus given ϕ a map in $\mathrm{add}(Q)$, its image under the functor induced by R is $R(\phi)$. Let α be some dimension vector, and ϕ a map in $\mathrm{add}(Q)$

$$\phi : \bigoplus_{v\in V} O(v)^{a(v)} \to \bigoplus_{v\in V} O(v)^{b(v)}$$

then for any representation R of dimension vector α, $R(\phi)$ is a $\sum_{v\in V} a(v)\alpha(v)$ by $\sum_{v\in V} b(v)\alpha(v)$ matrix. If $\sum_{v\in V} a(v)\alpha(v) = \sum_{v\in V} b(v)\alpha(v)$, we define a semi–invariant polynomial function $P_{\phi,\alpha}$ on $R(Q,\alpha)$, by

$$P_{\phi,\alpha}(p) = \det R_p(\phi).$$

We shall refer to these semi–invariants as determinantal semi–invariants in future. We will show that the determinantal semi–invariants span all semi–invariants (Theorem 2.3 below). To prove Thereom 2.3 we will use the classical symbolic method which was also used by Procesi to show that the invariants of matrices under conjugation are generated by traces [5]. Procesi's result is generalized in [4] where it is shown that invariants of quiver-representations are generated by traces of oriented cycles. Subsequently Donkin showed that suitable analogues of these results were valid in characteristic p [2, 3].

It was shown in [6] that the determinantal semi-invariants can be defined in terms of the representation theory of Q. This is reviewed in Section §3.

One corollary worth stating of this representation theoretic interpretation is the following: define a point p of $R(Q,\beta)$ to be semistable if some non–constant semi–invariant polynomial function does not vanish at p. Then we have:

Corollary 1.1. *In characteristic zero the point p of $R(Q,\beta)$ is semistable if and only there is some non–trivial (possibly infinite dimensional) representation T of Q such that* $\mathrm{Hom}(T,R_p)=\mathrm{Ext}(T,R_p)=0$.

For the proof we refer to the end of Section 3.

In the sequel we will frequently change the quiver $Q=(V,A)$ to another quiver $Q'=(V',A')$ which is connected to Q through an additive funcor.

$$s:\mathrm{add}(Q')\to\mathrm{add}(Q)$$

Through functoriality s will act on various objects associated to Q and Q'. We will list these derived actions below. In order to avoid having to introduce a multitude of adhoc notations we denote each of the derived actions also by s.

To start there is an associated functor

$$s:\mathrm{Rep}(Q)\to\mathrm{Rep}(Q'):R\mapsto R\circ s$$

If R has dimension vector α then $s(R)$ has dimension vector $s(\alpha)\overset{\mathrm{def}}{=}\alpha\circ s$. Put $\alpha'=s(\alpha)$. We obtain that s defines a map

$$s:R(Q,\alpha)\to R(Q',\alpha')$$

such that $s(R_p)=R_{s(p)}$. As usual there is a corresponding k-algebra homomorphism

$$s:S(Q',\alpha)\to S(Q,\alpha)$$

given by $s(f)=f\circ s$. Writing out the definitions one obtains:

$$s(P_{\phi',\alpha'})=P_{s(\phi'),\alpha} \tag{1.1}$$

Finally s defines a homomorphism

$$s:\mathrm{Gl}(\alpha)\to\mathrm{Gl}(\alpha')$$

which follows from functoriality by considering $\mathrm{Gl}(\alpha)$ as the automorphism group of $R_0(\alpha)$ in $\mathrm{Rep}(Q)$. One checks that for $g\in\mathrm{Gl}(\alpha)$, $p\in R(Q,\alpha)$ one has $s(g\cdot p)=s(g)\cdot s(p)$. It follows in particular that if f is a semi–invariant in $S(Q',\alpha')$ with character ψ' then $s(f)$ is a semi-invariant with character $s(\psi')\overset{\mathrm{def}}{=}\psi'\circ s$.

