GRADED MODULES OF GELFAND-KIRILLOV DIMENSION ONE OVER THREE-DIMENSIONAL ARTIN-SCHELTER REGULAR ALGEBRAS

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Abstract. Let A be a three dimensional Artin-Schelter regular algebra. We give a description of the category of finitely generated A-modules of Gelfand-Kirillov dimension one (modulo those of finite dimension over the ground field). The proof is based upon a result by Gabriel which says that locally finite categories can be described by module categories over topological rings.

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1. Introduction

Let us start with our main result. Afterwards we will indicate the application we have in mind. k is an algebraically closed field.

Theorem 1.1. *Let $\mathcal{A}$ be a k-linear locally noetherian Grothendieck category (that is, an abelian category which satisfies AB5 and has a family of noetherian generators). Let $G : \mathcal{A} \to \mathcal{A}$ be an autoequivalence and let $\eta : G \to \mathrm{id}_{\mathcal{A}}$ be a natural transformation such that $\eta(F)$ is surjective for every injective object in $\mathcal{A}$. Let $\mathcal{B}$ be the full subcategory of $\mathcal{A}$ consisting of objects $\mathcal{M}$ with $\eta(\mathcal{M}) = 0$ and let $\mathcal{C}_f$ be the full subcategory of $\mathcal{A}$ consisting of finite length objects whose composition factors lie in $\mathcal{B}$.*

Assume that every simple object in $\mathcal{B}$ has finite injective dimension in $\mathcal{A}$ and furthermore that there is a Cohen-Macaulay curve Y over k such that $\mathcal{B}$ is equivalent to $\mathrm{Qch}(Y)$, the category of quasi-coherent $\mathcal{O}_Y$-modules. For $x \in Y$ denote by $\mathcal{P}_x$ the object of $\mathcal{B}$ corresponding to x and define $\tau : Y \to Y$ by $G^{-1}(\mathcal{P}_x) = \mathcal{P}_{\tau x}$. Then we have the following.

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1. *$\mathcal{C}_f = \oplus_{z \in Y/\langle\tau\rangle} \mathcal{C}_{f,z}$, where $\mathcal{C}_{f,z}$ is the full subcategory of $\mathcal{C}_f$ consisting of objects whose Jordan-Holder quotients are given by $\mathcal{P}_y$ with $y \in O_\tau(z)$.*
2. *There is a category equivalence F between $\mathcal{C}_{f,z}$ and the category of finite dimensional right modules over a ring C_z. This ring C_z has the following form :*
   (a) *If $|O_\tau(z)| = \infty$ then C_z is given by the $\mathbb{Z} \times \mathbb{Z}$ lower triangular matrices with entries in $\hat{\mathcal{O}}_{Y,z}$. In this case z is regular on Y and thus we have $\hat{\mathcal{O}}_{Y,z} \cong k[[x]]$.*
   (b) *If $|O_\tau(z)| = n$ then C_z is given by a ring of $n \times n$ matrices of the form*

$$\begin{pmatrix} R & RU & \cdots & RU \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & RU \\ R & \cdots & \cdots & R \end{pmatrix}$$

where R is a complete local ring of the form

$$R = k\langle\langle x, y\rangle\rangle/(\psi)$$

with

$$\psi = yx - qxy + \text{higher terms} \tag{1.1}$$

for some $q \in k^*$, or

$$\psi = yx - xy - x^2 + \text{higher terms} \tag{1.2}$$

U is a regular normalizing element in $\operatorname{rad}(R)$ such that $R/(U) = \hat{\mathcal{O}}_{Y,z}$. If z is not fixed under τ then z is regular on Y and also $U \notin \operatorname{rad}^2(R)$.

3. *Let $I = \mathbb{Z}$ if $|O_\tau(z)| = \infty$ and $I = \mathbb{Z}/n\mathbb{Z}$ if $|O_\tau(z)| = n$. In this way the elements of C_z correspond to $I \times I$-matrices. For $i \in I$ let e_i be the corresponding diagonal idempotent. Then every finite dimension right C_z-representation W satisfies $W = \oplus_i We_i$.*
4. *Put $S_i = e_i C_z / \operatorname{rad}(e_i C_z)$. Then $F(\mathcal{P}_{\tau^i z}) = S_i$.*
5. *Define the following normal element N of C_z.*
   (a) *If $|O_\tau(z)| = \infty$ then N is given by the matrix whose entries are everywhere zero except on the lower subdiagonal where they are one.*
   (b) *If $|O_\tau(z)| < \infty$ then*

$$N = \begin{pmatrix} 0 & \cdots & 0 & U \\ 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

Let $\phi = N \cdot N^{-1}$ Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{f,z} & \xrightarrow{G} & \mathcal{C}_{f,z} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \operatorname{Mod}_r(C_z) & \xrightarrow{(-)_\phi} & \operatorname{Mod}_r(C_z) \end{array}$$

where $\operatorname{Mod}_r(C_z)$ denotes the category of right C_z-modules.

6. *If $\mathcal{M}$ is an object in $\mathcal{C}_{f,z}$ then one has the following commutative diagram.*

$$\begin{tikzcd} FG(\mathcal{M}) \arrow[r, "F(\eta(\mathcal{M}))"] \arrow[d, equal] & F(\mathcal{M}) \arrow[d, equal] \\ F(\mathcal{M})_\phi \arrow[r, "\cdot N"] & F(\mathcal{M}) \end{tikzcd}$$

7. *Let $\mathcal{C}_{f,z,Y}$ be the pullback of $\mathcal{C}_{f,z}$ in* $\mathrm{Qch}(Y)$. *Thus the objects of $\mathcal{C}_{f,z,Y}$ are the finite length object in* $\mathrm{Qch}(Y)$ *whose support is contained in the τ-orbit of z. Put $D_z = C_z/(N) = \prod_i \hat{\mathcal{O}}_{Y,\tau^i p}$. Let $\hat{(-)}_z$ be a shorthand for the product of the complection functors $\hat{(-)}_{\tau^i z}$. Then the following diagram is commutative.*

$$\begin{tikzcd} \mathcal{C}_{f,z,Y} \arrow[r, "\cong"] \arrow[d, "\hat{(-)}_z"'] & \mathcal{C}_{f,z} \arrow[d, "F"'] \\ \mathrm{Mod}_r(D_z) \arrow[r] & \mathrm{Mod}_r(C_z) \end{tikzcd}$$

From this theorem we can extract the following corollary.

Corollary 1.2. *If z is not a fixed point for τ then z is regular on Y.*

If we think of the curve Y as being embedded in a kind of non-commutative space $\mathcal{A}$ then Theorem 1.1 gives us some insight into the structure of $\mathcal{A}$ in a neigborhood of Y.

Now for the application. Recall that an (Artin-Schelter) regular algebra is a graded algebra $A = k + A_1 + \cdots$ with the following properties.

- A has polynomial growth.
- A has finite global dimension.
- A is "Gorenstein". That is, there is a number δ such that

$$\mathrm{Ext}^i_A(k_A, A) = \begin{cases} k & \text{if } i = \delta \\ 0 & \text{otherwise} \end{cases}$$

If A is generated in degree one and has global dimension three then according to [3] there are two possibilities :

- A has 2 generators in degree one and 2 relations in degree three.
- A has 3 generators in degree one and 3 relations in degree two.

Assume that A is a regular algebra of dimension three generated in degree one. In order to understand the geometrical properties of A, people have studied various classes of graded A-modules [5, 1]. In this note, in order to enhance our understanding of the graded module category of A, we will apply Theorem 1.1 to give a description of the category of finitely generated A-modules of Gelfand-Kirillov dimension one (modulo those of finite dimension over k).

We first introduce a few notions from the theory of graded rings. Assume that $A = k + A_1 + \cdots$ is a right noetherian graded ring. Let $\mathrm{Gr}(A)$ be the category of graded right A-modules. $\mathrm{Tors}(A) \subset \mathrm{Gr}(A)$ is the category of graded A-modules which are directed unions of right bounded ones. Following [2] we put $\mathrm{QGr}(A) = \mathrm{Gr}(A)/\mathrm{Tors}(A)$ and we let $\pi : \mathrm{Gr}(A) \to \mathrm{QGr}(A)$ be the quotient map. We denote by s the automorphism of $\mathrm{Gr}(A)$ which sends M to $M(1)$. s induces an automorphism on $\mathrm{QGr}(A)$ which we denote by the same letter. By $\mathrm{Proj}\, A$ we denote the couple $(\mathrm{QGr}(A), \pi A)$.

Let $(X, \sigma, \mathcal{L})$ be a triple where X is a noetherian scheme, $\sigma \in \operatorname{Aut}(X)$ and $\mathcal{L} \in \operatorname{Pic}(X)$. Define $B = \oplus_{n \in \mathbb{Z}} B_n$ with $B_n = \Gamma(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}})$. B is a graded ring with multiplication $a \cdot b = ab^{\sigma^m}$, $a \in B_m$, $b \in B_n$. We call B the twisted homogeneous coordinate ring associated to the triple $(X, \sigma, \mathcal{L})$ [6].

Let $\operatorname{Qch}(X)$ be the category of quasi-coherent $\mathcal{O}_X$-modules. We say that $\mathcal{L}$ is σ ample if the functor $s : \operatorname{Qch}(X) \to \operatorname{Qch}(X) : \mathcal{M} \mapsto \mathcal{M}^{\sigma^{-1}} \otimes \mathcal{L}^{\sigma^{-1}}$ has the property that for every coherent $\mathcal{M}$ one has $H^1(X, s^n\mathcal{M}) = 0$ and $s^n\mathcal{M}$ is generated by global sections for large n.

The key result of [6] is that we have functors

$$\begin{gathered} \operatorname{Gr}(B) \xrightarrow{\widetilde{(-)}} \operatorname{Qch}(X) \\ \operatorname{Qch}(X) \xrightarrow{\Gamma_*} \operatorname{Gr}(B) \end{gathered} \tag{1.3}$$

which, if $\mathcal{L}$ is σ-ample, factor through π to give inverse equivalences between $\operatorname{QGr}(B)$ and $\operatorname{Qch}(X)$.

