GROTHENDIECK GROUPS AND TILTING OBJECTS

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dedicated to Klaus Roggenkamp for his 60th birthday

Abstract. Let $\mathcal{C}$ be a connected noetherian hereditary abelian category with Serre functor over an algebraically closed field k, with finite dimensional homomorphism and extension spaces. Using the classification of such categories from [31], we prove that if $\mathcal{C}$ has some object of infinite length, then the Grothendieck group of $\mathcal{C}$ is finitely generated if and only if $\mathcal{C}$ has a tilting object.

Introduction

Let k be an algebraically closed field and $\mathcal{C}$ a hereditary abelian Ext-finite k-category. That $\mathcal{C}$ is hereditary means that $\operatorname{Ext}^i(-,-)$ vanishes for $i \geq 2$, and we say that $\mathcal{C}$ is Ext-finite if $\operatorname{Ext}^i(A,B)$ is finite dimensional over k for all A, B in $\mathcal{C}$ and all i. A central problem in the representation theory of artin algebras is to describe such $\mathcal{C}$ which have a tilting object. This is important in connection with the investigation of quasitilted algebras, as introduced in [19].

When a hereditary abelian Ext-finite k-category $\mathcal{C}$ has a tilting object, it is a consequence that the Grothendieck group $K_0(\mathcal{C})$ is free abelian of finite rank [19]. This suggests the problem of describing the $\mathcal{C}$ for which $K_0(\mathcal{C})$ is finitely generated (or free abelian of finite rank), and to decide to which extent having a finitely generated Grothendieck group implies the existence of a tilting object. Relating the existence of a tilting object to properties of the more widely known notion of Grothendieck group provides a better insight into the meaning of the condition of the existence of a tilting object. In particular, it is interesting to understand in terms of Grothendieck groups the special role the category $\operatorname{coh}\mathcal{X}$ of coherent sheaves on a weighted projective line [12] plays within the larger class of quotient categories of finitely generated graded modules over commutative noetherian isolated singularities of Krull dimension two, or within the class of categories $\operatorname{coh}\mathcal{O}$ of coherent modules over a sheaf $\mathcal{O}$ of hereditary orders over a smooth projective curve. The categories $\operatorname{coh}\mathcal{X}$ of coherent sheaves on a weighted projective line then correspond to the $\operatorname{coh}\mathcal{O}$ where the smooth projective curve is the projective line $\mathbb{P}^1$.

We will use the general classification results from [31] to solve the above problems under the additional hypotheses that $\mathcal{C}$ is noetherian and has a Serre functor (see section 1). The latter hypothesis is natural since it is a consequence of the existence of a tilting object. Both additional hypotheses are satisfied for the quotient categories mentioned above. Since the properties of $\mathcal{C}$ having a tilting object, a finitely generated Grothendieck group or a Serre functor are preserved under derived equivalence of hereditary categories [17, 31] our results apply more generally to the $\mathcal{C}$ with the additional hypothesis of having a Serre functor and being derived equivalent to a noetherian hereditary category.

Some of the results on Grothendieck groups of quotient categories proved in this paper are inspired by similar results for two-dimensional complete noetherian rings, used as a tool for classifying maximal orders of finite representation type in [30].

Section 1 is devoted to discussing background material from various sources, collected together for the benefit of the reader. In section 2 we describe the $\mathcal{C}$ with finitely generated Grothendieck group. A new criterion for an object to be a tilting object is given in section 3, which is interesting also in its own right. In section 4 we construct exceptional collections of modules over a hereditary order over a discrete valuation ring. This is used in section 5, along with the criterion from section 3, to construct a tilting object in the category $\operatorname{coh}\mathcal{O}$ of coherent modules over a sheaf $\mathcal{O}$ of hereditary orders over $\mathbb{P}^1$. In section 6 we give our main result on the connection between the existence of a tilting object and the Grothendieck group being finitely generated. Under our assumptions the conditions turn out to be equivalent if $\mathcal{C}$ is connected and has some object of infinite length. In section 7 we give some examples and comments.

We give an appendix proving directly the relationship between the category of coherent sheaves on a weighted projective line and the category $\operatorname{coh}\mathcal{O}$ of coherent modules over a sheaf $\mathcal{O}$ of hereditary orders over $\mathbb{P}^1$.

Hereditary abelian k-categories which are Ext-finite, noetherian and have a Serre functor were classified in [31]. It is also of interest to investigate hereditary abelian categories which do not satisfy the additional assumptions, for example with respect to when the Grothendieck group is finitely generated. In Appendix B we give some sources of examples of hereditary abelian categories.

1. Background

In this section we provide some background material from various sources, to provide a better understanding of how our work fits in.

1.1. **Tilting objects.** We will say that an abelian or triangulated category has finite homological dimension if between any two objects only a finite number of Ext are non-zero. If $\mathcal{C}$ is an abelian category then $\mathcal{C}$ has finite homological dimension if and only if this is the case for $D^b(\mathcal{C})$.

Let $\mathcal{A}$ be a triangulated category of finite homological dimension. If $T \in \mathcal{A}$, then add(T) is by definition the smallest additive category containing T which is closed under finite direct sums and summands. We say that T is a tilting object if $\operatorname{Ext}^i_{\mathcal{A}}(T,T) = 0$ for $i \neq 0$ and $\operatorname{add} T$ generates $\mathcal{A}$ (in the sense that $\mathcal{A}$ is the smallest subcategory of $\mathcal{A}$ containing $\operatorname{add} T$ which is closed under shifts and cones).

If $\mathcal{C}$ is an abelian category of finite homological dimension then $T \in \mathcal{C}$ is a tilting object if it is a tilting object in $D^b(\mathcal{C})$. This definition of a tilting object in $\mathcal{C}$ is equivalent to the usual notion of tilting module (of finite projective dimension) when $\mathcal{C}$ is the category $\operatorname{mod}\Lambda$ of finitely generated modules for an artin algebra of finite global dimension, as is seen directly or by using [9, 32]. For an Ext-finite hereditary abelian k-category $\mathcal{C}$ it is equivalent to the following definition used in [18] (reformulating conditions from [19, 16]): An object T in $\mathcal{C}$ is a tilting object if $\operatorname{Ext}^1(T,T) = 0$ and if $\operatorname{Hom}(T,X) = 0 = \operatorname{Ext}^1(T,X)$ implies $X = 0$. In general the definition in [19] is modelled on the definition of a tilting module of projective dimension at most one, and is hence different from ours when $\mathcal{C}$ is not hereditary.

When T is a tilting object in $\operatorname{mod}\Lambda$ for an artin algebra Λ of finite global dimension, there is an induced equivalence $D^b(\operatorname{mod}\Lambda) \to D^b(\operatorname{mod}\operatorname{End}(T)^{\mathrm{opp}})$ between bounded derived categories, and similarly if T is a tilting object in the category $\operatorname{coh} X$ of coherent sheaves on a smooth projective variety X [15, 32, 3]. This can easily be extended to the case of $\operatorname{coh}\mathcal{O}$ where $\mathcal{O}$ is a coherent $\mathcal{O}_X$-algebra locally of finite global dimension. The analogous result for tilting objects in Ext-finite hereditary abelian k-categories is given in [19].

In general if T is a tilting object in a triangulated category $\mathcal{A}$ one may expect an equivalence $\mathcal{A} \cong D^b(\operatorname{End}(T)^{\mathrm{opp}})$. There is recent work in this direction by Keller, using A_∞-categories, and building on [21]. In particular it follows from his results that if T is a tilting object in an Ext-finite abelian k-category $\mathcal{C}$ of finite homological dimension, then there is an equivalence between $D^b(\mathcal{C})$ and $D^b(\operatorname{End}(T)^{\mathrm{opp}})$.

It follows from this derived equivalence that when $\mathcal{C}$ is an Ext-finite abelian k-category of finite homological dimension having a tilting object T, the Grothendieck group $K_o(\mathcal{C})$ is isomorphic to $\mathbb{Z}^n$, where n is the number of nonisomorphic summands of T (see [19] for the hereditary case). Given T in $\mathcal{C}$ with $\operatorname{Ext}^i(T,T) = 0$ for $i > 0$, it is an important problem, open even for artin algebras, whether T having n nonisomorphic summands is sufficient for T to be a tilting object. It is known in the case of artin algebras when the projective dimension of T is at most one [6], and for $\mathcal{C}$ Ext-finite hereditary if $\mathcal{C}$ has some tilting object [16].