2. Semi–invariant polynomial functions

Next, we discuss the ring $S(Q,\alpha)$. $S(Q,\alpha)$ has two gradings, one of which is finer than the other. First of all, $S(Q,\alpha)$ may be graded by $\mathbb{Z}^A$ in the natural way since $R(Q,\alpha)=\times_{a\in A}{}^{\alpha(ia)}k^{\alpha(ta)}$. We call this the A–grading. On the other hand, $\mathrm{Gl}(\alpha)$ acts on $R(Q,\alpha)$ and hence $S(Q,\alpha)$ and so $\times_{v\in V}k^*$ acts on $S(Q,\alpha)$; we may therefore decompose $S(Q,\alpha)$ as a direct sum of weight spaces for the action of $\times_{v\in V}k^*$ which gives a grading by $\mathbb{Z}^V$. We call this the V–grading. The semi–invariants $P_{\phi,\alpha}$ are homogeneous with respect to the second grading though not the first. The first grading is induced by the natural action of $\times_{a\in A}k^*$ on $\mathrm{add}(Q)$. $\times_{a\in A}k^*$ acts on $\mathrm{add}(Q)$ by $(\dots,\lambda_a,\dots)(a)=\lambda_a a$ and according to (1.1) $gP_{\phi,\alpha}=P_{g\phi,\alpha}$ for $g\in\times_{a\in A}k^*$.

A standard Van der Monde determinant argument implies that the vector subspace spanned by determinantal semi–invariants in $S(Q,\alpha)$ is also the space spanned by the homogeneous components of the determinantal semi–invariants with respect to the $\mathbb{Z}^A$–grading. Thus it is enough to find the latter subspace. Given a character χ of $\times_{a\in A}k^*$, we define $P_{\phi,\alpha,\chi}$ to be the χ–component of $P_{\phi,\alpha}$.

We begin by describing the semi–invariants which are homogeneous with respect to the A–grading and linear in each component of $R(Q,\alpha)=\times_{a\in A}{}^{\alpha(ia)}k^{\alpha(ta)}$. We call these the homogeneous multilinear semi–invariants. We need temporarily another kind of semi–invariant. A path l in the quiver is an oriented cycle if $il=tl$. Associated to such an oriented cycle is an invariant Tr_l for the action of $\mathrm{Gl}(\alpha)$ on $R(Q,\alpha)$ defined by $\mathrm{Tr}_l(p)=\mathrm{Tr}(R_p(l))$ where Tr is the trace function.

Theorem 2.1. *The homogeneous multilinear semi–invariants of $R(Q,\alpha)$ are spanned by semi–invariants of the form $P_{\phi,\alpha,\chi}\prod_i\mathrm{Tr}_{l_i}$ where l_i are oriented cycles in the quiver.*

Proof. The semi–invariants are invariants for $\mathrm{Sl}(\alpha)=\times_{v\in V}\mathrm{Sl}_{\alpha(v)}(k)$ and conversely, $\mathrm{Sl}(\alpha)$–invariant polynomials that are homogeneous with respect to the V–grading are also semi–invariant for $\mathrm{Gl}(\alpha)$. We may therefore use Weyl's description of the homogeneous multilinear invariants for $\mathrm{Sl}_n(k)$ and hence for $\mathrm{Sl}(\alpha)$.

Given a homogeneous multilinear $\mathrm{Sl}(\alpha)$–invariant $f:\times_{a\in A}{}^{\alpha(ia)}k^{\alpha(ta)}\to k$, it factors as

$$\times_{a\in A}{}^{\alpha(ia)}k^{\alpha(ta)}\to\otimes_{a\in A}{}^{\alpha(ia)}k^{\alpha(ta)}\xrightarrow{\tilde f}k$$

for a suitable linear map $\tilde f$. Since

$${}^{\alpha(ia)}k^{\alpha(ta)}\cong{}^{\alpha(ia)}k\otimes k^{\alpha(ta)}$$

as $\mathrm{Sl}(\alpha)$–representation, we may write

$$\tilde f:\otimes_{a\in A}{}^{\alpha(ia)}k\otimes k^{\alpha(ta)}\to k.$$

Moreover, $\otimes_{a\in A}{}^{\alpha(ia)}k\otimes k^{\alpha(ta)}$ as $\mathrm{Sl}(\alpha)$–representation is a tensor product of covariant and contravariant vectors for $\mathrm{Sl}(\alpha)$. Thus we may re–write

$$\bigotimes_{a\in A}{}^{\alpha(ia)}k\otimes k^{\alpha(ta)}=\bigotimes_{v\in V}\left(\bigotimes_{a,ia=v}{}^{\alpha(v)}k\otimes\bigotimes_{a,ta=v}k^{\alpha(v)}\right)$$

and $\tilde f=\prod_{v\in V}\tilde f_v$ for $\mathrm{Sl}_{\alpha(v)}(k)$–invariant linear maps:

$$\tilde f_v:\bigotimes_{a,ia=v}{}^{\alpha(v)}k\otimes\bigotimes_{a,ta=v}k^{\alpha(v)}\to k.$$