Now let A be again a regular algebra of dimension three, generated in degree one. According to [4] A possesses a regular normalizing element g in degree three or four (depending on whether A has three or two generators) such that $B = A/(g)$ is a twisted homogeneous coordinate ring associated to a triple $(Y, \sigma, \mathcal{L})$ with Y a plane curve of arithmetic genus one and $\mathcal{L} = \mathcal{O}_Y(1)$.

Basic objects in the geometrical study of A are the so-called point modules. These are graded A-modules, generated in degree zero, which are one-dimensional over k in every degree. All such point modules are annihilated by g [4, 5] and hence it follows easily from the category equivalence (1.3) that they are of the form $P_x = \Gamma_*(k(x))$ for $x \in Y$. Put $\mathcal{P}_x = \pi P_x$. Let G be the autoequivalence of $\operatorname{Gr}(A)$ given by $- \otimes_A gA$ and denote by the same letter the induced autoequivalence on $\operatorname{QGr}(A)$. The natural transformation $\eta(\mathcal{M})$ is the obvious map $G(\mathcal{M}) \to \mathcal{M}$ obtained from the inclusion $gA \hookrightarrow A$.

It is now clear that the hypotheses for Theorem 1.1 are satisfied, whence we can apply that theorem in order to give a description of the category of A-modules of Gelfand-Kirilov dimension one modulo those of finite dimension.

Theorem 1.1 has also some significance for the study of "non-commutative surfaces" [7]. These are graded rings of Gelfand-Kirillov dimension three satisfying some suitable regularity properties. At present it is unclear why, but non-commutative surfaces which are sufficiently generic seem to posses a regular normalizing element of positive degree, such that factoring by it yields a twisted homogeneous coordinate ring of a Cohen-Macaulay curve (at least in high degree).

The proof of Theorem 1.1 is based upon a result by Gabriel [8] stating that locally finite categories are dual to pseudocompact rings. Pseudocompact rings are rings equipped a with an especially nice topology.

Sections 3-5 will be devoted to some generalities concerning pseudocompact rings. We are especially interested in the relationship between topological and non-topological properties of such rings. This should remove the anxiety some readers might experience when confronted with topological rings.

In sections 6,7 we give some classification theorems which are slightly more general than what we need for the proof of Theorem 1.1.

Finally in section 8 we give the proof of Theorem 1.1.

2. Notations and conventions

If C is a ring then $\mathrm{Mod}(C)$ refers to the category of left modules over C. The category of right modules is denoted by $\mathrm{Mod}_r(C)$. Note that in the introduction $\mathrm{Gr}(A)$ was used to denote the category of graded *right*-modules over a graded ring A. An unspecified module will always be a left module.

If M is a left module over a ring C and ϕ is an automorphism of C then ${}_\phi M$ is the left C-module which is equal to M as a set, but which has its multiplication twisted by $\phi : c \cdot m = \phi(c)m$. A similar notation is used for right modules.

3. Pseudocompact rings

Recall that a left topological module M over a topological ring A is *pseudocompact* if it is Hausdorf, complete and its topology is generated by left submodules of finite colength. A is said to be a *pseudocompact ring* if A is pseudocompact as a left A-module.

In the rest of this section A will be a pseudocompact ring. The category of pseudocompact modules over A is denoted by $\mathrm{PC}(A)$. It is an abelian category satisfying AB5* and AB3 [8]. Its dual category is a locally finite category. That is, a Grothendieck category possessing a set of generators of finite length.

Conversely assume that $\mathcal{C}$ is a locally finite category. If $M, N \in \mathcal{C}$ then the *natural* topology on $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is the linear topology generated by the subgroups of the form

$$\mathfrak{l}(S) = \{f : M \to N \mid f(S) = 0\}$$

where S runs through the objects of finite length in $\mathcal{C}$. The following result is proved in [8]

Theorem 3.1. *If E is an injective cogenerator for $\mathcal{C}$ then $A = \mathrm{End}_{\mathcal{C}}(E)$, equipped with the natural topology, is a pseudocompact ring, and the functor which sends $M \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(M, E)$ (with the natural topology) is an equivalence of categories between $\mathcal{C}$ and $\mathrm{PC}(A)^\circ$.*

One easy property of a linear topology will be used repeatedly below.

Lemma 3.2. *Assume that M is a topological group with a topology generated by subgroups and $L \subset M$ is an open subgroup. Then L is also closed and the quotient topology on M/L is discrete.*

Proof. L is the complement of the union of cosets of L in M which are not equal to L. Since this union is a union of open sets, it is itself open. Thus L is closed. L is the inverse image of $\bar{e}$ in M/L and hence $\{\bar{e}\}$ is open and closed in the quotient topology. $\square$

The following proposition records for further reference some of the properties of the forgetful functor $\mathrm{PC}(A) \to \mathrm{Mod}(A)$ [8].

Proposition 3.3. *The forgetful functor $\mathrm{PC}(A) \to \mathrm{Mod}(A)$ is faithful and commutes with kernels, cokernels and products. In particular it reflects isomorphism and exactness. If $M \in \mathrm{PC}(A)$ then the subobjects of M in $\mathrm{PC}(A)$ are in one-one correspondence with the subobjects of M in $\mathrm{Mod}(A)$ which are closed.*

Let $\mathrm{Fin}(A)$ be the full subcategory of $\mathrm{Mod}(A)$ consisting of objects which are of finite length and let $\mathrm{PCFin}(A)$ be its pullback in $\mathrm{PC}(A)$. A module of finite length carrying a linear topology can only be separated if its topology is discrete. So we conclude immediately that the forgetful functor

$$\mathrm{PCFin}(A) \to \mathrm{Fin}(A)$$

is fully faithful. The following lemma gives us more information on $\mathrm{PCFin}(A)$.

Lemma 3.4. 1. *An object in* $\mathrm{PC}(A)$ *is simple in* $\mathrm{PC}(A)$ *if and only if it is simple in* $\mathrm{Mod}(A)$.
2. *The objects in* $\mathrm{PCFin}(A)$ *are precisely the finite length objects in* $\mathrm{PC}(A)$.

Proof. 1. One direction is clear. For the other direction assume that $0 \neq S \in \mathrm{PC}(A)$ is simple in $\mathrm{PC}(A)$. We want to show that S is simple in $\mathrm{Mod}(A)$. Take $0 \neq x \in S$. Since S is Hausdorf there exists an open submodule $L \subset S$, not containing x. L is also closed (lemma 3.2) and thus it is pseudocompact if we give it the induced topology. Since S is simple in $\mathrm{PC}(A)$ we obtain $L = 0$. This implies that $S = S/L$ carries the discrete topology. But then every submodule of S is closed and thus is a subobject of S in $PC(A)$. Since S is simple in $\mathrm{PC}(A)$ there can be no non-trivial subobjects and thus S is simple in $\mathrm{Mod}(A)$.
2. This follows from 1. □

Let us say that $M \in \mathrm{PC}(A)$ is finitely generated in $\mathrm{PC}(A)$ if there is a surjective map $A^k \to M$ in $\mathrm{PC}(A)$ for some k.

Proposition 3.5. *Assume that* $M, N \in \mathrm{PC}(A)$, M *finitely generated. Then*

$$\mathrm{Hom}_{\mathrm{PC}(A)}(M, N) = \mathrm{Hom}_{\mathrm{Mod}(A)}(M, N)$$

Proof. If $m \in M$ then the map $a \mapsto am$ is continuous which yields

$$\mathrm{Hom}_{\mathrm{PC}(A)}(A, M) = \mathrm{Hom}_{\mathrm{Mod}(A)}(A, M)$$

This proves the proposition for $M = A$ and hence also for $M = A^k$.

Now assume M general. There is an exact sequence in $\mathrm{PC}(A)$

$$0 \to M' \to F \to M \to 0$$

with $F = A^k$. This yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{PC}(A)}(M, N) & \longrightarrow & \mathrm{Hom}_{\mathrm{PC}(A)}(F, N) & \longrightarrow & \mathrm{Hom}_{\mathrm{PC}(A)}(M', N) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{Mod}(A)}(M, N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Mod}(A)}(F, N) & \longrightarrow & \mathrm{Hom}_{\mathrm{Mod}(A)}(M', N)
\end{array}$$

The vertical maps are injective and the middle one is an isomorphism. It follows that the left map must be an isomorphism. □

Corollary 3.6. *An object in* $\mathrm{PC}(A)$ *is finitely generated in* $\mathrm{PC}(A)$ *if and only if it is finitely generated in* $\mathrm{Mod}(A)$.

Corollary 3.7. *A direct summand in* $\mathrm{Mod}(A)$ *of a finitely generated object in* $\mathrm{PC}(A)$ *is a direct summand in* $\mathrm{PC}(A)$. *In particular a finitely generated object in* $\mathrm{PC}(A)$ *is projective in* $\mathrm{PC}(A)$ *if and only if it is projective in* $\mathrm{Mod}(A)$.

Proof. If M is a finitely generated object in $\mathrm{PC}(A)$ then a direct summand of M is the image of an idempotent in $\mathrm{End}_{\mathrm{Mod}(A)}(M)$. The result now follows from Propositions 3.5, 3.3. □

Corollary 3.8. *If $M \in \mathrm{PC}(A)$ is finitely generated then a submodule $L \subset M$ is open if and only if $M/L \in \mathrm{PCFin}(A)$.*

Proof. If L is open then it is closed and of finite colength. Hence $M/L \in \mathrm{PCFin}(A)$. Conversely assume $M/L \in \mathrm{PCFin}(A)$. By Proposition 3.5 the quotient map $M \to M/L$ is continuous. Since M/L carries the discrete topology, $\bar{0} \in M/L$ is open and thus so is its inverse image L. □

Since $\mathrm{PC}(A)$ is the dual of a locally finite category, it has projective covers. The projective covers of the pseudocompact simples are the indecomposable projectives. Furthermore every projective in $\mathrm{PC}(A)$ is a product of such indecomposable projectives. By [8] the indecomposable projectives are of the form Ae, where e is a primitive idempotent in A.

Recall also from [8] that if $(e_i)_{i\in I}$ is a summable set of primitive, pairwise orthogonal idempotents with sum 1 then $A = \prod_{i\in I} Ae_i$ and every indecomposable projective in $\mathrm{PC}(A)$ is isomorphic to at least one Ae_i.