1.2. **Hereditary categories and weighted projective lines.** Hereditary abelian Ext-finite categories are of special interest in connection with quasitilted algebras, as introduced in [19]. The quasitilted algebras are by definition the endomorphism algebras $\operatorname{End}(T)^{\mathrm{opp}}$ when T is a tilting object in a hereditary category. Equivalently, an algebra Λ is quasitilted if and only if gl.dim.$\Lambda \leq 2$ and each indecomposable X in $\operatorname{mod}\Lambda$ has projective or injective dimension at most one [19]. Main examples of categories $\mathcal{C}$ are $\operatorname{mod}\Lambda$ where Λ is a hereditary artin algebra and $\operatorname{coh} X$ when X is a smooth projective curve. More generally there are the coherent sheaves on weighted projective lines [12], which we discuss next.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $n \geq 2$, be a finite number of points in $\mathbb{P}^1$, with $\lambda_1 = 0$, $\lambda_2 = \infty$, $\lambda_3 = 1$. For a sequence $e = (e_1, \dots, e_n)$ of positive integers consider the associated ring

$$R = k[x_1, \dots, x_n]/[(x_i^{e_i} - x_2^{e_2} + \lambda_i x_1^{e_1})]_{i \geq 3}. \tag{1.1}$$

Let H be the abelian group generated by $h_1, \dots, h_n$ and with relations $e_1 h_1 = \cdots = e_n h_n$. Then H is isomorphic to $\mathbb{Z} \oplus G$, where G is a finite group [12]. R is then a H-graded ring, and Geigle and Lenzing write $\operatorname{coh}\mathcal{X}$ for the hereditary category $\operatorname{gr}_H R/\text{finite length}$, where $\operatorname{gr}_H R$ denotes the category of finitely generated H-graded R-modules with degree zero homomorphisms. Geometrically $\operatorname{coh}\mathcal{X}$ can be viewed as the coherent sheaves over a (hypothetical) space $\mathcal{X}$ which is a generalization of $\mathbb{P}^1$. Hence Geigle and Lenzing call $\mathcal{X}$ a "weighted projective line".

The category $\operatorname{coh}\mathcal{X}$ is a noetherian hereditary abelian category with finite dimensional homomorphism and extension spaces, which has a tilting object T such that $\operatorname{End}_{\mathcal{H}}(T)^{\mathrm{opp}}$ is a canonical algebra in the sense of Ringel [33], and actually all canonical algebras are obtained this way. Like for tilting for finite dimensional algebras [15], there is also induced an equivalence of derived categories $D^b(\operatorname{coh}\mathcal{X}) \to D^b(\operatorname{End}_{\mathcal{H}}(T)^{\mathrm{opp}})$ [12]. This setup is used to give an alternative approach to the study of the module theory for canonical algebras, by first investigating the hereditary category $\operatorname{coh}\mathcal{X}$. The rings described by (1.1) are H-graded factorial and it is shown in [22] that this property characterizes those rings amongst the two-dimensional rings.

There are two main known sources of connected hereditary categories $\mathcal{C}$ with tilting object; the module categories of finite dimensional hereditary k-algebras and the categories $\operatorname{coh}\mathcal{X}$. In addition there are the hereditary abelian categories derived equivalent to them. It is conjectured that there are no more, and in fact this is proved in [25] for noetherian hereditary categories and more generally in [17, 18] under the assumption that $\mathcal{C}$ has at least one nonzero object of finite length, or at least one directing object, that is, an object which does not lie on a cycle of nonzero nonisomorphisms.

1.3. **Noetherian hereditary categories with Serre functor.** We recall some essential features of the classification of noetherian hereditary abelian Ext-finite k-categories with Serre functor. For further details we refer to [31].

Assume that $\mathcal{C}$ is an Ext-finite abelian k-category. A Serre functor for $\mathcal{C}$ is an auto-equivalence $F : D^b(\mathcal{C}) \to D^b(\mathcal{C})$ where $D^b(\mathcal{C})$ denotes the bounded derived category, such that for all A, B in $D^b(\mathcal{C})$ there are isomorphisms $\operatorname{Hom}(A,B) \xrightarrow{\cong} \operatorname{Hom}(B,FA)^*$ natural in A and B ($(-)^*$ is the k-dual). For A in $\mathcal{C}$ we then have $\operatorname{Hom}(A,B[i]) \simeq \operatorname{Hom}(B[i],FA)^*$ for all B in $\mathcal{C}$ and any integer i. Since FA is in $D^b(\mathcal{C})$, there is some integer i_0 such that $\operatorname{Hom}(B[i],FA) = 0$ for $i \geq i_0$ and any B in $\mathcal{C}$. Hence A has finite projective dimension, and consequently $\mathcal{C}$ has finite homological dimension.

For a hereditary abelian Ext-finite k-category $\mathcal{C}$ the existence of a Serre functor implies the existence of almost split sequences, and the converse holds if $\mathcal{C}$ has no non-zero projective or injective objects [31].

Let be $\mathcal{C}$ a connected category. It is proved in [31] that if $\mathcal{C}$ is a connected noetherian hereditary Ext-finite k-category with Serre functor then $\mathcal{C}$ has one of the following forms.

(i) A category $\operatorname{coh}\mathcal{O}$, where $\mathcal{O}$ is a sheaf of hereditary orders over a smooth projective curve.
(ii) $\mathcal{C}$ is $\operatorname{mod}\Lambda$ for a finite dimensional hereditary k-algebra Λ.
(iii) $\mathcal{C}$ is the category of nilpotent finite dimensional representations of the quiver $\tilde{A}_n$ with cyclic orientation with $n < \infty$.
(iv) $\mathcal{C}$ is derived equivalent to a hereditary category where all objects have finite length and having an infinite number of nonisomorphic simple objects.

The actual result in [31] also contains a precise classification of the categories in (iv). We also recall from [31] that for the categories in (i) there is an alternative description as follows.

(i') Categories of the form $\operatorname{qgr} S = \operatorname{gr} S/\text{finite length}$, where $\operatorname{gr} S$ denotes the category of finitely generated graded modules over a commutative noetherian $\mathbb{Z}$-graded domain $S = k + S_1 + S_2 + \ldots + S_i + \ldots$ of Krull dimension two which is finite over its center, where the S_i are finite dimensional over k, and S is an isolated singularity.

The categories $\operatorname{coh}\mathcal{X}$ have also as we have seen a similar description, as quotient categories starting with an H-graded ring, where H is not necessarily $\mathbb{Z}$. But it follows from [13][24] that one can assume that the rings are $\mathbb{Z}$-graded, so that the class $\operatorname{coh}\mathcal{X}$ is a subclass of the $\operatorname{qgr} S$.

1.4. **Classical hereditary orders.** In this section we collect some well-known properties of hereditary orders. We will loosely refer to a classical hereditary order as an order Λ in a central simple algebra A over a field K which is hereditary. Let R be the center of Λ. According to [34] R is a Dedekind ring. (This fact does not seem to be contained in exactly this form in [29]).

Assume that R is a discrete valuation ring with maximal ideal m. Then according to [29] the radical I of Λ is invertible. Furthermore by [29] there is an integer e such that $I^e = m\Lambda$, called the ramification index of Λ/R. It follows from the structure theory of hereditary orders in [29] that if $e = 1$ then Λ is maximal. If the converse is true then we say that A/K is unramified. This happens for example if $A = M_n(K)$.

If R is not a discrete valuation ring then by localizing one defines ramification indices e_P for the non-zero primes in R. By analyzing $\mathcal{D} = \operatorname{Hom}(\Lambda, R)$ it follows easily that Λ/R ramifies in only a finite number of primes.

2. Finitely generated Grothendieck groups

Let $\mathcal{C}$ be a noetherian Ext-finite hereditary abelian k-category with Serre functor, where k is an arbitrary field. In this section we describe which $\mathcal{C}$ have finitely generated Grothendieck group. For this we use the classification theorem from [31] in the form recalled in §1.3. The main problem we need to deal with is when the category $\operatorname{coh}\mathcal{O}$ of coherent modules over a sheaf $\mathcal{O}$ of hereditary orders over a smooth projective curve X has finitely generated Grothendieck group.

Let X be a regular connected curve over a field k. Let K be the function field of X and let A be a central simple algebra over K. Let $\mathcal{O}$ be a sheaf of hereditary orders in A over $\mathcal{O}_X$. Thus locally $\mathcal{O}$ is a hereditary order over a Dedekind ring (in the sense of [29]).

If $\mathcal{A}$ is a sheaf of rings on a topological space Z then we use the notation $\operatorname{coh}(\mathcal{A})$ for the category of coherent $\mathcal{A}$-modules and we write $K_0(\mathcal{A})$ for $K_0(\operatorname{coh}(\mathcal{A}))$.

Our first aim is to give some results on $K_0(\mathcal{O})$.