Roughly speaking Weyl [7] showed that there are 3 basic linear semi–invariant functions on tensor products of covariant and contravariant vectors for $\mathrm{Sl}_m(k)$. Firstly, there is the linear map from ${}^mk\otimes k^m$ to k given by $f(x\otimes y)=yx$; this is just the trace function on ${}^mk\otimes k^m\cong M_m(k)$. Secondly, there is the linear map from $\otimes_{i=1}^m{}^mk$ to k determined by $f(x_1\otimes\cdots\otimes x_m)=\det(x_1|\dots|x_m)$. The third case is similar to the second; there is a linear map from $\otimes_{i=1}^mk^m$ to k again given by the determinant. A spanning set for the linear semi–invariant functions on a general tensor product of covariant and contravariant vectors is constructed from these next. A spanning set for the $\mathrm{Sl}_m(k)$–invariant linear maps from $\otimes_B{}^m k \otimes \otimes_C k^m$ to k is obtained in the following way. We take three disjoint indexing sets I, J, K: we have surjective functions $\mu : B \to I \cup K$, $\nu : C \to J \cup K$ such that $\mu^{-1}(k)$ and $\nu^{-1}(k)$ have one element each for $k \in K$, and $\mu^{-1}(i)$ and $\nu^{-1}(j)$ have m elements each for $i \in I$, $j \in J$. We label this data by $\gamma = (\mu, \nu, I, J, K)$. To γ, we associate an $\mathrm{Sl}_m(k)$–invariant linear map

$$f_\gamma\left(\bigotimes_{b\in B} x_b \otimes \bigotimes_{C} y_c\right) = \prod_{k\in K} y_{\nu^{-1}(k)} x_{\mu^{-1}(k)} \prod_{i\in I} \det\left(x_{b_1}|\ldots|x_{b_m}\right) \prod_{j\in J} \det\begin{pmatrix} y_{c_1} \\ \vdots \\ y_{c_m}\end{pmatrix}$$

where $\{b_1, \ldots, b_m\} = \mu^{-1}(i)$, $\{c_1, \ldots, c_m\} = \nu^{-1}(j)$. Note that f_γ is determined only up to sign since we have not specified an ordering of $\mu^{-1}(i)$ or $\nu^{-1}(j)$.

A spanning set for $\mathrm{Sl}(\alpha)$–invariant linear maps from

$$\bigotimes_{v\in V}\left(\bigotimes_{a, ia=v}{}^{\alpha(v)} k \otimes \bigotimes_{a, ta=v} k^{\alpha(v)}\right)$$

is therefore determined by giving quintuples $(\mu, \nu, I, J, K) = \Gamma$ where

$$I = \dot{\bigcup_{v\in V}} I_v$$

$$J = \dot{\bigcup_{v\in V}} J_v$$

$$K = \dot{\bigcup_{v\in V}} K_v$$

and surjective maps

$$\mu : A \to I \cup K,$$
$$\nu : A \to J \cup K$$

where

$$\mu(a) \in I_{ia} \cup K_{ia},$$
$$\nu(a) \in J_{ta} \cup K_{ta},$$

$\mu^{-1}(k)$ and $\nu^{-1}(k)$ have one element each, $\mu^{-1}(i)$ and $\nu^{-1}(j)$ have $\alpha(v)$ elements each for $i \in I_v$ and $j \in J_v$. Then Γ determines data γ_v for each $v \in V$ and we define

$$f_\Gamma = \prod_{v\in V} f_{\gamma_v}.$$

We show that these specific semi–invariants lie in the linear span of the homogeneous components of determinantal semi–invariants. First, we treat the case where K is empty. Let $n = |A|$. We have two expressions for n:

$$n = \sum_{v\in V} |I_v|\alpha(v) = \sum_{v\in V} |J_v|\alpha(v).$$

To each arrow a, we have a pair $(\mu(a),\nu(a))$ associated. To this data, we associate a map in $\operatorname{add}(Q)$ in the following way. We consider a map

$$\Phi_\Gamma : \bigoplus_{v\in V} O(v)^{I_v} \to \bigoplus_{v\in V} O(v)^{J_v}$$

whose (i,j)–entry is

$$\sum_{a,(\mu(a),\nu(a))=(i,j)} a.$$

Given $p \in R(Q,\alpha)$, $R_p(\Phi_\Gamma)$ is an n by n matrix which we may regard as a partitioned matrix where the rows are indexed by I and the columns by J, there are $\alpha(v)$ rows having index $i \in I_v$, $\alpha(v)$ columns having index $j \in J_v$ and the block having index (i,j) is