Lemma 3.9. *The $(Ae_i)_{i\in I}$ are the projective covers in $\mathrm{Mod}(A)$ of the simple A-modules which are pseudocompact.*

Proof. This follows from the fact that $\mathrm{End}_{\mathrm{Mod}(A)}(Ae_i) = \mathrm{End}_{\mathrm{PC}(A)}(Ae_i)$ is local since the Ae_i are projective covers of simple modules in $\mathrm{PC}(A)$. □

From Proposition 3.3 it follows that M is noetherian in $\mathrm{PC}(A)$ if it satisfies the ascending chain condition on closed subobjects. Thus if M is noetherian in $\mathrm{Mod}(A)$ then it is noetherian in $\mathrm{PC}(A)$. We now show that the converse holds.

Proposition 3.10. *Every subobject in $\mathrm{Mod}(A)$ of a noetherian object in $\mathrm{PC}(A)$ is closed and hence lies in $\mathrm{PC}(A)$.*

Proof. Let M be a noetherian object in $\mathrm{PC}(A)$. Since the $(Ae_i)_i$ form a set of generators for $\mathrm{PC}(A)$, M is a quotient of a direct sum of a finite number of such Ae_i, and in particular is finitely generated.

Assume now that $N \subset M$ in $\mathrm{Mod}(A)$. We have to show that $\bar{N} = N$. Since $\bar{N}$ is a closed submodule of M it is also noetherian in $\mathrm{PC}(A)$. Therefore, without loss of generality, we may assume that $\bar{N} = M$.

We want to show that $N = M$. Assume that this is not so. Consider the partially ordered set

$$\mathcal{P} = \{N \subset N' \subsetneq M \mid N' \in \mathrm{Mod}(A)\}$$

Since M is finitely generated, $\mathcal{P}$ has a maximal element by Zorn's lemma. Again without loss of generality we may replace N by this maximal element. In that case M/N is simple. However M/N is not pseudocompact since otherwise by Proposition 3.5 $N = \ker(M \to M/N)$ would be pseudocompact and hence closed which is impossible because $\bar{N} = M \neq N$.

Let $\phi : Ae_i \to M$ be a non-zero map. Then either $\phi^{-1}(N) = Ae_i$ or $Ae_i/\phi^{-1}(N)$ is simple but not pseudocompact. The last case is impossible since by lemma 3.9 Ae_i has only one simple quotient, and this simple quotient is pseudocompact. Thus N contains the image of every ϕ and therefore $N = M$. □

Corollary 3.11. *An object in $\mathrm{PC}(A)$ is noetherian in $\mathrm{PC}(A)$ if and only if it is noetherian in $\mathrm{Mod}(A)$.*

Corollary 3.12. *Assume that $M \in \mathrm{PC}(A)$ is noetherian. Then the topology on M is the cofinite topology. That is, a submodule $L \subset M$ is open if and only if M/L has finite length.*

Proof. One direction is clear. For the other direction, let L be a submodule of M of finite colength. By Proposition 3.10 L is closed in M. Hence M/L is pseudocompact and since it is of finite length it carries the discrete topology. Thus $\bar{0} \in M/L$ is open, and so is its inverse image L. □

Let R, I, M be respectively a ring, an ideal in R and an R-module. Then the *I-adic* topology on R is the linear topology generated by the submodules of M of the form $I^n M$. In a pseudocompact ring A the Jacobson radical $\mathrm{rad}(A)$ is the common annihilator of the simple pseudocompact A-modules [8, dual of Prop IV.12]. The following is a reformulation of the previous corollary.

Corollary 3.13. *Assume that $M \in \mathrm{PC}(A)$ is noetherian. Then the topology on M is given by the $\mathrm{rad}(A)$-adic topology.*

Proof. It suffices to show that $M/\mathrm{rad}(A)M$ is a finite sum of simples. To prove this we may replace M by $M/\mathrm{rad}(A)M$ and A by $A/\mathrm{rad}(A)$. Then by [8, dual op Prop. IV.12] A is a product of endomorphism rings of vectorspaces over division rings and M is still a noetherian A-modulde. Now one shows by direct verification that M must be a finite direct sum of simples. □

Definition 3.14.
- We say that A is *locally noetherian* if the $(Ae_i)_{i\in I}$ are noetherian in $\mathrm{PC}(A)$.
- We say that A is noetherian if A is noetherian in $\mathrm{PC}(A)$.

Proposition 3.15. *Let A be locally noetherian and $M, N \in \mathrm{PC}(A)$. Assume that M is noetherian. Then*

$$\mathrm{Ext}^i_{\mathrm{PC}(A)}(M,N) = \mathrm{Ext}^i_{\mathrm{Mod}(A)}(M,N)$$

Proof. The case $i = 0$ follows from Propositions 3.5. For $i > 0$ we use an exact sequence

$$0 \to M' \to P \to M \to 0$$

in $\mathrm{PC}(A)$ with P a finite direct sum of Ae_i. Since A is locally noetherian, M' is also noetherian. The proposition now follows by degree shifting. □

Proposition 3.15 yields the following corollary.

Corollary 3.16. *Let $\mathrm{pc}(A)$ resp. $\mathrm{mod}(A)$ be the full subcategories of $\mathrm{PC}(A)$ resp. $\mathrm{Mod}(A)$ consisting of noetherian objects. Then the functor*

$$\mathrm{pc}(A) \to \mathrm{mod}(A)$$

is fully faithful and its essential image is closed under extensions. In particular $\mathrm{PCFin}(A)$ is closed under extensions inside $\mathrm{Fin}(A)$.

Proposition 3.17. *If A is locally noetherian and $M \in \mathrm{PC}(A)$ is noetherian then*

$$\operatorname{proj\,dim}_{\mathrm{Mod}(A)} M = \operatorname{proj\,dim}_{PC(A)} M$$

Now we discuss briefly automorphisms of pseudocompact rings.

Lemma 3.18. *Assume that A is a pseudocompact ring. If $\phi \in \operatorname{Aut}(A)$ then ϕ is continuous if and only if for every pseudocompact A-module S of finite length, we have that ${}_\phi S$ is pseudocompact.*

Proof. "$\Rightarrow$" One needs that the left multiplication on ${}_\phi S$ (with the discrete topology) is continuous. Since ϕ is assumed to be continuous, this is clear.

"$\Leftarrow$" Assume that $L \subset A$ is an open ideal. Then A/L is pseudocompact of finite length and hence ${}_\phi(A/L) \cong A/\phi^{-1}(L)$ is pseudocompact of finite length. Thus $\phi^{-1}(L)$ is open in A. □

Corollary 3.19. *If A is locally noetherian and $\phi \in \operatorname{Aut}(A)$ then ϕ is a homeomorphism.*

Proof. It suffices to show that ϕ is continuous. By lemma 3.18 and Corollary 3.16 we must show that if S is pseudocompact simple, then so is ${}_\phi S$. This is clear since $S = Ae/\operatorname{rad}(Ae)$ for some primitive idempotent e and thus ${}_\phi S = A\phi^{-1}(e)/\operatorname{rad}(A\phi^{-1}(e))$. □

To close this section we discuss noetherian pseudocompact rings.

Proposition 3.20. *Let A be a noetherian pseudocompact ring. Then the forgetful functor*

$$\operatorname{pc}(A) \to \operatorname{mod}(A) \tag{3.1}$$

is an equivalence of categories.

Proof. By Corollary 3.16 We only have to show that (3.1) is essentially surjective. Let $M \in \operatorname{mod}(A)$. Then M has a resolution

$$F_1 \xrightarrow{\phi} F_0 \to M \to 0$$

where the F_i are finitely generated free A-modules. By Proposition 3.5, $\phi \in \operatorname{Hom}_{\operatorname{PC}(A)}(F_1, F_0)$. Therefore $M = \operatorname{coker}\phi \in \operatorname{PC}(A)$. □

Proposition 3.21. *Assume that A is a noetherian pseudocompact ring. Put $J = \operatorname{rad}(A)$. Then*

1. *A/J is semisimple.*
2. *A is complete for the J-adic topology.*
3. *The topology on A coincides with the J-adic topology.*

Conversely if A is a left noetherian ring satisfying 1.,2. then A is pseudocompact when equipped with the J-adic topology.

Proof. "$\Leftarrow$" Since A is semisimple and J^n is finitely generated, all J^n/J^{n+1} are finite direct sums of simples. Hence A/J^n has finite length for all n. Therefore 2. implies that A is pseudocompact.

"$\Rightarrow$" By [8] A/J is a product of endomorphism rings of vectorspaces. Since A/J is also noetherian, it must be semisimple. This proves 1. Property 2. follows from 3. and 3. is precisely Corollary 3.13. □

Proposition 3.22. *Let A be a pseudocompact ring, $N \in \operatorname{rad}(A)$ a regular normalizing element. Assume that $M \in \operatorname{PC}(A)$ is such that M/NM is fintely generated. Then M is finitely generated.*

Proof. The key point is that the N-adic topology on M and A is finer than the given topology. Thus if $(f_i)_i$ is a Cauchy sequence for the N-adic topology then it is convergent in the given topology. The standard proof now applies. For completeness we give it below.

Let $t_1, \dots, t_n \in M$ be such that $\bar{t}_1, \dots, \bar{t}_n$ generate M/NM. We show that $t_1, \dots, t_n$ generate M.

Take $t \in M$. Then there exist $a_1^{(1)}, \dots, a_n^{(1)} \in A$, $t^{(1)} \in M$ such that

$$t - \sum_i a_i^{(1)} t_i = N t^{(1)}$$

Continuing this procedure we find $a_i^{(j)} \in A$ such that

$$t - \left(\sum_i a_i^{(1)} t_i + \sum_i N a_i^{(2)} t_i + \dots + \sum_i N^{p-1} a_i^{(p)} t_i \right) \in N^p M$$

and hence

$$t = \sum_i \sum_j (N^{j-1} a_i^{(j)}) t_i$$

Thus M is generated by $t_1, \dots, t_n$. □

From this we deduce the following.