Proposition 2.1. *Let $\mathcal{O}$ be as above. Let $x_1, \dots, x_t$ be the points in which $\mathcal{O}$ ramifies, and let $e_1, \dots, e_t$ be the corresponding ramification indices (see §1.4).*

Put

$$r = \sum_i (e_i - 1). \tag{2.1}$$

Let $\bar{\mathcal{O}}$ be a maximal order lying over $\mathcal{O}$. Then

$$K_0(\mathcal{O}) \cong K_0(\bar{\mathcal{O}}) \oplus \mathbb{Z}^r.$$

If k is algebraically closed then

$$K_0(\mathcal{O}) \cong K_0(\mathcal{O}_X) \oplus \mathbb{Z}^r.$$

Proof. The hereditary orders in A containing $\mathcal{O}$ form a partially ordered set which we will denote by $\mathcal{H}(\mathcal{O})$.

If $\mathcal{O}' \in \mathcal{H}(\mathcal{O})$ lies minimally over $\mathcal{O}$ then one proves exactly as in [30, Thm 1.14] that $K_0(\mathcal{O}) = K_0(\mathcal{O}') \oplus \mathbb{Z}$.

Let $\bar{\mathcal{O}}$ be a maximal order lying over $\mathcal{O}$. We deduce that $K_0(\mathcal{O}) \cong \mathbb{Z}^r \oplus K_0(\bar{\mathcal{O}})$, where r is the length of a maximal chain in $\mathcal{H}(\mathcal{O})$, starting in $\mathcal{O}$ and ending in $\bar{\mathcal{O}}$.

A local computation shows that r is given by the formula (2.1), which finishes the proof of the computation of $K_0(\mathcal{O})$.

If k is algebraically closed then by Tsen's theorem [8, p. 374] one has that $\bar{\mathcal{O}} \cong \operatorname{End}_{\mathcal{O}_X}(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle of rank n on X. Hence by Morita theory $K_0(\bar{\mathcal{O}}) \cong K_0(\mathcal{O}_X)$. □

Corollary 2.2. *Assume k is a algebraically closed. Then $K_0(\mathcal{O})$ is finitely generated if and only if X is an open subset of $\mathbb{P}^1$.*

Proof. By the previous proposition it suffices to prove this for $\mathcal{O} = \mathcal{O}_X$. Let $\bar{X}$ be the regular projective curve associated to the function field of X [10]. Then $\bar{X}$ is a regular compactification of X. In particular $\bar{X} - X$ is a finite number of points, whence by the localization sequence $K_0(\mathcal{O}_X)$ is finitely generated if and only if $K_0(\mathcal{O}_{\bar{X}})$ is finitely generated. Hence we may assume that X is projective. By [20, Ex. II.6.12, Rem. IV.4.10.4] one has $K_0(\mathcal{O}_X) \cong \mathbb{Z}^2 \oplus J(k)$ where $J(k)$ denotes the

k-points of the Jacobian of X. It is well-known that $J(k)$ is not finitely generated if $X \not\cong \mathbb{P}^1$ (for example because in that case $J(k)$ is non-trivial and divisible [26]). $\square$

Combining with §1.3 we now get the following main result of this section.

Theorem 2.3. *Let $\mathcal{C}$ be a connected noetherian* Ext*-finite hereditary abelian k-category with Serre functor where k is an algebraically closed field. Then $K_o(\mathcal{C})$ is finitely generated (free abelian) if and only if $\mathcal{C}$ has one of the following forms.*

1. $\operatorname{mod}\Lambda$ *where Λ is an indecomposable finite dimensional hereditary k-algebra.*
2. *Finite dimensional nilpotent representations over $\tilde{A}_n$ with n finite and cyclic orientation.*
3. $\operatorname{coh}\mathcal{O}$ *where $\mathcal{O}$ is a sheaf of hereditary $\mathcal{O}_X$-orders with $X = \mathbb{P}^1$.*

Proof. It is clear that the categories in 1. and 2. have finitely generated (free abelian) Grothendieck groups, and that $K_o(\mathcal{C})$ is not finitely generated if $\mathcal{C}$ is derived equivalent to a hereditary category $\mathcal{C}'$ with all objects of finite length and an infinite number of nonisomorphic simple objects. In view of §1.3 the proof is completed by using Corollary 2.2. $\square$

3. A criterion for deciding if an object is a tilting object.

The aim of this section is to give a criterion for an object of projective dimension at most one in an abelian category to be a tilting object. For this we need to recall some results on semiorthogonal pairs in triangulated categories from [4, 5].

Let $\mathcal{A}$ be a triangulated category and let $\mathcal{B}$, $\mathcal{C}$ be two strict (= closed under isomorphisms) full triangulated subcategories of $\mathcal{A}$. The pair $(\mathcal{B},\mathcal{C})$ is said to be a *semi-orthogonal pair* if $\operatorname{Hom}_{\mathcal{A}}(B,C)=0$ for $B\in\mathcal{B}$ and $C\in\mathcal{C}$. Define

$$\mathcal{B}^{\perp} = \{A\in\mathcal{A} \mid \forall B\in\mathcal{B} : \operatorname{Hom}_{\mathcal{A}}(B,A)=0\}.$$

The category ${}^{\perp}\mathcal{C}$ is defined similarly.

If $\mathcal{S}$ is a class of objects in $\mathcal{A}$ then the (triangulated) category generated by $\mathcal{S}$ is the smallest subcategory of $\mathcal{A}$ which is closed under shifts, cones, and isomorphisms.

The following result is a slight variation of the statement of [4, Lemma 3.3.1] (see also [5, §1]).

Lemma 3.1. *The following conditions are equivalent for a semi-orthogonal pair $(\mathcal{B},\mathcal{C})$.*

1. *$\mathcal{B}$ and $\mathcal{C}$ generate $\mathcal{A}$.*
2. *For every $A\in\mathcal{A}$ there exists a distinguished triangle $B\to A\to C$ with $B\in\mathcal{B}$ and $C\in\mathcal{C}$.*
3. *$\mathcal{C}=\mathcal{B}^{\perp}$ and the inclusion functor $i_*:\mathcal{B}\to\mathcal{A}$ has a right adjoint $i^!:\mathcal{A}\to\mathcal{B}$.*
4. *$\mathcal{B}={}^{\perp}\mathcal{C}$ and the inclusion functor $j_*:\mathcal{C}\to\mathcal{A}$ has a left adjoint $j^*:\mathcal{A}\to\mathcal{C}$.*

If one of these conditions holds then the functors $i^!, j^*$ are exact and the triangles in 2. are (for a fixed A) unique up to unique isomorphism. They are necessarily of the form

$$i_*i^!A \to A \to j_*j^*A \tag{3.1}$$

where the maps are obtained by adjointness from the identity maps $i^!A\to i^!A$ and $j^*A\to j^*A$. In particular triangles as in 2. are functorial.

If any of the conditions of the previous lemma holds then we say that $(\mathcal{B},\mathcal{C})$ is a semi-orthogonal decomposition of $\mathcal{A}$.

Now for the rest of this section let k be a field. All categories (abelian or triangulated) will be k-linear with finite dimensional Hom's and Ext's and have finite homological dimension.

Let $\mathcal{A}$ be a triangulated or abelian category. For an object T in $\mathcal{A}$ we denote by $T^\perp$ the full subcategory of $\mathcal{A}$ whose objects are the C in $\mathcal{A}$ with $\operatorname{Ext}^i(T,C)=0$ for all i. We say that an object $T\in\mathcal{A}$ is exceptional if $\operatorname{Ext}^i_{\mathcal{A}}(T,T)=0$ for $i>0$ and $\operatorname{End}_{\mathcal{A}}(T)$ is a (finite dimensional) division algebra. A sequence of exceptional objects $T_1,\dots,T_n$ is an exceptional collection if $\operatorname{Ext}^*_{\mathcal{A}}(T_i,T_j)=0$ for $j>i$. An exceptional collection is strongly exceptional if $\operatorname{Ext}^t_{\mathcal{A}}(T_i,T_j)=0$ for $t>0$ and all i,j.

The following is proved in [5].

Lemma 3.2. *Assume that $T_1,\dots,T_n$ is an exceptional collection in a triangulated category $\mathcal{A}$. Let $\mathcal{B}$ be the triangulated subcategory of $\mathcal{A}$ generated by $T_1,\dots,T_n$ and put $T=\oplus_i T_i$. Then $\mathcal{A}$ has a semi-orthogonal decomposition given by $(\mathcal{B},\mathcal{B}^\perp)=(\mathcal{B},T^\perp)$.*

Proof. For the convenience of the reader we repeat the proof. We have to show that $\mathcal{A}$ is generated by $\mathcal{B}$ and $T^\perp$. Let $\mathcal{B}_1$ be the full subcategory of $\mathcal{A}$ consisting of objects isomorphic to finite direct sums of the form $\oplus_j T_1[j]^{a_j}$. Then $\mathcal{B}_1$ is a strict triangulated subcategory of $\mathcal{A}$. This can be deduced from the fact that the formation of triangles in $\mathcal{A}$ is compatible with direct sums [37, Cor. II.1.2.5]. Sending $A\in\mathcal{A}$ to $\oplus_i \operatorname{Ext}^i(T_1,A)\otimes_D T_1[-i]$ defines a right adjoint to the inclusion $\mathcal{B}_1\hookrightarrow\mathcal{A}$. This yields a semi-orthogonal decomposition of $\mathcal{A}$ given by $(\mathcal{B}_1,\mathcal{B}_1^\perp)=(\mathcal{B}_1,T_1^\perp)$. In particular $\mathcal{A}$ is generated by $\mathcal{B}_1$ and $T_1^\perp$. Now we repeat this construction with $T_2\in T_1^\perp$. So if $\mathcal{B}_2$ is the full subcategory of $\mathcal{A}$ consisting of objects isomorphic to finite direct sums of the form $\oplus_j T_2[j]^{a_j}$ then we have that $T_1^\perp$ is generated by $\mathcal{B}_2$ and $(T_1\oplus T_2)^\perp$. Continuing this procedure we find that $\mathcal{A}$ is generated by $\mathcal{B}_1,\dots\mathcal{B}_n$ and $T^\perp$. This finishes the proof. □

We point out the following consequence of Lemma 3.2.