$$\sum_{a:(\mu(a),\nu(a))=(i,j)} R_p(a).$$

We claim that $f_\Gamma = P_{\Phi_\Gamma,\alpha,\chi}$ up to sign where $\chi((\lambda_a)_a) = \prod_a \lambda_a$. To prove this it will be convenient to define a new quiver $Q' = (V', A')$ whose vertices are given by $I \cup J$ and whose edges are the same as those of Q. The initial and terminal vertex of $a \in A' = A$ are given by $(\mu(a),\nu(a))$. On Q' we define data Γ' which is defined by the same quintuple (μ,ν,I,J,K) as Γ, but with has different decompositions $I = \bigcup_{i\in V'} I_i$, $J = \bigcup_{j\in V'} J_j$. In fact

$$I_i = \begin{cases} \{i\} & \text{if } i \in I \\ \emptyset & \text{otherwise} \end{cases} \qquad \text{and} \qquad J_j = \begin{cases} \{j\} & \text{if } j \in J \\ \emptyset & \text{otherwise} \end{cases}$$

We define a functor $s : \operatorname{add}(Q') \to \operatorname{add}(Q)$ by $s(a) = a$, $s(O(i)) = O(v)$, $s(O(j)) = O(w)$ where $a \in A' = A$, $i \in I_v$, $j \in J_w$.

Since Q and Q' have the same edges, the action of $\times_{a\in A} k^*$ on $\operatorname{add}(Q)$ lifts canonically to an action of $\times_{a\in A} k^*$ on $\operatorname{add}(Q')$. Put $\alpha' = s(\alpha)$. Using (1.1) we then find $s(P_{\Phi_{\Gamma'},\alpha'}) = P_{s(\Phi_{\Gamma'}),\alpha} = P_{\Phi_\Gamma,\alpha}$ and similarly $s(P_{\Phi_{\Gamma'},\alpha',\chi}) = P_{\Phi_\Gamma,\alpha,\chi}$. Finally one also verifies that $s(f_{\Gamma'}) = f_\Gamma$.

Hence to prove that $f_\Gamma = \pm P_{\Phi_\Gamma,\alpha,\chi}$ we may replace the triple (Q,α,Γ) by (Q',α',Γ'). We do this now.

In order to prove that $f_\Gamma = \pm P_{\Phi_\Gamma,\alpha,\chi}$, we need only check that the two functions agree on the image of $W = \times_{a\in A} {}^{\alpha(ia)}k \times k^{\alpha(ta)}$ in $\otimes_{a\in A} {}^{\alpha(ia)}k \otimes k^{\alpha(ta)}$.

Let $\psi : \mathrm{Gl}(\alpha) \to k^*$ be the character given by

$$\psi((A_v)_{v\in V}) = \prod_{i\in I} \det A_i \cdot \prod_{j\in J} (\det A_j)^{-1}$$

Then one checks that both f_Γ and $P_{\Phi_\Gamma,\alpha}$ are semi-invariants on W with character ψ. Now we claim that on W we have $f_\Gamma = P_{\Phi_\Gamma,\alpha}$, up to sign. To prove this we use the fact that $\mathrm{Gl}(\alpha)$ has an open orbit on W.

For vertices $i \in I$ and $j \in J$, we let $\{a_{i,1},\ldots,a_{i,\alpha(i)}\}$ and $\{a_{j,1},\ldots,\alpha_{j,\alpha(j)}\}$ be the sets of arrows incident with i and j respectively. So

$$\times_{a\in A} {}^{\alpha(ia)}k \times k^{\alpha(ta)} = \times_{i\in I} \left(\times_{l=1}^{\alpha(i)} {}^{\alpha(i)}k \right) \times \times_{j\in J} \left(\times_{m=1}^{\alpha(j)} k^{\alpha(j)} \right).$$

We take the point p whose (i,l)–th entry is the l–th standard column vector in ${}^{\alpha(i)}k$; that is its nth entry is δ_{nl} and whose (j,m)–th entry is the m–th standard row vector in $k^{\alpha(j)}$. The $\mathrm{Gl}(\alpha)$-orbit of this point is open in W. Hence to show that $f_\Gamma = \pm P_{\Phi_\Gamma,\alpha}$ on W, it suffices to do this in the point p.

Now $f_\Gamma(p) = \pm 1$. Furthermore we have

$$R_p(\Phi_\gamma)_{(i,l),(j,m)} = \begin{cases} 1 & \text{if } a_{il} = a_{jm} \\ 0 & \text{otherwise} \end{cases}$$

In particular $R_p(\Phi_\Gamma)$ is a permutation matrix and thus $P_{\Phi_\Gamma,\alpha}(p) = \pm 1$. Hence indeed $f_\Gamma = \pm P_{\Phi_\Gamma,\alpha}$ (on W). Note that the non-zero entries of $R_p(\Phi_\Gamma)$ are naturally indexed by A.