Proposition 3.23. *Let A be a pseudocompact ring, $N \in \mathrm{rad}(A)$ a regular normalizing element. Assume that $M \in \mathrm{PC}(A)$ is such that M/NM is noetherian. Then M is noetherian.*

Proof. This is again a standard proof.

Let $T \subset M$ be an arbitrary submodule. We want to show that T is finitely generated. We define first

$$T^{\mathrm{sat}} = \{ t \in M \mid \exists k : N^k t \in T \}$$

Obviously $(T^{\mathrm{sat}})^{\mathrm{sat}} = T^{\mathrm{sat}}$ and $T^{\mathrm{sat}} \cap N^p M = N^p T^{\mathrm{sat}}$. Since $T^{\mathrm{sat}}/NT^{\mathrm{sat}} \subset M/NM$ we find that $T^{\mathrm{sat}}/NT^{\mathrm{sat}}$ is finitely generated and hence so is T^{sat} by the previous proposition.

Since T^{sat} is finitely generated, it follows from the definition of T^{sat} that there exists a k such that $N^k T^{\mathrm{sat}} \in T$. Thus $T^{\mathrm{sat}} \cap N^k M = N^k T^{\mathrm{sat}} \subset T$. Since $N^k T^{\mathrm{sat}}$ is finitely generated it now suffices to show that $T/(T^{\mathrm{sat}} \cap N^k M)$ is finitely generated. Now clearly, $M/N^n M$ is noetherian and hence so is the subobject $T/(T^{\mathrm{sat}} \cap N^k M)$. □

4. A matrix representation for pseudocompact rings

If A is an arbitrary ring, M a left A-module and $(e_i)_{i=1,\dots,n}$ a finite set of pairwise orthogonal idempotents with sum 1 then it is classical that A is isomorphic to the matrix ring

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}$$

with $A_{ij} = e_i A e_j$ and M is isomorphic to the set of column vectors $(M_1, \dots, M_n)^t$ with $M_j = e_j M$.

An element a in A is sent to the matrix $(e_i a e_j)_{ij}$ and an element m of M is sent to $(e_j m)_j$. It is clear how to extend this result to the pseudocompact situation.

Lemma 4.1. *Assume that $(e_i)_{i\in I}$ is a summable set of orthogonal idempotents in a pseudocompact ring A such that $\sum_i e_i = 1$. Let M be a pseudocompact A-module and put $A_{ij} = e_i A e_j$, $M_i = e_i M$. Then A is isomorphic to the ring of doubly infinite matrices $(a_{ij})_{ij} \in (A_{ij})_{ij}$ with summable columns and M is isomorphic to the set of summable column vectors $(m_i)_i \in (M_i)_i$. The isomorphisms are given by the maps*

$$(a_{ij})_{ij} \mapsto \sum_{ij} a_{ij}$$
$$(m_i)_i \mapsto \sum_i m_i$$

Note that this lemma only says something about the ring structure on A and the module structure on M, but nothing on the topology.

Below we give the A_{ij} the topology induced from A and M_i the topology induced from M. Since $A_{ij} = e_i A e_j$ is clearly closed in A, it is complete. A similar result is true for M_i. Furthermore the topology is linear (given by abelian subgroups). We also have multiplication mappings

$$A_{ij} \times A_{jk} \to A_{ik}$$
$$A_{ij} \times M_j \to M_i$$

and since these are induced from the multiplication on A and M they are continuous. This makes A_{ii} into a topological ring, A_{ij} into a topological A_{ii}-A_{jj}-bimodule and M_i into a left topological A_{ii}-module.

Lemma 4.2.
- *A_{ii} is a pseudocompact ring and A_{ij} is a pseudocompact A_{ii}-module.*
- *M_i is a pseudocompact A_{ii}-module.*

Proof. It suffices to prove 2. Indeed if we take $M = Ae_j$ (and afterwards $j = i$) then we obtain part 1.

Let $L \subset M$ be an open submodule. If T is an A_{ii} submodule of $e_i M$ containing $L \cap e_i M = e_i L$ then $\tilde{T} = AT + L$ is an A-submodule of M containing L, and furthermore $\tilde{T} \cap e_i M = T$. This yields that the length of $e_i M/(e_i M \cap L)$ is bounded by that of M/L. Since the topology on $e_i M$ is induced from that on M we deduce that $e_i M$ is pseudo-compact. □

Unfortunately it is not in general true that A and M carry the induced topology from the product topologies on $\prod_{ij}(A_{ij})$ on $\prod_i M_i$. A counter example is given by the endomorphism ring of an infinite dimensional vector space.

Under some mild extra hypotheses this defect can be repaired. Note that the Ae_i are pseudocompact projectives. Hence they are products of indecomposable pseudocompact projectives.

Proposition 4.3. *Let $(e_i)_{i\in I}$ be as in the previous lemma. Assume that every indecomposable pseudocompact projective is a summand of at most a finite number* *of Ae_i. Then as topological spaces.*

$$A = \prod_{i,j} A_{ij} \tag{4.1}$$

$$M = \prod_i M_i \tag{4.2}$$

Proof. We certainly have $A = \prod_j Ae_j$. Hence it suffices to prove (4.2). We have an inclusion

$$M \subset \prod_i M_i \tag{4.3}$$

which is given by the product of the maps $M \to M_i : m \mapsto e_i m$ These maps are continuous and hence the inclusion is also continuous.

We now show that the topology on M is courser than the induced topology for the inclusion (4.3). Let $L \subset M$ be an open submodule. Since M/L has finite length the hypotheses imply that $\operatorname{Hom}_A(Ae_i, M/L)$ is non-zero for at most a finite number of i. Since $\operatorname{Hom}(Ae_i, M/L) = e_i(M/L)$ we deduce that for almost all i, $M_i = e_i M \subset L$. Hence $\prod_i (M_i \cap L)$ is open in $\prod_i M_i$. Observing that $M \cap \prod_i (M_i \cap L) \subset L$ finishes the proof.

The proof we have just given also shows that if $(m_i)_i \in \prod M_i$ then $(m_i)_i$ is summable in M. Sending $(m_i)_i$ to $\sum_i m_i$ defines an inverse to the inclusion (4.3). A similar result holds for A. □

5. Global dimension

In this section A is a pseudocompact ring. We define

$$\operatorname{gl}\dim A = \sup_{M \in \mathrm{PC}(A)} \operatorname{proj}\dim_{PC(A)}(M)$$

Note that by Proposition 3.17 we often have $\operatorname{proj}\dim_{\mathrm{PC}(A)} M = \operatorname{proj}\dim_{\mathrm{Mod}(A)} M$. Therefore, if no confusion can arise, we make no distinction between those two types of projective dimension, and we simply write $\operatorname{proj}\dim M$.

Lemma 5.1. *We also have*

$$\operatorname{gl}\dim(A) = \sup_{\substack{S \in \mathrm{PC}(A) \\ S \text{ simple}}} \operatorname{proj}\dim_{\mathrm{PC}(A)} S$$

Proof. By Theorem 3.1 it suffices to prove the dual statement for locally finite categories. So assume that $\mathcal{C}$ is such a category and $\operatorname{inj}\dim S \le n$ for every simple S in $\mathcal{C}$. Hence for every finite length module F one also has $\operatorname{inj}\dim F \le n$. If $M \in \mathcal{C}$ is arbitrary then by definition M is a direct limit of finite length objects. By the proof [9, Thm 1.10.1] monomorphisms into injectives can be constructed in a functorial way and hence so can injective resolutions. Taking the direct limit of the injective resolutions of the subobjects of finite length of M yields and injective resolution of M of length $\le n$ ($\mathcal{C}$ is locally noetherian and hence a direct limit of injectives is injective). □

The following result is very classical.

Proposition 5.2. *Let $N \in \operatorname{rad}(A)$ be a regular normalizing element in A. Assume that A is locally noetherian. Then*

$$\operatorname{gl}\dim A \le \operatorname{gl}\dim A/(N) + 1$$

Proof. Let $\operatorname{gl}\dim A/(N) = p$. We have to show that $\operatorname{proj}\dim S \le p+1$ for every pseudocompact simple or what amounts to the same

$$\operatorname{proj}\dim \operatorname{rad}(P) \le p$$

where P runs through the indecomposable projectives in $\mathrm{PC}(A)$. This follows form lemma 5.3 below. □

Lemma 5.3. *Assume that A is locally noetherian and let L be a noetherian pseudocompact A-module which is N-torsion free. Then*

$$\operatorname{proj}\dim L = \operatorname{proj}\dim_{A/(N)} L/NL$$

Proof. By degree shifting one reduces to the case where L/NL is projective over $A/(N)$. In that case the result follows by an appropriate version of Nakayama's lemma. □

The following type of result seems to be referred to less often.

Proposition 5.4. *Assume that A is a locally noetherian and $N \in \operatorname{rad}(A)$ is a regular normalizing element such that for every indecomposable pseudocompact projective one has $NP \not\subset \operatorname{rad}^2(P)$. Then*

$$\operatorname{gl}\dim A/(N) + 1 \le \operatorname{gl}\dim A \tag{5.1}$$

Proof. This is an immediate generalization of the proof by Serre that local rings of finite global dimension are regular. Let $\phi = N \cdot N^{-1}$. By Corollary 3.19 ϕ is a homeomorphism.

Let P be an indecomposable pseudocompact projective over A with cosocle S. We have an inclusion

$${}_{\phi^{-1}}S \cong NP/N\operatorname{rad}(P) \hookrightarrow \operatorname{rad}(P)/N\operatorname{rad}(P) \tag{5.2}$$

Now we also have $NP \cap \operatorname{rad}^2(P) = N\operatorname{rad}(P)$ and thus there is an inclusion

$$NP/N\operatorname{rad}(P) \hookrightarrow \operatorname{rad}(P)/\operatorname{rad}^2(P) \tag{5.3}$$

$\operatorname{rad}(P)/\operatorname{rad}^2(P)$ is a finite sum of simples and hence this inclusion splits. From the commutative diagram

$$\begin{tikzcd} NP/N\operatorname{rad}(P) \arrow[r] \arrow[d, equal] & \operatorname{rad}(P)/N\operatorname{rad}(P) \arrow[d] \\ NP/N\operatorname{rad}(P) \arrow[r] & \operatorname{rad}(P)/\operatorname{rad}^2(P) \end{tikzcd}$$

we deduce that (5.2) is also split. Thus

$$\begin{aligned} \operatorname{proj}\dim_A S - 1 &= \operatorname{proj}\dim_A \operatorname{rad}(P) \\ &= \operatorname{proj}\dim_{A/(N)} \operatorname{rad}(P)/N\operatorname{rad}(P) \qquad \text{(lemma5.3)} \\ &\ge \operatorname{proj}\dim_{A/(N)} {}_{\phi^{-1}}S \end{aligned}$$

Taking the supremum over all S yields (5.1). □

6. A classification problem

Let I be either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$. In this section we aim to classify the following data.