Corollary 3.3. *Let $T_1,\cdots,T_n$ be a strongly exceptional collection in a triangulated category $\mathcal{A}$ satisfying the above assumptions. Then T is a tilting object if and only if $T^\perp=0$.*

We shall need that semi-orthogonal decompositions behave nicely with respect to Grothendieck groups.

Lemma 3.4. *Assume that $(\mathcal{B},\mathcal{C})$ is a semi-orthogonal decomposition for a triangulated category $\mathcal{A}$. Then $K_0(\mathcal{A})\cong K_0(\mathcal{B})\oplus K_0(\mathcal{C})$.*

Proof. The inclusions $\mathcal{B},\mathcal{C}\subset\mathcal{A}$ define a map $K_0(\mathcal{B})\oplus K_0(\mathcal{C})\to K_0(\mathcal{A})$. An inverse to this map is given by sending $[A]$ to $[i^!A]\oplus[j^*A]$ (see Lemma 3.1 for notations). □

Lemma 3.5. *Assume that $\mathcal{A}$ is a triangulated category, and let $T_1,\dots,T_n\in\mathcal{A}$ be an exceptional collection. Put $T=\oplus_i T_i$. Then $K_0(\mathcal{A})\cong\mathbb{Z}^n\oplus K_0(T^\perp)$.*

Proof. Using the same method as in Lemma 3.2 we find inductively using Lemma 3.4 that $K_0(\mathcal{A})=\oplus_i K_0(\mathcal{B}_i)\oplus K_0(T^\perp)$. Now it is easy to see that sending $\oplus_j T_i[j]^{a_j}$ to $\sum_j(-1)^j a_j$ defines an isomorphism $K_0(\mathcal{B}_i)\cong\mathbb{Z}$. This proves what we want. □

If $\mathcal{B}$ is an abelian category and $B\in\mathcal{B}$ then we will say that B has projective dimension $\le r$ if $\operatorname{Ext}^s_{\mathcal{B}}(B,-)=0$ for $s>r$.

The above results have a counterpart for abelian categories provided we work with exceptional objects of projective dimension ≤ 1. This follows from the following lemma.

Lemma 3.6. *Assume that $\mathcal{B}$ is an abelian category and let T be an object in $\mathcal{B}$ of projective dimension ≤ 1. If an object in $D^b(\mathcal{B})$ is (right) perpendicular to T then so is its homology. In particular $D^b_{T^\perp_{\mathcal{B}}}(\mathcal{B})=T^\perp_{D^b(\mathcal{B})}$.*

Proof. Let $B \in T^{\perp}_{D^b(\mathcal{B})}$ and assume that n is maximal such that $H^n(B) \neq 0$. Then there is a triangle

$$\tau_{<n}B \to B \to H^n(B)[-n] \to \tag{3.2}$$

Applying $\operatorname{Hom}(T,-)$ yields injections $\operatorname{Ext}^i(T, H^n(B)) \hookrightarrow \operatorname{Hom}(T, (\tau_{<n}B)[n+1+i])$. Since for $i \geq 0$ the non-trivial homology of $(\tau_{<n}B)[n+1+i]$ occurs in degrees ≤ -2 and since the projective dimension of T is less than or equal to 1 it follows that $\operatorname{Hom}(T, (\tau_{<n}B)[n+1+i]) = 0$ for $i \geq 0$. In particular $\operatorname{Ext}^i(T, H^n(B)) = 0$ for $i \geq 0$. Since trivially $\operatorname{Ext}^i(T, H^n(B)) = 0$ for $i < 0$ it follows that $H^n(B) \in T^{\perp}$. But then it follows from (3.2) that $\tau_{<n}B \in T^{\perp}$. Repeating this procedure with B replaced by $\tau_{<n}B$ eventually yields that the homology of B is in $T^{\perp}$. $\square$

As a corollary one obtains a proof of the following standard result.

Corollary 3.7. *Let T and $\mathcal{B}$ be as in the previous lemma. Then $T^{\perp}_{\mathcal{B}}$ is an abelian category.*

Proof. If $f : A \to B$ is a map in $T^{\perp}_{\mathcal{B}}$ then one has to show that $\ker f$, $\operatorname{coker} f \in T^{\perp}_{\mathcal{B}}$. Since the complex represented by f clearly lies in $T^{\perp}_{D^b(\mathcal{B})}$, this follows from the previous lemma. $\square$

As a consequence of Lemma 3.6 we obtain the following result on Grothendieck groups.

Corollary 3.8. *Assume that $\mathcal{B}$ is an abelian category. Assume that $T_1, \dots, T_n \in \mathcal{B}$ is an exceptional collection consisting of objects of projective dimension ≤ 1. Let $T = \oplus_i T_i$. Then $K_0(\mathcal{B}) \cong \mathbb{Z}^n \oplus K_0(T^{\perp})$.*

Proof. By Lemmas 3.5 and 3.6 one has

$$\begin{aligned} K_0(\mathcal{B}) &\cong K_0(D^b(\mathcal{B})) \\ &\cong \mathbb{Z}^n \oplus K_0(T^{\perp}_{D^b(\mathcal{B})}) \\ &\cong \mathbb{Z}^n \oplus K_0(D^b_{T^{\perp}_{\mathcal{B}}}(\mathcal{B})) \\ &\cong \mathbb{Z}^n \oplus K_0(T^{\perp}_{\mathcal{B}}) \end{aligned}$$

We now get the main result of this section.

Corollary 3.9. *Assume that $\mathcal{B}$ is an abelian category of finite Krull dimension. Let $T = \oplus_{i=1}^n T_i$ be as in Corollary 3.8, but assume is addition that $\operatorname{Ext}^1(T_i, T_j) = 0$ for all i, j. If $n = \operatorname{rk} K_0(\mathcal{B})$ then T is a tilting object in $\mathcal{B}$.*

Proof. By Lemma 3.2 we have to show $T^{\perp} = 0$. By Lemma 3.6 it follows that $T^{\perp}$ is equal to $D^b_{T^{\perp}_{\mathcal{B}}}(\mathcal{B})$. So it is sufficient to show that $\mathcal{H} \overset{\text{def}}{=} T^{\perp}_{\mathcal{B}} = 0$.

By Corollary 3.8 one has $\operatorname{rk} K_0(\mathcal{H}) = 0$. Since $\mathcal{H}$ is an abelian subcategory of $\mathcal{B}$, it also has finite Krull dimension. In particular if $\mathcal{H} \neq 0$ there is a quotient category $\mathcal{C}$ of $\mathcal{H}$ which has finite length. Selecting a simple object in $\mathcal{C}$ yields a rank function on $\mathcal{H}$ which is non-trivial. Hence $\operatorname{rk} K_0(\mathcal{H}) > 0$. This yields a contradiction. $\square$

It would be interesting to know if for a nonzero hereditary abelian k-category $\mathcal{B}$ with finite dimensional homomorphism and extension spaces we must have $K_0(\mathcal{B}) \neq 0$.

4. Strongly exceptional collections for hereditary orders over discrete valuation rings

In order to construct a tilting object in the category $\operatorname{coh}\mathcal{O}$ of coherent modules for a hereditary order $\mathcal{O}$ over $\mathbb{P}^1$ we need to produce some exceptional collections of modules for hereditary orders over discrete valuation rings. We start by recalling some properties for such orders. For simplicity we restrict ourselves to hereditary orders contained in a matrix ring since that is the only case we will need.

Let R be a discrete valuation ring and let m be its maximal ideal. Furthermore let K be the quotient field of R and put $A = M_n(K)$. Let Δ be a hereditary order in A in the sense of [29].