To prove that $f_\Gamma = \pm P_{\Phi_\Gamma,\alpha,\chi}$ on W it is now sufficient to prove that $P_{\Phi_\Gamma,\alpha} = P_{\Phi_\Gamma,\alpha,\chi}$ on W. To this end we lift the $\times_{a\in A} k^*$ action on $R(Q,\alpha)$ to W by defining $(\lambda_a)_a \cdot (x_{ia}, y_{ta})_a = (\lambda_a x_{ia}, y_{ta})_a$ (this is just some convenient choice).

Now we have to show that $P_{\Phi_\Gamma,\alpha}$ is itself homogeneous with character χ when restricted to W. Since the action of $\times_{a\in A} k^*$ commutes with the $\mathrm{Gl}(\alpha)$-action it suffices to do this in the point p. Now if we put $q = (\lambda_a)_a \cdot p$ then $R_q(\Phi_\Gamma)$ is obtained from $R_p(\Phi_\Gamma)$ by multiplying by λ_a for all $a \in A$ the non-zero entry in $R_p(\Phi_\Gamma)$ indexed by a. Therefore $P_{\Phi_\Gamma,\alpha}(p)$ is multiplied by $\prod_a \lambda_a$. This proves what we want.

It remains to deal with the case where K is non–empty. Roughly speaking one of two things happens here; if we have two distinct arrows $a = \nu^{-1}(k)$ and $b = \mu^{-1}(k)$ for some $k \in K$ then this element of K corresponds to replacing a and b by their composition; if on the other hand $a = \nu^{-1}(k) = \mu^{-1}(k)$ then $ia = ta$ and this element of K corresponds to taking the trace of a.

We associate a quiver $Q(A,K)$ with vertex set A and arrow set K; given $k \in K$, $ik = \nu^{-1}(k)$, $tk = \mu^{-1}(k)$. This quiver has very little to do with the quiver Q; it is a temporary notational convenience. The connected components of $Q(A,K)$ are of three types: either they are oriented cycles, open paths or isolated points. The vertices of components of the first type are arrows of Q that also form an oriented cycle, those of the second type are arrows that compose to a path in the Q (which can in fact also be an oriented cycle); the isolated points we shall treat in the same way to the second type.

We label the oriented cycles $\{L_l\}$ and the open paths $\{M_m\}$. To an oriented cycle L in $Q(A,K)$, we associate the invariant Tr_{p_L} where p_L is the path around the oriented cycle in Q. This is independent of our choice of starting point on the loop. To the open path M in $Q(A,K)$, we associate the path p_M, the corresponding path in Q.

We consider the adjusted quiver Q_K with vertex set V and arrow set $\{p_{M_m}\}$ where ip_M and tp_M are defined as usual. Define the functor $s : \mathrm{add}(Q_K) \to \mathrm{add}(Q)$ as follows: on vertices s is the identity, and on edges $s(p_{M_m}) = p_{M_m}$.

If $p_{M_m} = a_{m,1} \ldots a_{m,d}$, we define

$$\mu(p_{M_m}) = \mu(a_{m,1})$$
$$\nu(p_{M_m}) = \nu(a_{m,d}).$$

Then

$$\mu : \{p_{M_m}\} \to I$$
$$\nu : \{p_{M_m}\} \to J$$

are surjective functions giving data Γ_K on Q_K. One checks directly that

$$f_\Gamma = s(f_{\Gamma_K})\prod_L \mathrm{Tr}_{p_L} = s(P_{\phi_{\Gamma_K},\alpha,\chi_K})\prod_L \mathrm{Tr}_{p_L}$$
$$= P_{s(\phi_{\Gamma_K}),\alpha,\chi}\prod_L \mathrm{Tr}_{p_L}$$

for χ_K and χ of weight 1 on each arrow for the quivers Q_K and Q respectively, which completes our proof. □

If l is an oriented cycle in the quiver then $\mathrm{Tr}(R_p(l))$ lies in the linear span of $\det(I+\lambda R_p(l))$ for varying $\lambda \in k$ and $\det(I+\lambda R_p(l)) = P_{I+\lambda l,\alpha}(p)$ Also

$$P_{\phi,\alpha}P_{\mu,\alpha} = P_{\left(\begin{smallmatrix}\phi & 0\\ 0 & \mu\end{smallmatrix}\right),\alpha}$$

so we may deduce the following corollary.