(A) A pseudocompact ring A with a summable set of primitive orthogonal idempotents $(e_i)_{i\in I}$ such that $\sum_i e_i = 1$ and $Ae_i \not\cong Ae_j$ for $i \neq j$.

(B) A regular normalizing element $N \in \operatorname{rad}(A)$, inducing a homeomorphism $\phi = N \cdot N^{-1}$ such that $\phi(e_i) = e_{i+1}$ and such that the image of the $(e_i)_i$ becomes central in $B = A/(N)$.

The solution to this classification problem is the following.

Proposition 6.1. 1. *If $|I| = \mathbb{Z}$ then A is isomorphic to the ring $T_I(R)$ of lower triangular $I \times I$-matrices with entries in a local pseudocompact ring R. The topology on $T_I(R)$ is the product topology. Under the isomorphism the e_i correspond to the diagonal idempotents and N corresponds to the matrix in which every entry is zero except those on the lower subdiagonal, which are one.*

2. *If $|I| = \mathbb{Z}/n\mathbb{Z}$ then A is isomorphic to a ring of $n \times n$-matrices of the form*

$$\begin{pmatrix} R & UR & \cdots & \cdots & UR \\ \vdots & R & UR & \cdots & UR \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & UR \\ R & R & \cdots & \cdots & R \end{pmatrix} \tag{6.1}$$

where R is a local pseudocompact ring and U is a normalizing element in $\operatorname{rad}(R)$ inducing a homeomorphism $U \cdot U^{-1}$. The topology on (6.1) is the product topology. Under the isomorphism of A with (6.1) the $(e_i)_i$ correspond to the diagonal idempotents and N corresponds to the matrix

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & U \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{6.2}$$

Proof. It is clear that the rings exhibited in 1.,2. and the corresponding N, $(e_i)_{i\in I}$ satisfy (A)(B), so we only have to be concerned with the converse. To simplify the notations we put $P_i = Ae_i$ and S_i will be the unique simple quotient of P_i.

Recall that by Proposition 4.3 we have a matrix form $A = (A_{ij})_{ij}$ and we also have $N = \sum e_{i+1}Ne_i = \sum N_{i+1i}$. Since N is regular we have injections

$$A_{ij} \xrightarrow{\cdot N_{jj-1}} A_{i,j-1} \tag{6.3}$$

$$A_{ij} \xrightarrow{N_{i+1i}\cdot} A_{i+1,j} \tag{6.4}$$

and furthermore since A/NA is diagonal (6.3)(6.4) are isomorphisms for $i \neq j-1$.

Left and right multiplication by N are continuous. Furthermore since A is pseudocompact it follows that the quotient topology on AN, for right multiplication with N, coincides with the induced topology. We claim that that is also the case for left multiplication with N. Indeed left multiplication by N is the composition $A \xrightarrow{\cdot N} A \xrightarrow{\phi} A$ The fact that ϕ is a homeomorphism shows what we want.

Since (6.3)(6.4) are restrictions from left and right multiplication by N they are continuous. Furthermore the topology on the image coincides with the induced topology. This means in particular that if $i \neq j-1$ then they are homeomorphisms. The fact that N is normalizing also implies

$$N_{i+1,i}A_{ij} = A_{i+1,j+1}N_{j+1,j}$$

We will first consider the case $|I| < \infty$. We define for $i = 0, \dots, n-1$, $N_i = N_{i,i-1} \cdots N_{1,0}$ and maps

$$\theta_{ij} : A_{ij} \to A_{00} : a \mapsto N_i^{-1} a N_j$$

Note that $A_{ij}N_j \subset A_{i0}$ and hence N_i defines a homeomorphism $A_{00} \to A_{i0}$. Thus it makes sense to use N_i^{-1}.

Clearly if $a \in A_{ij}$, $b \in A_{jk}$ then $\theta_{ij}(a)\theta_{jk}(b) = \theta_{ik}(ab)$ and hence $\theta = (\theta_{ij})_{ij}$ defines an inclusion of A into $M_n(A_{00})$. We want to understand its image.

If $i \geq j$ then θ_{ij} is a homeomorphism and thus $\theta_{ij}(A_{ij}) = A_{00}$. Hence we look at the case $i < j$. We have

$$\theta_{ij}(A_{ij}) = N_i^{-1}A_{ij}N_j = A_{0j}N_j = A_{0,n-1}N_{n-1,0} = A_{00}U$$

with $U = N_{0,n-1}N_{n-1,n-2} \cdots N_{1,0}$. It is easy to see that $UA_{00} = A_{00}U$ and thus U is a regular normalizing element in A_{00}. Hence (putting $R = A_{00}$)

$$\theta(A) = \begin{pmatrix} R & UR & \cdots & \cdots & UR \\ \vdots & R & UR & \cdots & UR \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & UR \\ R & R & \cdots & \cdots & R \end{pmatrix}$$

From the definition of U it is clear that $U \cdot U^{-1}$ is a homeomorphism.

One computes

$$\theta(N)_{ij} = N_i^{-1}N_{ij}N_j = \begin{cases} 0 & \text{if } i \neq j+1 \\ 1 & \text{if } i = j+1,\ j \neq n-1 \\ U & \text{if } i = 0, j = n-1 \end{cases}$$

If $U \notin \operatorname{rad}(R)$ then it is easily seem that $N \notin \operatorname{rad}(A)$ and thus (B) would be violated.

Let us now consider the case $|I| = \infty$. The following three lemmas are standard.

Lemma 6.2. $\operatorname{Ext}^i_A(S_i, S_j) = 0$ *for* $i \neq j, j-1$ *(the hypothesis* $|I| = \infty$ *is not used here).*

Proof. Assume $\operatorname{Ext}^1_A(S_i, S_j) \neq 0$. Then there is a non-trivial extension

$$0 \to S_j \to F \to S_i \to 0 \tag{6.5}$$

There are now two possibilities

- $NF = 0$. In this case (6.5) is an extension as B-modules and thus $i = j$.

- $NF \neq 0$. In this case multiplication by N defines a non-trivial map $S_i \to {}_\phi S_j$ and since S_i, ${}_\phi S_j$ are simple, this map must be an isomorphism. Thus $S_i = {}_\phi S_j$. Now ${}_\phi S_j$ is a simple quotient of ${}_\phi Ae_j = Ae_{j-1}$. Thus ${}_\phi S_j = S_{j-1}$ and we find $i = j - 1$. □

Lemma 6.3. *Assume $|I| = \infty$. Let M be a finite length module in* $\mathrm{mod}(A)$ *with composition factors among the $(S_j)_j$. Assume $M/\operatorname{rad}(M) = S_i$. Then the composition factors of M are of the form S_k, $k \geq i$.*

Proof. We prove this by induction on the length of M. Let $S_t \subset M$ be a simple submodule. By induction the subquotients of M/S_t are of the form S_k, $k \geq i$.

Hence if $t < i$, then it follows from lemma 6.2 that $\mathrm{Ext}^1(M/S_t, S_t) = 0$ and thus $M = M/S_t \oplus S_t$. In particular S_t is a simple quotient of M, different from S_i, contradicting the hypotheses. □

Lemma 6.4. *Assume $|I| = \infty$. Then* $\mathrm{Hom}(P_i, P_j) = 0$ *for $i < j$.*

Proof. Assume there is a non-zero map $P_i \xrightarrow{\psi} P_j$. Since P_j is separated there exists an open submodule $L \subsetneq P_j$ such that $P_i/\psi^{-1}(L)$ is not zero. P_j/L has finite length and is equal to S_j modulo its radical. $P_i/\psi^{-1}(L)$ is a subobject of P_j/L and since S_i is a quotient of $P_i/\psi^{-1}(L)$ it follows that S_i is a subquotient of P_j/L. This implies that $i \geq j$ by lemma 6.3 and we are done. □

Since $\mathrm{Hom}_A(P_i, P_j) = e_i A e_j = A_{ij}$ this last lemma implies that in the case $|I| = \infty$ the matrix form for A is lower triangular.

For every (i, j), $i \geq j$ there is a homeomorphism

$$\theta_{ij} : A_{ij} \to A_{00}$$

obtained by composing homeomorphisms of the form (6.3)(6.4). One checks that θ_{ij} is uniquely determined in this way. By a verification as in the case $|I| < \infty$ (but somewhat more complicated) one also shows that θ_{ij} is compatible with multiplication and $\theta_{i+1,i}(N_{i+1,i}) = 1$.

Thus A is isomorphic to the ring of lower triangular matrices with entries in $R = A_{00}$ and N has the required form. This finishes the proof of Proposition 6.1. □

We now exhibit when pseudocompact rings as in Proposition 6.1 are locally noetherian and have finite global dimension.

Proposition 6.5. *Let A be a pseudocompact ring as in Proposition 6.1. Then A is locally noetherian if and only if R is noetherian. Furthermore if R is noetherian then*

$$\operatorname{gl\,dim} A = \begin{cases} \operatorname{gl\,dim} R + 1 & \text{if } |I| = \infty \\ \operatorname{gl\,dim} R/(U) + 1 & \text{if } 2 \leq |I| < \infty \\ \operatorname{gl\,dim} R & \text{if } |I| = 1 \end{cases} \tag{6.6}$$

Proof. We have

$$A/(N) = \begin{cases} R/(U)^I & \text{if } |I| < \infty \\ R^I & \text{if } |I| = \infty \end{cases} \tag{6.7}$$

The condition for A to be noetherian then follows from Proposition 3.23.