Thus there exist strictly positive integers $n_1, \dots, n_t$ such that $n = n_1 + \cdots + n_t$ and such that Δ is isomorphic to

$$\begin{pmatrix} R_{n_1\times n_1} & m_{n_1\times n_2} & \cdots & \\ R_{n_2\times n_1} & R_{n_2\times n_2} & \cdots & \\ \vdots & \vdots & \ddots & \\ & & & R_{n_t\times n_t} \end{pmatrix} \tag{4.1}$$

Here $(-)_{a\times b}$ is a shorthand for $M_{a\times b}(-)$. Strictly speaking this is proved in [29] only in the case that R is complete, but as is remarked in [29, bottom of p. 364] the result remains valid in the case we consider.

For $i = 1, \dots, t$ put $p_i = \sum_{j\le i} n_j$, $q_i = \sum_{j>i} n_j$. Also let P_i be the i'th indecomposable projective for Δ.

Thus by definition

$$P_i = \begin{pmatrix} m_{p_{i-1}} \\ R_{q_{i-1}} \end{pmatrix}$$

where $(-)_a$ now stands for $M_{a\times 1}(-)$.

Clearly $P_1 \supset P_2 \supset \cdots \supset P_t$. Set $S_i = P_1/P_i$ for $i = 2, \dots, t$ and for the same indices put

$$\Delta_i = \begin{pmatrix} R_{p_{i-1}\times p_{i-1}} & m_{p_{i-1}\times q_{i-1}} \\ R_{q_{i-1}\times p_{i-1}} & R_{q_{i-1}\times q_{i-1}} \end{pmatrix}$$

The Δ_i are submaximal orders containing Δ and S_i is a simple Δ_i-module.

Using this observation one proves:

Lemma 4.1. *One has*

$$\Delta_i \otimes_\Delta S_j = \begin{cases} 0 & \text{if } i > j \\ S_i & \text{if } i \le j \end{cases}$$

Proof. Since one has $\Delta_i \otimes_\Delta P_1 = \Delta_i \otimes_\Delta \Delta_i \otimes_{\Delta_i} P_1 = \Delta_i \otimes_{\Delta_i} P_1 = P_1$ it follows that

$$\Delta_i \otimes_\Delta S_j = P_1/\Delta_i P_j$$

It is now easy to see that

$$\Delta_i P_j = \begin{cases} P_1 & \text{if } i > j \\ P_i & \text{if } i \le j \end{cases}$$

which yields the result. $\square$

Corollary 4.2. $\operatorname{Ext}^1_\Delta(S_j, S_i) = 0$, *and*

$$\operatorname{Hom}_\Delta(S_j, S_i) = \begin{cases} 0 & \text{if } i > j \\ R/m & \text{if } i \le j \end{cases}$$

Proof. One has

$$\operatorname{Ext}^p_\Delta(S_j,S_i)=\operatorname{Ext}^p_{\Delta_i}(\Delta_i\otimes_\Delta S_j,S_i)=\begin{cases}0 & \text{if } i>j\\ \operatorname{Ext}^p_{\Delta_i}(S_i,S_i) & \text{if } i\le j\end{cases}$$

It is easy to see that $\operatorname{Hom}(S_i,S_i)=R/m$ and one verifies directly that $\operatorname{Ext}^1_{\Delta_i}(S_i,S_i)=0$. □

Corollary 4.3. *$(S_i)_{i=2,\dots,t}$ is a strongly exceptional collection for Δ.*

One also has

Lemma 4.4. $\operatorname{Ext}^p_\Delta(S_j,P_1)=0$ *for all p.*

Proof. Put $\Gamma=M_n(R)$. Then one has $\operatorname{Ext}^p_\Delta(S_j,P_1)=\operatorname{Ext}^p_\Gamma(\Gamma\otimes_\Delta S_j,P_1)=0$ □

5. Existence of tilting objects for hereditary orders on $\mathbb{P}^1$

In this section X will be $\mathbb{P}^1$ for an algebraically closed field k. Let K be the function field of X and let $A=M_n(K)$. Let $\mathcal{O}$ be a sheaf of hereditary orders in A. Thus locally $\mathcal{O}$ is a hereditary order over a Dedekind ring (in the sense of [29]).

To compute global Ext's in $\operatorname{coh}(\mathcal{O})$ below we will use the fact that [14, Prop. II.5.3]

$$\operatorname{RHom}_{\mathcal{O}}(A,B)=R\Gamma(X,\mathrm{R}\mathcal{H}om_{\mathcal{O}}(A,B)) \tag{5.1}$$

together with the fact that $\mathrm{R}\mathcal{H}om_{\mathcal{O}}(A,B)$ can be computed locally. That is, if $x\in X$ then $\mathrm{R}\mathcal{H}om_{\mathcal{O}}(A,B)_x=\operatorname{RHom}(A_x,B_x)$.

Let $\bar{\mathcal{O}}$ be a maximal order in A lying over $\mathcal{O}$. By Tsen's theorem there exists a vector bundle $\mathcal{E}$ of rank n on X such that $\bar{\mathcal{O}}=\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$.

Let the notations be as in Proposition 2.1.

Fix i and put $R_i=\mathcal{O}_{X,x_i}$, $\Delta_i=\mathcal{O}_{x_i}$. Then Δ_i is isomorphic to an order of the form (4.1) with $t=e_i$. We choose this isomorphism in such a way that it extends to an isomorphism between $\bar{\mathcal{O}}_{x_i}$ and $M_n(R_i)$. Let us write S_{ij} $(i=1,\dots t, j=2,\dots,e_i)$ for the finite length Δ_i-modules which were denoted by S_j in Section 4. We consider the S_{ij} as $\mathcal{O}$-modules.

Since $\operatorname{coh}\mathcal{O}_X$ has a tilting object, for example given by $\mathcal{O}_X\oplus\mathcal{O}_X(-1)$, the same is true for $\bar{\mathcal{O}}$ by Morita theory. We will take $\bar{T}=\mathcal{E}\oplus\mathcal{E}(-1)$ as a tilting object in $\operatorname{coh}(\bar{\mathcal{O}})$. By the choice of the local isomorphisms, $\mathcal{E}_{x_i}$ will correspond to the projective Δ-module denoted by P_1 in the previous section.

Proposition 5.1. *$T=\bigoplus_{ij}S_{ij}\oplus\mathcal{E}\oplus\mathcal{E}(-1)$ is a tilting object in $\operatorname{coh}(\mathcal{O})$.*

Proof. Since $K_0(\mathcal{O}_{\mathbb{P}^1})\cong\mathbb{Z}^2$ it follows from Proposition 2.1 that the number of summands of T is equal to the rank of $K_0(\mathcal{O})$.

We want to show that the summands of T are a strongly exceptional collection. To do this we have to compute the $\operatorname{Ext}^*(-,-)$ between the summands of T. We first compute the $\mathrm{R}\mathcal{H}om$'s using Corollary 4.2 and lemma 4.4. The result is as follows.

(5.2)

	S_{kl}	$\mathcal{E}$	$\mathcal{E}(-1)$
S_{ij}	$*$	0	0
$\mathcal{E}$	$\mathcal{O}_{x_k}$	$\mathcal{O}_X$	$\mathcal{O}_X(-1)$
$\mathcal{E}(-1)$	$\mathcal{O}_{x_k}$	$\mathcal{O}_X(1)$	$\mathcal{O}_X$

For the square marked '$*$' we have (using Corollary 4.2)

$$\mathrm{R}\mathcal{H}om(S_{ij},S_{kl})=\begin{cases}0 & \text{if } i\neq k\\ 0 & \text{if } i=k \text{ and } l>j\\ \mathcal{O}_{x_i} & \text{otherwise}\end{cases}$$

It now follows immediately from (5.1) that Ext^* is zero between the summands of T. For the Hom's we find:

(5.3)

	S_{kl}	$\mathcal{E}$	$\mathcal{E}(-1)$
S_{ij}	$*$	0	0
$\mathcal{E}$	k	k	0
$\mathcal{E}(-1)$	k	k^2	k

with the '$*$' entry given by

$$\mathrm{Hom}(S_{ij}, S_{kl}) = \begin{cases} 0 & \text{if } i \neq k \\ 0 & \text{if } i = k \text{ and } l > j \\ k & \text{otherwise} \end{cases}$$

So it follows in particular that T is defined by a strongly exceptional collection. We are now done by Corollary 3.9. □

6. Finitely generated Grothendieck groups and existence of tilting objects.

In this section we combine our previous results to get our desired connection between existence of tilting objects and the Grothendieck group being finitely generated.

The main result of this paper is the following.

Theorem 6.1. *Let $\mathcal{C}$ be a connected noetherian hereditary abelian* Ext*-finite k-category with Serre functor, where k is an algebraically closed field. Then the following are equivalent.*

(a) *$K_o(\mathcal{C})$ is finitely generated.*
(b) (i) *$\mathcal{C}$ has a tilting object or*
 (ii) *$\mathcal{C}$ is the category of finite dimensional representations of the quiver $\tilde{A}_n$ with cyclic orientation for some $n < \infty$.*

Proof. $(b) \Rightarrow (a)$. We have already pointed out that $(b)(i)$ implies (a) [19, I.4.6], and $(b)(ii)$ implies (a) is obvious.