Corollary 2.2. *Homogeneous multilinear semi–invariants lie in the linear span of determinantal semi–invariants.*

Since we have dealt with the multilinear case, standard arguments in characteristic 0 apply to the general case.

Theorem 2.3. *In characteristic* 0*, the semi–invariant polynomial functions for the action of* $\mathrm{Gl}(\alpha)$ *on* $R(Q,\alpha)$ *are spanned by determinantal semi–invariants.*

Proof. It is enough to consider semi–invariants homogeneous with respect to the A–grading of weight χ where $\chi((\lambda_a)_a) = \prod_a \lambda_a^{m_a}$.

We may also assume that no component of the A–grade on the semi–invariant is 0 since we may always restrict to a smaller quiver. To χ, we associate a new quiver Q_χ with vertex set V and arrow set A_χ where $A_\chi = \bigcup_{a\in A}\{a_1,\dots,a_{m_a}\}$ where $ia_i = ia$, $ta_i = ta$.

We have functors

$$\sigma : \mathrm{add}(Q) \to \mathrm{add}(Q_\chi)$$
$$\sigma(a) = \sum a_i$$
$$\pi : \mathrm{add}(Q_\chi) \to \mathrm{add}(Q)$$
$$\pi(a_i) = a.$$

Given a semi–invariant f for $R(Q,\alpha)$, we define $\tilde f$ to be the χ'–component of $\sigma(f)$ where $\chi'((\lambda_{a_i})_{a,i}) = \prod_{a,i}\lambda_{a_i}$. So $\tilde f$ is homogeneous multilinear. Then one checks that $\pi(\tilde f) = \prod_a m_a! \cdot f$.

However, by the previous corollary, $\tilde f$ lies in the linear span of determinantal semi–invariants; therefore

$$\tilde f = \sum_i \lambda_i P_{\phi_i,\alpha}$$

and so

$$\begin{aligned}\prod_{a\in A} m_a!\cdot f &= \pi(\tilde{f})\\ &= \sum_i \lambda_i \pi(P_{\phi_i,\alpha})\\ &= \sum_i \lambda_i P_{\pi(\phi_i),\alpha} \qquad \text{(by (1.1))}\end{aligned}$$

Hence, in characteristic 0, semi–invariants homogeneous with respect to the A–grading and hence all semi–invariants lie in the linear span of the determinantal semi–invariants as required. □

To finish this section we will give a a slightly more combinatorial interpretation of our proof of Theorem 2.3.

Let us say that a pair (Q,α) is *standard* if one of the following holds.

1. Q consists of one vertex with one loop.
2. Q is bipartite with vertex set $I\cup J$ with all initial vertices in I and all terminal vertices in J. Furthermore if $v\in V=I\cup J$ then v belongs to exactly $\alpha(v)$ edges.

To a standard pair (Q,α) we associate a standard semi–invariant as follows. If Q is a loop then we take $\mathrm{Tr}(R_p(a))$ where a is the loop. If Q is bipartite then we take $P_{\phi,\alpha}$ where ϕ is given by

$$\phi : \oplus_{i\in I} O(i) \to \oplus_{j\in J} O(j)$$

such that the (i,j) component of ϕ is given by the sum of arrows from i to j.

If (Q,α) is arbitrary then we define the standard semi–invariants of $S(Q,\alpha)$ as those semi–invariants which are of the form $s(f)$ for some functor

$$s : \mathrm{add}(Q') \to \mathrm{add}(Q)$$

such that $(Q', s(\alpha))$ is standard and such that f is the corresponding standard semi–invariant.

The follows corollary can now easily be obtained from our proof of Theorem 2.3.

Corollary 2.4. *In characteristic zero the semi–invariants in $S(Q,\alpha)$ are generated by the standard semi–invariants.*

3. Interpretation in terms of representation theory

This section is mainly a review of some results of [6] which provided the motivation for the current paper. We prove Corollary 1.1.

It is standard that the category $\mathrm{Rep}(Q)$ is equivalent to the category of right modules over kQ, the *path algebra* of Q. We will identify a representation with its corresponding kQ-module.

With every vertex $v\in Q$ corresponds canonically an idempotent e_v in kQ given by the empty path. We denote by P_v the representations of Q associated to $e_v kQ$. Clearly this is a projective object in $\mathrm{Rep}(Q)$.