The statement about global dimension is clear in the case that $|I| = 1$. For $|I| > 1$ one verifies that N satisfies the hypotheses of Propositions 5.2,5.4. Then (6.6) follows from (6.7) □

7. More classification

In this section we classify rings R satisfying

(C) R is local, complete and contains an algebraically closed field k, isomorphic to its residue field.

(D) Let m be the maximal ideal of R. We require that m contains a regular normalizing element U such that $R/(U)$ is a commutative noetherian Cohen-Macaulay local ring of Krull dimension one.

(E) $\operatorname{proj\,dim}_R R/m < \infty$.

The solution to this classification problem is as follows.

Proposition 7.1. *Assume that R satisfies (C)(D)(E) above. Then*

$$R \cong k\langle\langle x, y\rangle\rangle/(\phi) \tag{7.1}$$

where

$$\phi = yx - qxy + \text{higher order terms} \tag{7.2}$$

for some $q \in k^*$ or

$$\phi = yx - xy - x^2 + \text{higher order terms} \tag{7.3}$$

Conversely every such ring satisfies (C)(D)(E).

Proof. Let us first show that a ring R of the form (7.1) with ϕ of the form (7.2)(7.3) does indeed satisfy (C)(D)(E).

It is clear that (C) is satisfied. For (E) observe that we have $\phi = ux + vy$ for some $u, v \in m$ such that $(\bar{u}, \bar{v})$ form a basis of m/m^2. This means that we have a complex

$$0 \to R \xrightarrow{(u\ v)} R^2 \xrightarrow{\binom{x}{y}} R \to R/m \to 0 \tag{7.4}$$

and we have to show that this complex is exact. We filter R with the m-adic filtration. For this filtration it is easy to see that

$$\operatorname{gr} R = k[x, y]/(\theta) \tag{7.5}$$

where θ consists of the quadratic part of ϕ.

The exactness of (7.4) now follows from the exactness of

$$0 \to \operatorname{gr} R \xrightarrow{(\bar{u}\ \bar{v})} (\operatorname{gr} R)^2 \xrightarrow{\binom{\bar{x}}{\bar{y}}} \operatorname{gr} R \to R/m \to 0 \tag{7.6}$$

which is standard. This proves (E).

Now let us consider (D). We assume that R is not commutative since otherwise (D) is trivial. From (7.5) it follows that R is a domain, so every element of R is regular. Put $U = [y, x]$. We claim that U is normalizing. This was independently observed by Artin and Stafford. Assume first that we are in case (7.2). One computes

$$\begin{aligned} Ux &= qxU + [x, \gamma] \\ Uy &= q^{-1}yU + [y, \gamma] \end{aligned} \tag{7.7}$$

where γ represents the non-quadratic terms of ϕ. Now clearly

$$\begin{aligned} [x,\gamma] &= \sum_i u_i U v_i \\ [y,\gamma] &= \sum_i u_i' U v_i' \end{aligned} \tag{7.8}$$

for appropriate $u_i, v_i, u_i', v_i' \in R$. Substituting (7.8) into (7.7) and then substituting the resulting equations repeatedly into themselves yields the formulas

$$\begin{aligned} Ux &= (qx + \cdots)U \\ Uy &= (q^{-1}y + \cdots)U \end{aligned}$$

Thus U is a normalizing element.

Case (7.3) is treated similarly, starting from

$$\begin{aligned} Ux &= xU + [x,\gamma] \\ Uy &= (y+2x)U + [x,\gamma] + [y,\gamma] \end{aligned}$$

Since $R/[y,x] = k[[x,y]]/(\phi)$ is clearly Cohen-Macaulay of Krull dimension one, we have shown that R satisfies (D).

Now we prove the converse. Note that by (D), R is automatically left and right noetherian.

Step 1. $\operatorname{proj\,dim}_R R/m = 2$.

Proof. We have

$$\begin{aligned} \operatorname{proj\,dim}_R R/m &= 1 + \operatorname{proj\,dim}_R m \\ &= 1 + \operatorname{proj\,dim}_{R/(N)} m/Nm \qquad \text{(lemma 5.3)} \end{aligned}$$

In particular $\operatorname{proj\,dim}_{R/(N)} m/Nm$ is finite. Since $R/(N)$ is commutative of Krull dimension one this implies $\operatorname{proj\,dim}_{R/(N)} m/Nm \leq 1$. Thus $\operatorname{proj\,dim}_R R/m \leq 2$.

Assume that the projective dimension of R/m is strictly less than 2. It cannot be 0, hence it must be one. This means that there is a resolution

$$0 \to R^n \to R \to R/m \to 0$$

which easily yields that R is the completion of a free k-algebra in n variables. If $n > 1$ then R is not noetherian and if $n = 1$ then R is a discrete valuation ring and hence (D) not satisfied. $\square$

Step 2. The minimal resolution of R/m looks like

$$0 \to R \xrightarrow{(u\ v)} R^2 \xrightarrow{\binom{x}{y}} R \to R/m \to 0$$

where $(\bar{x}, \bar{y})$, $(\bar{u}, \bar{v})$ form bases for m/m^2.

Proof. The minimal resolution of R/m looks like

$$0 \to R^b \xrightarrow{g} R^a \xrightarrow{f} R \to R/m \to 0 \tag{7.9}$$

Tensoring with $R/(N)$ and taking ranks yields that $a = b+1$.

We have

$$\begin{aligned} \operatorname{Ext}^i_R(R/m, R) &= \operatorname{Ext}^{i-1}_{R/(U)}(R/m, R/(U)) \\ &= 0 \qquad \text{for } i \leq 1 \end{aligned}$$

and thus by dualizing (7.9) we find a minimal resolution of $\mathrm{Ext}^2_R(R/m, R)$ as right R-module :

$$0 \to R \xrightarrow{f^t} R^a \xrightarrow{g^t} R^b \to \mathrm{Ext}^2_R(R/m, R) \to 0 \tag{7.10}$$

Now $\mathrm{Ext}^2_R(R/m, R)$ is annihilated by m and thus $\dim_k \mathrm{Ext}^2_R(R/m, R) = b$. Hence if $b \neq 1$ then we see that (7.10) decomposes as a direct sum of subcomplexes. But then so does the dual complex (7.9), which is impossible since this is a minimal projective resolution of a simple R-module. We conclude that $b = 1$, $a = 2$.

We now find that the minimal resolution of R/m looks like (7.9) with $(\bar{x}, \bar{y})$ a basis for m/m^2. Since the dual complex of (7.9) is a minimal resolution of R/m (as right module) we find that $(\bar{u}, \bar{v})$ is also a basis for m/m^2. □

We can now conclude the proof of Proposition 7.1. From Step 2 it follows that R is as in (7.1) with $\phi = ux + vy$. It is now easy to see that ϕ can be put in one of the standard forms (7.2)(7.3). □

Proposition 7.2. *Let R, m, U be as in Proposition 7.1. Then $R/(U)$ is regular iff $U \notin m^2$.*

Proof. If $U \in m/m^2$ then by Proposition 5.4 $R/(U)$ is regular. Conversely assume that $R/(U)$ is regular. Then $1 = \dim_k (m/(U))/(m/(U))^2 = m/((U)+m^2)$, whence $U \notin m^2$. □

Remark 7.3. This result is false in higher dimension. Consider for example

$$R = k\langle\langle x, y\rangle\rangle([x, [x, y]], [y, [x, y]])$$

Then $R/([x, y]) = k[[x, y]]$ is regular, but $[x, y] \in \mathrm{rad}^2(R)$.

8. Proof of Theorem 1.1

We start by discussing things a bit more generally. Let $\mathcal{A}$ be a Grothendieck category, $G : \mathcal{A} \to \mathcal{A}$ an autoequivalence and $\eta : G \to \mathrm{id}_{\mathcal{A}}$ a natural transformation such that

$$\eta(G(A)) = G(\eta(A)) \tag{8.1}$$

for all $A \in \mathcal{A}$. Define

$$\mathcal{B} = \{A \in \mathcal{A} \mid \eta(A) = 0\}$$

Then the following properties are easily verified.

Lemma 8.1. 1. *$\mathcal{B}$ is closed under subquotients, direct sums and direct products (and hence under limits and colimits).*
2. *$\mathcal{B}$ is closed under G, G^{-1} and if $A \in \mathcal{A}$ then $\ker, \operatorname{coker} \eta(A) \in \mathcal{B}$.*
3. *Let $i_* : \mathcal{B} \to \mathcal{A}$ be the inclusion functor. The functors $i^!, i^* : \mathcal{A} \to \mathcal{B}$ defined by*

$$i^!(A) = \ker(A \xrightarrow{\eta(G^{-1}(A))} G^{-1}(A))$$
$$i^*(A) = \operatorname{coker}(G(A) \xrightarrow{\eta(A)} A)$$

are respectively the right and the left adjoint to i_*.

Remark 8.2. The condition (8.1) is not automatic. A counter example is given by $\mathcal{A} = \mathrm{Mod}(A)$ with $A = k \oplus V$, where V is a k-vectorspace such that $V^2 = 0$ in A. For G we take $M \mapsto {}_{\psi}M$ for some $\psi \in \mathrm{GL}(V)$, which we extend in the obvious way to A. To define η we take $v \in V$, not ψ-invariant and we define $\psi : {}_{\psi}A \to A$ as the bimodule map which sends 1 to v. Then we put $\eta(-) = \phi \otimes_A -$. In this case $G(\eta(A)) \neq \eta(G(A))$, and in particular $\mathcal{B}$ is not G-invariant.

Nevertheless (8.1) holds in the case we are interested in as the following lemma shows.

Lemma 8.3. *Assume that for all injectives $E \in \mathcal{A}$ we have that $\eta(E)$ is surjective. Then* (8.1) *holds.*

Proof. We have that ηG is a natural transformation $G^2 \to G$. Applying this to the map $G^{-1}\eta G : GE \to E$ we get a commutative diagram (using $E = G(G^{-1}(E))$)

$$\begin{tikzcd}
G^2E \arrow[r, "\eta(GE)"] \arrow[d, "\eta(GE)"'] & GE \arrow[d, "\eta(E)"'] \\
GE \arrow[r, "G^{-1}\eta(GE)"'] & E
\end{tikzcd}$$

Applying this diagram with E injective and using the surjectivity hypothesis we find that $G^{-1}(\eta(GE)) = \eta(E)$.