$(a) \Rightarrow (b)$. Assume that $K_o(\mathcal{C})$ is finitely generated. If $\mathcal{C} = \operatorname{mod}\Lambda$ for a finite dimensional hereditary k-algebra Λ, then $\mathcal{C}$ has a tilting object. If $\mathcal{C} = \operatorname{coh}\mathcal{O}$ where $\mathcal{O}$ is a sheaf of hereditary orders over $\mathbb{P}^1$, it follows from Proposition 5.1 that $\mathcal{C}$ has a tilting object. Hence we are done using Theorem 2.3. □

Actually, the following related result is also of interest.

Theorem 6.2. *Let $\mathcal{C}$ be a connected noetherian* Ext*-finite hereditary category which has no projectives or injectives and which has an object which is not of finite length. Then the following are equivalent.*

1. *$\mathcal{C}$ has a tilting object.*
2. *$\mathcal{C}$ is derived equivalent to a finite dimensional algebra.*
3. *$\mathcal{C}$ has almost split sequences and $K_0(\mathcal{C})$ is finitely generated.*
4. *$\mathcal{C}$ is of the form $\mathrm{coh}(\mathcal{O})$ where $\mathcal{O}$ is a sheaf of hereditary $\mathcal{O}_{\mathbb{P}^1}$-orders.*
5. *$\mathcal{C}$ is of the form $\operatorname{coh}\mathbb{X}$ for a weighted projective line $\mathbb{X}$.*

Proof. 1. $\Rightarrow$ 2. When $\mathcal{C}$ has a tilting object T, it follows from [19, I, Th. 4.6] that $\mathcal{C}$ is derived equivalent to the finite dimensional algebra $\mathrm{End}(T)^{\mathrm{opp}}$.

2. $\Rightarrow$ 3. Since the hereditary category $\mathcal{C}$ is derived equivalent to a finite dimensional algebra Λ, it follows that Λ must have finite global dimension. Hence $\operatorname{mod}\Lambda$, and consequently $\mathcal{C}$, has a Serre functor [15]. Then it follows that $\mathcal{C}$ has almost split sequences [31].

Since $K_0(\operatorname{mod}\Lambda)$ is finitely generated, it follows that $K_o(\mathcal{C})$ is finitely generated because this property is an invariant of derived equivalence.

3. $\Rightarrow$ 4. Since $\mathcal{C}$ has almost split sequences and no nonzero projectives or injectives, it follows that $\mathcal{C}$ has a Serre functor [31]. Since $\mathcal{C}$ has some object of infinite length, it follows from Theorem 2.3 that $\mathcal{C}$ is of the form $\operatorname{coh}(\mathcal{O})$ where $\mathcal{O}$ is a sheaf of hereditary $\mathcal{O}_{\mathbb{P}^1}$-orders.

4. $\Rightarrow$ 1. This follows from Proposition 5.1.

5. $\Leftrightarrow$ 1. That $\operatorname{coh}\mathcal{X}$ for a weighted projective line $\mathcal{X}$ has a tilting object follows from [12], and 1. $\Rightarrow$ 5. follows from [25]. □

In an appendix we give for completeness an independent proof of 4. $\Leftrightarrow$ 5, hence providing a proof of Theorem 6.2 without using [25].

7. Examples and comments

In this section we give some examples and comments without proofs, related to the material in this paper.

We start by pointing out how to obtain some concrete examples of categories $\operatorname{qgr} S$. Translation quivers $\mathbb{Z}\Delta$, where Δ is an extended Dynkin diagram occur as AR-quivers for the graded reflexive modules over invariant rings $S = k[X,Y]^G$ where G is a finite group and k is an algebraically closed field of characteristic zero (see [2] for the definition of AR-quiver). The corresponding mesh category for $\mathbb{Z}\Delta$ is then a full subcategory of $\operatorname{qgr} S$ whose objects have no nonzero summands of finite length. We obtain a (finite) basis for $K_0(\operatorname{qgr} S)$ by considering vertices given by a "section". Actually such a set of vertices corresponds to a tilting object. For two-dimensional Z-graded rings S' of finite (graded) representation type we have that $\operatorname{qgr} S'$ is equivalent to some $\operatorname{qgr} k[X,Y]^G$. The rings $k[X,Y]^G$ are Gorenstein. It is however not true for a two-dimensional isolated singularity in general that there is a commutative Gorenstein ring S with $\operatorname{qgr} S'$ equivalent to $\operatorname{qgr} S$(see [24]).

While there is a lot of analogy with the work in [30], we note that there are also some differences. In the complete case, for the rings Λ of finite representation type, considered as orders, the rank n of $K_0(q(\Lambda))$ gives information on how far Λ is from being a maximal order (of finite representation type). In this case there was a chain $\Lambda = \Lambda_1 \subset \cdots \subset \Lambda_n$ of orders with Λ_n maximal and such that there is no refinement of the chain. Then $K_0(q(\Lambda))$ has rank 1, when Λ is a maximal order, and all commutative Λ of finite type are maximal orders. In the graded case however, given S with rank $\operatorname{qgr}(S) = n$, there is no corresponding chain of graded orders ending up with $k[X,Y]$.

We point out that if $\mathcal{C}$ is a hereditary abelian k-category with all objects of finite length, then $\mathcal{C}$ does not necessarily have almost split sequences (or Serre functor). For example, this is the case if $\mathcal{C}$ is the category of holonomic modules over the first Weyl algebra (see [28]). And it follows from [35] that it holds for the category of finite dimensional representations over k of a finite connected quiver having oriented cycles, but which is not equal to a single oriented cycle.

Appendix A. Hereditary orders and weighted projective lines

In this appendix we will show directly that $\operatorname{coh}(\mathcal{O})$ for $\mathcal{O}$ a classical hereditary order over $\mathbb{P}^1$ is equivalent to $\operatorname{coh}\mathcal{X}$ for a weighted projective line $\mathcal{X}$ and furthermore we will show that every weighted projective line appears in this way. As was said before this can be deduced from Theorem 6.2 together with [25].

We follow the methods of [1], except that we consider gradings by rank one abelian groups which can have torsion.

To formalize this let $\mathcal{D}$ be an abelian category and let $O \in \mathcal{D}$ be an object. In addition let $(t_h)_{h\in H}$ be a family of autoequivalences of $\mathcal{D}$ indexed by a group H

and for any pair $h_1, h_2 \in H$ assume there are given natural isomorphisms $\eta_{h_1,h_2} : t_{h_1} t_{h_2} \to t_{h_1 h_2}$ satisfying the cocycle condition

$$\eta_{h_1 h_2, h_3}(\eta_{h_1,h_2} \cdot t_{h_3}) = (t_{h_1} \cdot \eta_{h_2,h_3})\eta_{h_1,h_2 h_3} \tag{A.1}$$

The data (t_h), (η_{h_1,h_2}) can be used to put a H-graded ring structure on

$$\Gamma^*(O) = \bigoplus_{h \in H} \operatorname{Hom}(t_h^{-1} O, O)$$

as well as a H-graded $\Gamma^*(O)$-module structure on

$$\Gamma^*(M) = \bigoplus_{h \in H} \operatorname{Hom}(t_h^{-1} O, M)$$

From now on we will assume that H is a finitely generated abelian group of rank one. We fix a non-torsion element z in H. Associated to H there is a surjective map $\phi : H \to \mathbb{Z}$, unique up to sign. We fix the sign by imposing $\phi(z) > 0$. If U is a H-graded abelian group then we say that U has right bounded grading if $U_h = 0$ for $\phi(h) \gg 0$. If R is a noetherian H-graded ring then we define $\operatorname{qgr}(R) = \operatorname{gr}(R)/\operatorname{tors}(R)$ where $\operatorname{tors}(R)$ consists of the right bounded modules.

The following result is an easy extension of [1, Thm 4.5].

Proposition A.1. *Let $\mathcal{D}$ be a noetherian* Ext*-finite abelian category and let $O \in \mathcal{D}$. Let $(t_h)_{h \in H}$ be a system of autoequivalences as above and assume that $(\mathcal{D}, O, t_z)$ is an ample triple in the sense of* [1]*. Then $R = \Gamma^*(O)$ is noetherian and the functor Γ^* defines an equivalence between $\mathcal{D}$ and* $\operatorname{qgr}(R)$.

Now let $X = \mathbb{P}^1$, $K = k(X)$ and let $\mathcal{O}$ be a sheaf of hereditary $\mathcal{O}_X$-orders in $A = M_n(K)$.