For any representation R we have $\mathrm{Hom}(P_v,R)\cong Re_v = R(v)$ and furthermore $\mathrm{Hom}(P_v,P_w) = e_w kQ e_v$. Now $e_w kQ e_v$ is a vector space spanned by the paths having initial vertex w and terminal vertex v. Hence in fact

$$\mathrm{Hom}_{\mathrm{Rep}(Q)}(P_v,P_w) = \mathrm{Hom}_{\mathrm{add}(Q)}(O(w),O(v))$$

In other words if we denote by $\mathrm{proj}(Q)$ the additive category generated by the $(P_v)_{v\in V}$ then $\mathrm{proj}(Q)$ is equivalent to the opposite category of $\mathrm{add}(Q)$.

Now recall the following:

Lemma 3.1. $\mathrm{proj}(Q)$ *is equivalent to the category of finitely generated projective* kQ*-modules.*

Only in the case of quivers with oriented cycles, there is something to prove here; one must show that the finitely generated projective modules kQ of the quiver are all direct sums of P_v's. This may be proved in a similar way as the fact that over a free algebra all projective modules are free [1].

Given a map

$$\gamma : \bigoplus_{v\in V} P_v^{b(v)} \to \bigoplus_{v\in V} P_v^{a(v)},$$

we denote by

$$\hat{\gamma} : \bigoplus_{v\in V} O(v)^{a(v)} \to \bigoplus_{v\in V} O(v)^{b(v)}$$

the corresponding map in $\mathrm{add}(Q)$; similarly, for μ a map in $\mathrm{add}(Q)$, $\hat{\mu}$ is the corresponding map in $\mathrm{proj}(Q)$.

In order to link the semi–invariant polynomial functions $P_{\phi,\alpha}$ with the representation theory of Q, we recall some facts and definitions about the category of representations of a quiver. First of all, Ext^n vanishes for $n > 1$. Given dimension vectors α and β, we define the Euler inner product by

$$\langle \alpha, \beta \rangle = \sum_{v\in V} \alpha(v)\beta(v) - \sum_{a\in A} \alpha(ia)\beta(ta).$$

If R and S are representations of dimension vector α and β respectively, then

$$\dim \mathrm{Hom}(R, S) - \dim \mathrm{Ext}(R, S) = \langle \alpha, \beta \rangle. \tag{3.1}$$

Given representations R and S, we say that R is left perpendicular to S and that S is right perpendicular to R if and only if $\mathrm{Hom}(R, S) = 0 = \mathrm{Ext}(R, S)$; Given a representation R we define the right perpendicular category to R, $R^\perp$, to be the full subcategory of representations that are right perpendicular to R and the left perpendicular category to R, ${}^\perp R$ is defined to be the full subcategory of representations that are left perpendicular to R. It is not hard to show that $R^\perp$ and ${}^\perp R$ are exact hereditary abelian subcategories of $\mathrm{Rep}(Q)$. In [6] the first author even shows that if Q has no oriented cycles and R has an open orbit in $R(Q, \alpha)$ then $R^\perp$ is given by the representations of a quiver without oriented cycles and with $|V| - s$ vertices where s is the number of non-isomorphic indecomposable summands of R. A similar result holds for ${}^\perp R$.

If R and S are finite dimensional and R is left perpendicular to S then it follows from (3.1) that $\langle \underline{\dim}\, R, \underline{\dim}\, S \rangle = 0$. The converse problem is interesting: suppose that we have a representation R of dimension vector α and a dimension vector β such that $\langle \alpha, \beta \rangle = 0$, what are the conditions on a point $p \in R(Q, \beta)$ in order that R_p lies in $R^\perp$?

This problem was discussed and solved by the first author in [6] in the case that Q has no oriented cycles. Let us assume this for a moment. We start with a minimal projective resolution of R:

$$0 \to \bigoplus_v P_v^{b(v)} \xrightarrow{\theta} \bigoplus_v P_v^{a(v)} \to R \to 0.$$

By the above discussion $\theta = \hat{\phi}$ for some map $\phi : \bigoplus_v O(v)^{a(v)} \to \bigoplus_v O(v)^{b(v)}$ in $\mathrm{add}(Q)$. Applying $\mathrm{Hom}(-, R_p)$ yields a long exact sequence

(3.2)

$$0 \to \mathrm{Hom}(R, R_p) \to \mathrm{Hom}(\oplus_v P_v^{a(v)}, R) \xrightarrow{\mathrm{Hom}(\hat{\phi}, R_p)} \mathrm{Hom}(\oplus_v P_v^{b(v)}, R) \to \mathrm{Ext}(R, R_p) \to 0$$

and clearly $\mathrm{Hom}(\hat{\phi}, R_p) = R_p(\phi)$. The condition $\langle \alpha, \beta \rangle = 0$ translates into $\sum_{v \in A}((a(v) - b(v))\beta_v = 0$, so that $R_p(\phi)$ is in fact represented by a square matrix.