Now let $A \in \mathcal{A}$ be arbitrary and let

$$0 \to A \to E \to F$$

be an injective resolution of A. This yields commutative diagrams.

$$\begin{tikzcd}
0 \arrow[r] & GA \arrow[r] & GE \arrow[r] & GF \\
0 \arrow[r] & G^2A \arrow[r] \arrow[u, "G(\eta(A))"] & G^2E \arrow[r] \arrow[u, "G(\eta(E))"] & G^2F \arrow[u, "G(\eta(V))"]
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & GA \arrow[r] & GE \arrow[r] & GF \\
0 \arrow[r] & G^2A \arrow[r] \arrow[u, "\eta(G(A))"] & G^2E \arrow[r] \arrow[u, "\eta(G(E))"] & G^2F \arrow[u, "\eta(G(F))"]
\end{tikzcd}$$

The fact that the rightmost squares of these diagrams are commutative yields the result in general. $\square$

Now let $\mathcal{D} \subset \mathcal{B}$ be a G, G^{-1}-stable localizing subcategory (that is, closed under subquotients, extensions and direct sums) and define $\mathcal{D}^\infty$ as the full subcategory of $\mathcal{A}$ consisting of objects A having an ascending filtration $(F_iA)_{i=0,\dots,n}$ such that

$$F_0A = 0, \qquad F_{n+1}A/F_nA \in \mathcal{D}, \qquad A = \bigcup_n F_nA \tag{8.2}$$

If there is such a filtration with $F_nA = F_{n+1}A = \cdots$ then we say that $A \in \mathcal{D}^n$. Note that $\mathcal{D}^n = \mathcal{B}^n \cap \mathcal{D}^\infty$.

If $A \in \mathcal{A}$ then there is a maximal filtration $(R_nA)_n$ on A satisfying the first two properties in (8.2) with $\mathcal{D} = \mathcal{B}$. This filtration is given by

$$R_nA = \ker(A \xrightarrow{\eta^n} G^{-n}A)$$

An object $A \in \mathcal{A}$ is in $\mathcal{B}^n$ if $R_n A = A$ and it is in $\mathcal{B}^\infty$ if $\bigcup_n R_n A = A$. A is in $\mathcal{D}^\infty$ if in addition $R_{n+1}A/R_nA \in \mathcal{D}$.

We also consider the descending filtration on A given by

$$L_n A = \operatorname{im}(G^n A \xrightarrow{\eta^n} A)$$

This filtration satisfies $L_nA/L_{n+1}A \in \mathcal{B}$. If $A \in \mathcal{B}_m$ then $L_nA \subset R_{m-n}A$.

Proposition 8.4. 1. *$\mathcal{D}^\infty$ is a localizing subcategory in $\mathcal{A}$.*

2. *Assume that $\mathcal{A}$ is locally noetherian. If $\mathcal{D}$ is closed under injective hulls in $\mathcal{B}$ then $\mathcal{D}^\infty$ is closed under injective hulls in $\mathcal{A}$.*

Proof. 1. Only the closedness under extensions is not immediately clear. Let

$$0 \to D_1 \to A \xrightarrow{\psi} D_2 \to 0$$

be an extension such that $D_1, D_2 \in \mathcal{D}^\infty$.

We consider four cases.

(a) If $D_1, D_2 \in \mathcal{D}^n$ for some n then there is nothing to prove.

(b) Assume $D_1 \in \mathcal{D}^n$ for some n. Let F be a filtration on $\mathcal{D}_2$ satisfying (8.2). Then $A = \bigcup_n \psi^{-1}(F_nD_2)$. Since by (a) all $\psi^{-1}(F_nD_2)$ are in $\mathcal{D}^\infty$, we conclude that this is also true for A.

(c) Assume $D_2 \in \mathcal{D}^n$ for some n. Then $L_nA \subset D_1$ and hence $L_nA \in \mathcal{D}^\infty$. The exact sequence

$$0 \to G^n R_n A \to G^n(A) \xrightarrow{\eta^n} L_n A \to 0$$

combined with (b) shows what we want.

(d) Assume now that $D_{1,2}$ are general. Using (c) we can now use the same reasoning as in (b) to finish the proof.

2. This assertion can be split into two parts.

(a) $\mathcal{B}^\infty$ is closed under injective hulls in $\mathcal{A}$. To prove this let $B \hookrightarrow A$ be an essential extension with $B \in \mathcal{B}^\infty$ and $A \in \mathcal{A}$. We have to show that $A \in \mathcal{B}^\infty$. We may clearly assume that A/B contains no subobject in $\mathcal{B}^\infty$. Assume first that A is noetherian. In that case $B \in \mathcal{B}^n$ for some n. From the exact sequence

$$0 \to B \to A \xrightarrow{\eta^n} G^{-n}(L_nA) \to 0$$

we deduce that $A/B \cong G^{-n}(L_nA)$. Hence L_nA contains no subobject in $\mathcal{B}^\infty$. Thus $L_nA \cap B = 0$ and hence $L_nA = 0$. This yields $A = B$ and we are through.

Now assume that A is general. By hypothesis $A = \bigcup_{i\in I} A_i$ where the A_i are noetherian. By looking at the pairs $(B \cap A_i, A_i)$ we find that $A_i \in \mathcal{B}^\infty$. Hence $A \in \mathcal{B}^\infty$.

(b) $\mathcal{D}^\infty$ is closed under injective hulls in $\mathcal{B}^\infty$. To prove this assume that $D \hookrightarrow B$ is an essential extension with $D \in \mathcal{D}^\infty$ and $B \in \mathcal{B}^\infty$. Since $B = \bigcup_{n\in\mathbb{Z}} R_nB$, by considering all the pairs $(R_nB \cap D, R_nB)$, we may reduce to the case $B \in \mathcal{B}^n$. We then use induction on n. If $n = 1$ then $B \in \mathcal{B}$, $D \in \mathcal{D}$ and the result follows from the hypotheses on $\mathcal{D}$.

Assume now $n > 1$. We have the standard exact sequence

$$0 \to R_1B \to B \xrightarrow{\eta} G^{-1}(L_1B) \to 0 \tag{8.3}$$

Clearly $R_1 B \in \mathcal{B}$, $L_1 B \in \mathcal{B}^{n-1}$. Looking at the pairs $(R_1 B \cap D, R_1 B)$ and $(L_1 B \cap D, L_1 B)$ and induction reveals that $R_1 B, L_1 B \in \mathcal{D}^\infty$. Hence from (8.3) we deduce that $B \in \mathcal{D}^\infty$. □

From here on we assume that $\mathcal{A}$ is locally noetherian. Let $(T_i)_{i \in J}$ be the simple objects in $\mathcal{B}$. It is easy to see that these are also the simple objects in $\mathcal{B}^\infty$. Define $t : J \to J$ by $G^{-1}(T_i) = T_{ti}$. Clearly t is a permutation of J.

We let $\mathcal{D}$, $\mathcal{C}$ be the minimal localizing subcategories of $\mathcal{B}$ and $\mathcal{B}^\infty$ containing $(T_i)_{i \in J}$. Clearly $\mathcal{C} = \mathcal{D}^\infty$. For $i, j \in J$ we write $i \sim_{\mathcal{D}} j$, $i \sim_{\mathcal{C}} j$ if T_i, T_j are respectively in the same connected component of $\mathcal{D}$ and $\mathcal{C}$. With a reasoning similar to lemma 6.2 one shows that

$$i \sim_{\mathcal{C}} j \Rightarrow \exists p \in \mathbb{Z} : i \sim_{\mathcal{D}} t^p j \tag{8.4}$$

ILet $K \subset J$ be a union of equivalence classes for $\sim_{\mathcal{D}}$, stable under t, t^{-1}. By (8.4) K is then also a union of equivalence classes for $\sim_{\mathcal{C}}$.

We denote by $\mathcal{D}_K, \mathcal{C}_K$ the minimal localizing subcategories of $\mathcal{D}$ and $\mathcal{C}$ containing $(T_i)_{i \in K}$. Clearly

$$\begin{aligned} \mathcal{C} &= \oplus_{(K \in J/\sim_{\mathcal{C}})} \mathcal{C}_K \\ \mathcal{D} &= \oplus_{(K \in J/\sim_{\mathcal{D}})} \mathcal{D}_K \end{aligned} \tag{8.5}$$

Let E_i be the injective hull of T_i in $\mathcal{C}$. Put $E_K = \oplus_{i \in K} E_i$. Then E_K is an injective cogenerator of $\mathcal{C}_K$. The injective hull of T_i in $\mathcal{D}$ is given by $F_i = R_1 E_i$. We also put $F_K = \oplus_{i \in K} F_i$.

Proposition 8.5. *Assume that $\eta(E_K)$ is surjective. Let $C_K = \operatorname{End}_{\mathcal{C}}(E_K)$, $D_K = \operatorname{End}_{\mathcal{D}}(F_K)$ with the natural topology (as in Theorem 3.1). Then there is a regular normalizing element $N \in \operatorname{rad}(C_K)$ with the following properties.*

1. *$D_K = C_K/(N)$ as pseudocompact rings.*
2. *Put $\phi = N \cdot N^{-1}$. Let $e_i \in C_K$ be the idempotent corresponding to the projection $E_K \to E_i$. Then $\phi(e_i) = e_{ti}$.*
3. *Let $U \in \mathcal{C}_K$. There is an isomorphism as C_K-modules*
$$p : {}_\phi \operatorname{Hom}(U, E_K) \to \operatorname{Hom}(GU, E_K)$$
which is functorial in U.
4. *There is a commutative diagram.*
$$\begin{tikzcd} \operatorname{Hom}(U, E_K) \arrow[r, "N\cdot"] \arrow[d, equal] & {}_\phi \operatorname{Hom}(U, E_K) \arrow[d, "p"] \\ \operatorname{Hom}(U, E_K) \arrow[r, "\operatorname{Hom}(\eta(U), E)"] & \operatorname{Hom}(GU, E_K) \end{tikzcd} \tag{8.6}$$
5. *ϕ is a homeomorphism.*

Proof. By construction we have an exact sequence

$$0 \to F_K \to E_K \xrightarrow{\eta} G^{-1}(E_K) \to 0$$

Applying $\operatorname{Hom}_{\mathcal{C}}(-, E_K)$ and using the fact that $\operatorname{Hom}_{\mathcal{C}}(F_K, E_K) = \operatorname{Hom}_{\mathcal{D}}(F_K, F_K)$ we obtain an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{C}}(G^{-1} E_K, E_K) \xrightarrow{r} C_K \xrightarrow{s} D_K \to 0 \tag{8.7}$$

Here $r(f) = f \circ \eta$, $s(g) = f \mid F_K$. If U is a finite length object in $\mathcal{D}$ then one checks that $s^{-1}(\mathfrak{l}_D(U)) = \mathfrak{l}_C(U)$ and hence s is continuous.