Let $x_1, \dots, x_t \in X$ be the set of ramification points of $\mathcal{O}$. Since the analysis of the cases $t = 0$, $t = 1$ and $t \geq 2$ is somewhat different we consider the case $t \geq 2$ first. Afterwards we discuss the other cases. Fix an arbitrary point x in X distinct from $x_1, \dots, x_t$ and let $(f_i)_{i=1,\dots,t}$ be rational functions with divisor $-(x_i) + (x)$.

Let I_i be fractional $\mathcal{O}$-ideals in A defined by the condition

$$(I_i)_y = \begin{cases} (\operatorname{rad} \mathcal{O}_{x_i})^{-1} & \text{if } y = x_i \\ \mathcal{O}_y & \text{otherwise} \end{cases}$$

From this definition we obtain canonical isomorphisms (as fractional ideals)

$$I_i^{e_i} \to I_j^{e_j} : x \to x f_j / f_i \tag{A.2}$$

We let H be the abelian group of rank one generated by the elements $h_1, \dots, h_t$, subject to the relations $e_i h_i = e_j h_j$ and we put $z = h_1 + \cdots + h_t$.

Every $h \in H$ has a unique representation of the form $a_1 h_1 + \cdots + a_t h_t$ with $0 \leq a_i < e_i$ for $i > 1$. We define $I_h = I_1^{a_1} \cdots I_t^{a_t}$. From (A.2) we obtain canonical isomorphisms $\zeta_{h_1,h_2} : I_{h_1} I_{h_2} \to I_{h_1+h_2}$.

Associated to the fractional ideals I_h there are autoequivalences t_h on $\operatorname{coh}(\mathcal{O})$ given by $I_h \otimes -$. The ζ_{h_1,h_2} define natural isomorphisms $\eta_{h_1,h_2} : t_{h_1} t_{h_2} \to t_{h_1+h_2}$ satisfying the cocycle condition (A.1).

Now let $\bar{\mathcal{O}}$ be a maximal order overlying $\mathcal{O}$. As usual $\bar{\mathcal{O}} = \mathcal{E}nd(\mathcal{E})$ for some vector bundle $\mathcal{E}$ on X. It follows from [31, Ch IV] that the triple $(\operatorname{coh}(\mathcal{O}), \mathcal{E}, t_z)$ is ample. Hence if we take $O = \mathcal{E}$ in the above notations and we put $R = \Gamma^*(\mathcal{E})$ then R is a noetherian H-graded ring and Γ^* defines an equivalence between $\operatorname{coh}(\mathcal{O})$ and $\operatorname{qgr}(R)$.

Our next aim will be to show that R is in fact a weighted projective line. Unfortunately the autoequivalences η_{h_1,h_2} clutter up our computations rather badly. Therefore we will first give a more elegant description of R.

Let D be the graded ring defined by

$$D = A[u_1, u_1^{-1}, \dots, u_t, u_t^{-1}]/(f_i u_i^{e_i} = f_j u_j^{e_j})]$$

D is clearly H-graded by putting $\deg u_i = h_i$.

Let $\mathcal{A}$ be the graded order in D defined by

$$\mathcal{A} = \bigoplus_{p_1, \dots, p_t \in \mathbb{Z}} (I_1 u_1)^{p_1} \cdots (I_t u_t)^{p_t}$$

Now it is not hard to see that

$$R = \operatorname{Hom}(\mathcal{E}, \mathcal{A} \otimes_{\mathcal{O}} \mathcal{E})$$

We will determine the structure of R explicitly. A local computation shows that R is equal to

$$\bigoplus_{p_1, \dots, p_t \in \mathbb{N}} \Gamma(X, \mathcal{O}_X([p_1/e_1]x_1 + \cdots + [p_t/e_t]x_t)) u_1^{p_1} \cdots u_t^{p_t}$$

where $[\alpha]$ denotes the biggest integer not bigger than α.

We first claim that R is generated by $u_1, \dots, u_t$. By the relations in D it follows that

$$\mathcal{O}_X([p_1/e_1]x_1 + \cdots + [(p_i + e_i)/e_i]x_i + \cdots + [p_j/e_j]x_j + \cdots + [p_t/e_t]x_t) u_1^{p_1} \cdots u_i^{p_i+e_i} \cdots u_j^{p_j} \cdots u_t^{p_t}$$
$$= \mathcal{O}_X([p_1/e_1]x_1 + \cdots + [p_i/e_i]x_i + \cdots + [(p_j + e_j)/e_j]x_j + \cdots + [p_t/e_t]x_t) u_1^{p_1} \cdots u_i^{p_i} \cdots u_j^{p_j+e_j} \cdots u_t^{p_t}$$

as subsheaves of D. Hence to show that every section of $\mathcal{O}_X([p_1/e_1]x_1 + \cdots + [p_t/e_t]x_t]) u_1^{p_1} \cdots u_t^{p_t}$ is a linear combination of products of the u_i's, it suffices to do so in the case that $p_i < e_i$ for $i \geq 2$. So below we make this assumption.

Write $p_1 = q_1 + a e_1$ where $q_1 < e_1$. We then have $\mathcal{O}_X([p_1/e_1]x_1 + \cdots + [p_t/e_t]x_t]) = \mathcal{O}_X(a x_1)$. Let $b_1, \cdots, b_t \in \mathbb{N}$ be such that $b_1 + \cdots + b_t = a$. Using the relations in D we find that

$$u_1^{q_1 + b_1 e_1} u_2^{p_2 + b_2 e_2} \cdots u_t^{p_t + b_t e_t} = f_1^{b_2 + \cdots + b_t} / (f_2^{b_2} \cdots f_t^{b_t}) u_1^{p_1} u_2^{p_2} \cdots u_t^{p_t}$$

The divisor of $f_1^{b_2 + \cdots + b_t} / (f_2^{b_2} \cdots f_t^{b_t})$ is equal to $-a(x_1) + \sum_{i=1}^t b_i(x_i)$. Hence these rational functions clearly generate the global sections of $\mathcal{O}_X(a x_1)$, which is what we had to show.

We now claim that up to changing $f_1, (f_i)_{i \geq 3}$ by a scalar we have the following relations in R :

$$u_i^{e_i} - u_2^{e_2} + \lambda_i u_1^{e_1} = 0 \qquad (i \geq 3) \tag{A.3}$$

where the $(\lambda_i)_{i \geq 3}$ are suitable scalars with $\lambda_3 = 1$.

Rewriting $u_2^{e_2}$ and $u_i^{e_i}$ in terms of u_1 it follows that the relation (A.3) is equivalent to the existence of a linear dependence

$$f_1/f_i - f_1/f_2 + \lambda_i = 0 \tag{A.4}$$

Now the divisors of f_1/f_i, f_1/f_2 and 1 are respectively given by $-(x_1) + (x_i)$, $-(x_1) + (x_2)$ and 0. In particular these three rational functions are all sections of $\mathcal{O}_X(x_1)$. Since $\mathcal{O}_X(x_1)$ has degree one, it follows that there has to be at least a linear dependence

$$\alpha f_1/f_i + \beta f_1/f_2 + \gamma = 0 \tag{A.5}$$

Furthermore, inspecting divisors, it is easily seen that α, β, γ must all be non-zero. Dividing (A.5) by $-\beta$ and changing f_i by a scalar yields (A.4). To make λ_3 equal to 1 we finish by changing f_1 by a suitable scalar.

At this point we know that R is a quotient of the "weighted projective line"

$$k[u_1, \dots, u_t]/(u_i^{e_i} - u_2^{e_2} + \lambda_i u_1^{e_1}) \tag{A.6}$$

However a straightforward computation reveals that R and the ring defined by (A.6) have the same Hilbert series. Hence they are isomorphic. This concludes our analysis of the case $t \geq 2$.

We will now discuss the other cases. First let $t = 1$. We define R as above. Now $\dim R_i = 1$ for $i < e_1$ and $\dim R_e = 2$. Let $v \in R_2 \setminus k u_1^e$. We leave it as an exercise to the reader to check that $R \cong k[u_1, v]$. Hence $\mathrm{coh}(\mathcal{O})$ is again described by a weighted projective line.

The case $t = 0$ is even more trivial. In that case $\mathcal{O}$ is Morita equivalent to $\mathcal{O}_{\mathbb{P}^1}$. So $\mathrm{coh}(\mathcal{O})$ is in fact described by the ordinary projective line!

To finish we show that one can get all weighted projective lines from hereditary orders. It suffices to do this in the case $t > 2$. It is convenient to choose an affine coordinate system on $\mathbb{P}^1$ in such a way that $x_1 = \infty$, $x_2 = 0$, $x_3 = 1$. Then up to a scalar we have

$$f_1(z) = z - x, \qquad f_i(z) = \frac{z - x}{z - x_i} \text{ for } i > 1$$

Computing the λ_i explicitly with the above procedure we find $\lambda_i = x_i$. This shows what we want.