In [6] the first author defined $P_{R,\beta}(p) = \det R_p(\phi)$. It is not hard to see that $P_{R,\beta}$ is independent of the choice of θ and furthermore that it is a polynomial on $R(Q, \beta)$.

Hence if we define

$$V(R, \beta) = \{p \in R(Q, \beta) \mid R \perp R_p\}$$

then it follows from (3.2) that

$$V(R, \beta) = \{p \in R(Q, \beta) \mid P_{R,\beta}(p) \neq 0\}$$

In other words, $V(R, \beta)$ is either trivial or the complement of a hypersurface.

If there is an exact sequence in $\mathrm{Rep}(Q)$

$$0 \to R_1 \to R \to R_2 \to 0$$

with $\langle \underline{\dim}\, R_1, \beta \rangle = \langle \underline{\dim}\, R_2, \beta \rangle = 0$ then clearly $P_{R,\beta} = P_{R_1,\beta} P_{R_2,\beta}$. This provides some motivation for the main result of [6] which we state below.

Theorem 3.2. *Assume that Q has no oriented cycles and let S be a representation with dimension vector β which has an open orbit in $R(Q, \beta)$. Let $R_1, \ldots, R_t$ be the simple objects of ${}^\perp S$. Then the ring of semi–invariants for $R(Q, \beta)$ is a polynomial ring in the generators $P_{R_i,\beta}$.*

Now let us go back to the general case. Thus we allow that Q has oriented cycles. In this case kQ may be infinite dimensional, so that it is not clear how to define the minimal resolution of a representation R. Therefore we will take the map $\phi : \bigoplus_v O(v)^{a(v)} \to \bigoplus_v O(v)^{b(v)}$ as our fundamental object and we put $P_{\phi,\beta}(p) = \det R_p(\phi)$ (provided $\sum_{v \in A}(a(v) - b(v))\beta_v = 0$).

To make the link with the discussion above we need that $\hat{\phi}$ is injective. What happens if $\hat{\phi}$ is not injective? Then there is a non-trival kernel

$$0 \to P \to \bigoplus_v P_v^{b(v)} \xrightarrow{\hat{\phi}} \bigoplus_v P_v^{a(v)}$$

By the fact that kQ is hereditary and lemma 3.1 $P \cong \oplus_v P_v^{c(v)}$. Furthermore, again because kQ is hereditary P is a direct summand of $\oplus_v P_v^{b(v)}$. Now using the equivalence with $\mathrm{add}(Q)$ we find that $\hat{\phi}$ is not injective if and only if $\oplus_v O(v)^{b(v)}$ has a direct summand in $\mathrm{add}(Q)$ which is in the kernel of ϕ. Let ϕ' be the restriction of ϕ to the complementary summand. Then we find

$$P_{\phi,\beta} = \begin{cases} P_{\phi',\beta} & \text{if } \forall : v \in \mathrm{Supp}\,\beta : c(v) = 0 \\ 0 & \text{otherwise} \end{cases}$$

So for the purposes of semi-invariants we may assume that $\hat{\phi}$ is injective, which is what we will do below. If Q has no oriented cycles then it follows that $P_{\phi,\beta} = P_{\operatorname{cok}\hat{\phi},\beta}$. In general we find (using (3.2)):

$$P_{\phi,\beta}(p) \neq 0 \Leftrightarrow \det R_p(\phi) \neq 0 \Leftrightarrow R_p \in \operatorname{cok}\hat{\phi}^{\perp} \tag{3.3}$$

Note however that $\operatorname{cok}\hat{\phi}$ may be infinite dimensional.

Proof of corollary 1.1. Since the determinantal semi–invariant polynomial functions span all semi–invariant polynomial functions, there must exist some $\phi \in \operatorname{add}(Q)$ with the properties $P_{\phi,\beta}(p) \neq 0$, $P_{\phi,\beta}$ is not constant and $\hat{\phi}$ is injective. Then $\operatorname{cok}\hat{\phi} \in {}^{\perp}R_p$. If $\operatorname{cok}\hat{\phi} = 0$ then by (3.3) $P_{\phi,\beta}$ is nowhere vanishing, and hence constant by the Nullstellensatz. This is a contradiction, whence we may take $T = \operatorname{cok}\hat{\phi}$. □

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