Now choose isomorphisms $\mu_i : G^{-1}(E_i) \to E_{ti}$ and let $\mu = \oplus_{i\in K}\mu_i$. The map which sends h to $h \circ \mu$ defines an isomorphism $C_K \to \mathrm{Hom}_{\mathcal{C}}(G^{-1}E_K, E_K)$. Put $N = \mu \circ \eta$, as element of C_K. Then (8.7) becomes an exact sequence

$$0 \to C_K \xrightarrow{\cdot N} C_K \to D_K \to 0$$

from which we deduce that N is regular and normalizing.

The simple pseudocompact C_K modules are of the form $\mathrm{Hom}_{\mathcal{C}}(T_i, E_K)$ and if $f : T_i \to E_K$ is a map in $\mathcal{C}$ then f has its image in F_K and thus is annihilated by η. Hence $Nf = \mu\eta f = 0$ and thus $N \in \mathrm{rad}(C)$.

We now show that N satisfies 2. e_i is the composition of the projection $p_i : E_K \to E_i$ and the injection $q_i : E_i \to E_K$. The fact that $Ne_iN^{-1} = e_{ti}$ now follows from the following commutative diagram.

$$\begin{tikzcd}
E_K \arrow[r, "p_i"] \arrow[d, "\eta(G^{-1}E_K)"'] & E_i \arrow[r, "q_i"] \arrow[d, "\eta(G^{-1}E_i)"'] & E_K \arrow[d, "\eta(G^{-1}E_K)"'] \\
G^{-1}(E_K) \arrow[r, "G^{-1}(p_i)"] \arrow[d, "\mu"'] & G^{-1}(E_i) \arrow[r, "G^{-1}(q_i)"] \arrow[d, "\mu_i"'] & G^{-1}(E_K) \arrow[d, "\mu"'] \\
E_K \arrow[r, "p_{ti}"] & E_{ti} \arrow[r, "q_{ti}"] & E_K
\end{tikzcd}$$

Now we prove 3. Define the map

$$p : \mathrm{Hom}(U, E_K) \to \mathrm{Hom}(GU, E_K) : f \mapsto G(\mu^{-1}f)$$

We investigate the behaviour of p with respect to left multiplication by an element g of C_K. We find $p(gf) = G(\mu^{-1}gf) = G(\mu^{-1}g\mu\mu^{-1}f) = G(\mu^{-1}g\mu)p(f)$. Now we look at the following commutative diagram

$$\begin{tikzcd}
E_K \arrow[r, "G(\mu^{-1}g\mu)"] \arrow[d, "\eta"'] & E_K \arrow[d, "\eta"'] \\
G^{-1}(E_K) \arrow[r, "\mu^{-1}g\mu"] \arrow[d, "\mu"'] & G^{-1}(E_K) \arrow[d, "\mu"'] \\
E_K \arrow[r, "g"] & E_K
\end{tikzcd}$$

From this commutative diagram we deduce that $G(\mu^{-1}g\mu) = N^{-1}gN = \phi^{-1}(g)$. So we conclude that to make p a map of C_K-modules, it suffices to twist $\mathrm{Hom}(U, E_K)$ by ϕ.

Now we prove 4. The commutativity of (8.6) amounts to the identity $G(\mu^{-1}Nf) = \eta(U)f$ for f in $\mathrm{Hom}(U, E_K)$. Since $G(\mu^{-1}Nf) = G(\eta(G^{-1}E_K)f) = \eta(E_K)G(f)$ this follows from the fact that η is a natural transformation.

Finally we note that 5. follows from lemma 3.18 and 3. $\square$

Now we specialize to the situation of Theorem 1.1. Thus $\mathcal{B} = \mathrm{Qch}(Y)$ for a Cohen-Macaulay curve Y and $J = Y$. Furthermore $T_x = \mathcal{P}_x$, $t = \tau$. It is also clear that $x \sim_{\mathcal{D}} y \iff x = y$ and thus the equivalence classes for $\sim_{\mathcal{D}}$ are singletons. From (8.4) it follows that the equivalence classes for $\sim_{\mathcal{C}}$ are given by the τ-orbits. Finally we have for $K \subset Y$

$$D_K = \prod_{x \in K} \hat{\mathcal{O}}_{Y,x} \tag{8.8}$$

With these data, the proof of Theorem 1.1 is now a simple matter of translation, using the results in §6,7, ...

Proof of Theorem 1.1. 1. This follows from (8.5).

2.,3. By Theorem 3.1 the functor F° given by $\mathcal{M} \mapsto \operatorname{Hom}(\mathcal{M}, E_K)$ defines an equivalence between the dual of $\mathcal{C}_{f,z}$ and the category of left pseudocompact modules of finite length over the ring $C_z = C_K$ where K is the τ-orbit of z. By Proposition 8.5, $D_K = C_K/(N)$ for $N \in \operatorname{rad}(C_K)$ such that $\phi = N \cdot N^{-1}$ is a homeomorphism and such that $\phi(e_y) = e_{\tau y}$. Thus C_K satisfies the hypotheses of Proposition 6.1. From that proposition it follows that we can put C_K and N in the required matrix forms and that we have $R = \hat{\mathcal{O}}_{Y,z}$ if $|K| = \infty$ and $R/(U) = \hat{\mathcal{O}}_{Y,z}$ if $|K| < \infty$. To find out the exact form of R, we first note that by (8.8) D_K is locally noetherian and hence so is C_K by Proposition 3.23. Furthermore by Proposition 8.4 every object in $\mathcal{C}_K$ has finite injective dimension. Thus C_K has finite global dimension, and by Propositions 6.5,5.2 this means that R has finite global dimension. Hence the hypotheses for Proposition 7.1 are satisfied and thus R does indeed have the form (1.1) or (1.2).

Now note that if $2 \le |I| < \infty$ then Proposition 6.5 actually tells us that $\operatorname{gl\,dim} \hat{\mathcal{O}}_{Y,z} < \infty$. Thus z is regular on Y. Also by Proposition 7.2 this implies that $U \notin \operatorname{rad}^2(R)$.

The essential image of F° is given by the pseudocompact left C_z-modules of finite length. From Proposition 4.3 it follows that such modules correspond precisely to the finite dimensional left modules over C_z satisfying $V = \oplus_i e_i V$.

Under the duality $V \mapsto V^*$ such modules correspond to the finite dimensional right C_K-modules W satisfying $W = \oplus_i W e_i$.

We now claim that in fact every finite dimensional C_z representation is pseudocompact. This is clear if $|O_\tau(z)| < \infty$, so assume $|O_\tau(z)| = \infty$. In that case the statement depends upon the fact that $\operatorname{card}(k) = \infty$. Clearly we may reduce to the case that W is simple. Then W is annihilated by the Jacobson radical of A, which according to [8] is precisely given by the common annihilator of the pseudocompact simple modules. In other words $\operatorname{rad}(C_z)$ is given by the lower triangular matrices, having only non-units on the diagonal. Thus W is a $\prod_i k$-module. A finite dimensional simple module over a commutative k-algebra is clearly one dimensional. Hence $\dim W = 1$ and we have a corresponding character $\chi : \prod_i k \to k$. Choose $a = (a_i)_i \in \prod_i k$ in such a way that $a_i \neq a_j$ if $i \neq j$. Then there exist $b \in k$ such that $\chi(a-b) = 0$. But $a_i - b = 0$ for at most one i, and hence the ideal generated by $a - b$ is either improper or the kernel of the projection map $\operatorname{pr}_i : \prod_i k \to k$. The first case is clearly impossible and the second case implies that χ is given by projection on the i'th factor. Hence W is pseudocompact.

Putting $F(\mathcal{M}) = \operatorname{Hom}(\mathcal{M}, E_K)^*$ finishes the proof of 2.,3.

4. Since $F^\circ(\mathcal{P}_{\tau^i x})$ is by construction the i'th simple module of C_z, it is given by $C_z e_i / \operatorname{rad}(C_z e_i)$. Hence $F(\mathcal{P}_{\tau^i x}) = (C_z e_i / \operatorname{rad}(C_z e_i))^* = e_i C_z / \operatorname{rad}(e_i C_z)$.

5. This amounts to the construction of a natural isomorphism between $(F\mathcal{M})_\phi$ and $FG(\mathcal{M})$ for $\mathcal{M} \in \mathcal{C}_{f,z}$. Since $(F\mathcal{M})_\phi = (\operatorname{Hom}_{\mathcal{C}}(\mathcal{M}, E_K)^*)_\phi = {}_\phi \operatorname{Hom}_{\mathcal{C}}(\mathcal{M}, E_K)^*$ and $FG(\mathcal{M}) = \operatorname{Hom}(G\mathcal{M}, E_K)^*$, we can use p^* with p as in Proposition 8.5.3.
6. This diagram can be obtained by dualizing (8.6).
7. Let $\mathcal{M} \in \operatorname{Qch}(Y)$. Then $F\mathcal{M} = \operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, E_K)^* = \operatorname{Hom}_{\operatorname{Qch}(Y)}(\mathcal{M}, F_K)^* = \prod_i \operatorname{Hom}_{\operatorname{Qch}(Y)}(\mathcal{M}, F_i)^*$ where as before F_i is the injective hull of $k(\tau^i z)$ in $\operatorname{Qch}(Y)$. It follows from Matlis duality that $\operatorname{Hom}_{\operatorname{Qch}(Y)}(\mathcal{M}, F_i)^*$ is the completion of $\mathcal{M}$ at $\tau^i z$. □

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