Appendix B. Examples of hereditary abelian categories

In this appendix we give some sources of examples of hereditary abelian categories, which are usually not Ext-finite. These are inspired by [36].

Let R be a noetherian ring of Krull dimension $n \geq 0$, finitely generated as a module over a central subring C. Denote by $\mathrm{Mod}\, R$ the category of R-modules and as before by $\mathrm{mod}\, R$ the subcategory of finitely generated R-modules. For $i \geq -1$, let $\mathcal{C}_i$ be the subcategory of $\mathrm{mod}\, R$ whose objects have Krull dimension at most i, and let $\tilde{\mathcal{C}}_i$ be the subcategory of $\mathrm{Mod}\, R$ whose objects are direct limits of objects in $\mathcal{C}_i$. (We define $\mathcal{C}_{-1} = \tilde{\mathcal{C}}_{-1} = (0)$.) Let $\mathrm{qmod}_i(R) = \mathrm{mod}\, R/\mathcal{C}_i$ and $\mathrm{QMod}_i(R) = \mathrm{Mod}\, R/\tilde{\mathcal{C}}_i$ be the corresponding quotient categories in the sense of [14]. These are abelian categories. Similarly we consider the case when S is a $\mathbb{Z}$-graded noetherian ring finitely generated over k and finitely generated as a module over a central subring C, such that $S_i = 0$ for i small enough. We make the similar definitions starting with the category $\mathrm{Gr}\, S$ of graded S-modules with degree zero homomorphisms, and the subcategory $\mathrm{gr}\, S$ of finitely generated modules. The corresponding quotient categories will be denoted by $\mathrm{QGr}_i(S)$ and $\mathrm{qgr}_i(S)$. *Below when we work in the graded case all objects will be implicitly considered to be graded, unless otherwise specified.*

We can now prove the following, which gives some classes of hereditary categories.

Proposition B.1. *Let R be a noetherian ring of Krull dimension $n \geq 0$ with the above assumptions and notation. Assume that C satisfies $\mathrm{Kdim}\, C/P + \mathrm{ht}\, P = n$ for every prime ideal P in C. Then the following conditions are equivalent.*

(a) $\mathrm{QMod}_i(R)$ *is nonzero hereditary.*
(b) $\mathrm{qmod}_i(R)$ *is nonzero hereditary.*
(c) *Either $i = n - 2$ and $\mathrm{gl.dim}\, R_P \leq 1$ for any prime ideal P in C of height at most 1 or $i = n - 1$ and $\mathrm{gl.dim}\, R_P \leq 1$ for any prime ideal P in C of height 0.*

Proof. That(a) and (b) are equivalent follows from [31, Proposition A3]. To prove the other equivalences we first review some generalities. First of all if M is an R-module, then by [11, p430, Cor. 2] the Krull dimension of M as R-module is equal to the Krull dimension of M as C-module.

Furthermore we claim that $M \in \tilde{\mathcal{C}}_i$ if and only if $M_P = 0$ for all $P \in \mathrm{Spec}\, C$ such that $\mathrm{Kdim}\, C/P < i$ (or equivalently, if and only if $M_P = 0$ for all P such that $\operatorname{ht} P > n - i$). To see this we may assume that M is finitely generated. Then it follows from the theory of associated primes that M has a finite filtration (as C-module) with subquotients of the form C/Q with $Q \in \operatorname{Spec} C$. The claim is now an immediate verification.

The subcategory $\tilde{\mathcal{C}}_i$ of $\operatorname{Mod} R$ is a localizing subcategory. Since R is a noetherian ring which is finitely generated as a module over its center, $\tilde{\mathcal{C}}_i$ is closed under injective envelopes [11, p. 431]. Denoting by $T\colon \operatorname{Mod} R \to \operatorname{Mod} R/\tilde{\mathcal{C}}_i$ the associated quotient functor we have that T preserves injective objects and injective envelopes by [31, Proposition A4]. So if $0 \to M \to I_0 \to I_1 \to \cdots$ is a minimal injective resolution in $\operatorname{Mod} R$, then $0 \to T(M) \to T(I_0) \to T(I_1) \to \cdots$ is a minimal injective resolution in $\tilde{\mathcal{H}}_i$. A similar reasoning shows that for each prime ideal P in C, we have that $0 \to M_P \to (I_0)_P \to (I_1)_P \to \cdots$ is a minimal injective resolution in $\operatorname{Mod} R_P$.

(c) $\Rightarrow$ (a). Assume that (c) holds, and consider for M in $\operatorname{Mod} R$ a minimal injective resolution $0 \to M \to I_0 \to I_1 \to \cdots$ in $\operatorname{Mod} R$, and the induced minimal injective resolution $0 \to M_P \to (I_0)_P \to (I_1)_P \to \cdots$ for a prime ideal P in C.

If $i = n-2$ and $\operatorname{gl.dim} R_P \le 1$ for $\operatorname{ht} P \le 1$, we get $(I_j)_P = 0$ for $j \ge 2$ and $\operatorname{ht} P \le 1$, and hence I_j is in $\tilde{\mathcal{C}}_{n-2}$ for $j \ge 2$, so that $T(I_j) = 0$. Then we have an injective resolution $0 \to T(M) \to T(I_0) \to T(I_1) \to 0$ in $\operatorname{QMod}_{n-2}(R)$, which shows $\operatorname{id}_{\operatorname{QMod}_{n-2}(R)} T(M) \le 1$. Since T is essentially surjective, it follows that $\operatorname{gl.dim} \operatorname{QMod}_{n-2}(R) \le 1$.

If $i = n-1$ and $\operatorname{gl.dim} R_P \le 1$ for $\operatorname{ht} P = 0$, we get that I_j is in $\tilde{\mathcal{C}}_{n-1}$ for $j \ge 2$, so that $0 \to T(M) \to T(I_0) \to T(I_1) \to 0$ is an injective resolution of $T(M)$ in $\operatorname{QMod}_{n-1}(R)$. Hence we have $\operatorname{gl.dim} \operatorname{QMod}_{n-1}(R) \le 1$.

(a) $\Rightarrow$ (c). Assume now that $\operatorname{QMod}_i(R)$ is hereditary, and let $0 \to M \to I_0 \to I_1 \to \cdots$ be a minimal injective resolution of some M in $\operatorname{Mod} R$. Since then $0 \to T(M) \to T(I_0) \to T(I_1) \to \cdots$ is a minimal injective resolution in $\operatorname{QMod}_i(R)$, it follows that $T(I_j) = 0$ for $j > 1$. Then I_j is in $\tilde{\mathcal{C}}_i$ for $j > 1$, so that we have $(I_j)_P = 0$ when $\operatorname{ht} P < n-i$. Thus we have the exact sequence $0 \to M_P \to (I_0)_P \to (I_1)_P \to 0$ in this case, and hence $\operatorname{gl.dim} R_P \le 1$. Since for a noetherian ring R which is finitely generated as a module over its center we have $\operatorname{Kdim} R_P \le \operatorname{gl.dim} R_P$ [7], and furthermore trivially $\operatorname{Kdim} C_P \le \operatorname{Kdim} R_P$, it follows that $n-i-1 \le 1$, so that $n-i \le 2$. Hence when $\operatorname{QMod}_i(R) \ne 0$, we must have $i = n-2$ or $i = n-1$. In the first case we have $\operatorname{gl.dim} R_P \le 1$ for $\operatorname{ht} P \le 1$, and in the second case $\operatorname{gl.dim} R_P \le 1$ for $\operatorname{ht} P = 0$. □

In the case of graded rings S with the assumptions listed in the above, we consider graded prime ideals P in the center C and graded localizations S_P and graded global dimension. Using that also in this case $\tilde{\mathcal{C}}_i$ is closed under injective envelopes [27], the proof of Proposition B.1 is easily adapted to give the following.

Proposition B.2. *Let S be a Z-graded ring of Krull dimension $n \ge 0$ satisfying the standard assumptions, and with the previous notation. Suppose that C satisfies $\operatorname{Kdim} C/P + \operatorname{ht} P = n$ for every graded prime ideal P. Then the following are equivalent.*

(a) $\operatorname{QGr}_i(S)$ *is nonzero hereditary.*
(b) $\operatorname{qgr}_i(S)$ *is nonzero hereditary.*
(c) *Either $i = n-2$ and $\operatorname{gl.dim} S_P \le 1$ for any graded prime ideal P in C of height at most 1 or $i = n-1$ and $\operatorname{gl.dim} S_P \le 1$ for any graded prime ideal P in C of height 0.*

We also state the following special case.

Corollary B.3. *Let $S = C$ be a Z-graded commutative domain of Krull dimension 2 satisfying the standard assumptions. Then* $\mathrm{qgr}(S)$ *is hereditary if and only if S is an isolated singularity.*

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