A TRANSLATION PRINCIPLE FOR THE FOUR-DIMENSIONAL SKLYANIN ALGEBRAS

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Abstract. In this paper we prove a translation principle for the central quotients of four-dimensional Sklyanin algebras, which is analogous to the translation principle for semi-simple Lie algebras. In the course of the proof we construct an "elliptic" analog of the sheaf of differential operators on $\mathbb{P}^1$. The translation principle may be used to construct the fat points of a four-dimensional, non-PI, Sklyanin algebra.

Contents

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1. Introduction

In this introduction we let the ground field F be algebraically closed of characteristic zero.

The classical translation principle states that certain primitive quotients of enveloping algebras of semi-simple Lie algebras are Morita equivalent. To give a more precise discussion we will, for simplicity, assume that we work with $\mathfrak{sl}_2$.

The defining equations for $\mathfrak{sl}_2 = FA+FH+FB$ are $[H,A]=2A$, $[H,B]=-2B$, $[A,B]=H$. The Casimir element $C=2AB+2BA+H^2$ generates the center of $U(\mathfrak{sl}_2)$ and the maximal primitve quotients are given by

$$U^\lambda = U(\mathfrak{sl}_2)/(C-\lambda(\lambda+2))$$

The translation principle states that U^λ and $U^{\lambda'}$ are Morita equivalent if $\lambda-\lambda'\in\mathbb{Z}$ and $\lambda,\lambda'\neq -1$.

The standard proof of this result is through the Bernstein-Beilinson theorem [6]. For simplicty, assume that $\lambda,\lambda'\in\mathbb{N}$. Denote by $\mathcal{D}^\lambda$ the sheaf of differential operators on $\mathcal{O}_{\mathbb{P}^1}(\lambda)$. Then $U^\lambda=\Gamma(\mathbb{P}^1,\mathcal{D}^\lambda)$. The remarkable fact proved by Bernstein and Beilinson (in much greater generality) is that there is a category equivalence between U^λ-mod and the category of quasi-coherent $\mathcal{D}^\lambda$-modules. This equivalence is given by

$$U^\lambda\text{-mod} \underset{\Gamma(\mathbb{P}^1,-)}{\overset{\mathcal{D}^\lambda\otimes_{U^\lambda}-}{\rightleftarrows}} \mathcal{D}^\lambda\text{-qch} \tag{1.1}$$

On the other hand it is easy to prove that

$$\mathcal{D}^{\lambda'} = \mathcal{O}_{\mathbb{P}^1}(\lambda'-\lambda)\otimes_{\mathcal{O}_{\mathbb{P}^1}}\mathcal{D}^\lambda\otimes_{\mathcal{O}_{\mathbb{P}^1}}\mathcal{O}_{\mathbb{P}^1}(\lambda-\lambda')$$

and thus $\mathcal{O}(\lambda-\lambda')\otimes_{\mathcal{O}_{\mathbb{P}^1}} -$ defines a category equvalence between $\mathcal{D}^\lambda$-qch and $\mathcal{D}^{\lambda'}$-qch. Combining this with (1.1) yields the sought equivalence between U^λ-mod and $U^{\lambda'}$-mod.

In this paper we will define a similar translation principle for central quotients of four-dimensional Sklyanin algebras. The latter were defined by Sklyanin in [14][15] in connection with his work on quantum inverse scattering theory. They are noetherian graded algebras of the form

$$F\oplus A_1\oplus A_2\oplus\cdots \tag{1.2}$$

with Hilbert series $1/(1-t)^4$, having good homological properties [18][22]. In [12][13] Odeskii and Feigin showed that one could define Sklyanin algebras of arbitrary dimension, starting from a triple $(E,\sigma,\mathcal{L})$ where E is a smooth elliptic curve, σ is a translation and $\mathcal{L}$ is line bundle of degree r where r is the GK-dimension of the associated algebra. A similar construction in dimension three was independently given in [3][4]. That the higher dimensional Sklyanin algebras have the expected properties was shown in [22].

It was noticed in various papers that the four-dimensional Sklyanin algebras and $U(\mathfrak{sl}_2)$ have very similar properties [8][16]. In fact, to have a complete analogy it is better not to work with $U(\mathfrak{sl}_2)$ but with the homogenized version $H(\mathfrak{sl}_2) = F[A,B,H,t]$ where $[H,A]=2tA$, $[H,B]=-2tB$, $[A,B]=2tH$, $[t,-]=0$. $H(\mathfrak{sl}_2)$ may be graded by giving all generators degree 1.

The center of $H(\mathfrak{sl}_2)$ is generated by t and C, where the latter is considered as an element of degree 2 in the obvious way. The homogenized version of U^λ is given by

$$H^\lambda = H(\mathfrak{sl}_2)/(C - \lambda(\lambda+2)t^2)$$

The fact that $H(\mathfrak{sl}_2)$ and A are graded as in (1.2) means that they are amenable to the techniques of non-commutative projective geometry as developed in [2] by Artin and Zhang (see [5] for a precursor).

Artin and Zhang associate to a left noetherian F-algebra B, graded as in (1.2) a non-commutative projective scheme $\operatorname{proj} B$. To define it, put

$$\begin{aligned} B\text{-gr} &:= \text{finitely generated graded } B\text{-modules} \\ \text{tors} &:= \text{finite dimensional graded } B\text{-modules} \\ B\text{-qgr} &:= B\text{-gr}/\text{tors} \end{aligned}$$

B-gr and B-qgr are equipped with a natural auto-equivalence, given by the shift functor $s : M \mapsto M(1)$, and B itself defines a distinguished object of B-qgr, denoted by $\mathcal{O}$. Then by definition the projective scheme associated to B is

$$\operatorname{proj} B := (B\text{-qgr}, s, \mathcal{O})$$

In geometric language, s is called the polarization and $\mathcal{O}$ the structure sheaf.

For the purposes of this paper let us define

$$\operatorname{proj}^\circ B := (B\text{-qgr}, s)$$

Then the translation principle for $U(\mathfrak{sl}_2)$ may be quite easily extended to $H(\mathfrak{sl}_2)$ as follows : assume $\lambda, \lambda' \in F$, $\lambda - \lambda' \in \mathbb{Z}$ and $\lambda, \lambda' \neq -1$ then $\operatorname{proj}^\circ H^\lambda$ and $\operatorname{proj}^\circ H^{\lambda'}$ are equivalent. That is : H^λ-qgr and $H^{\lambda'}$-qgr are equivalent and this equivalence commutes with s. See Appendix A.

It is precisely this result that we will generalize to four-dimensional Sklyanin algebras. So let A be such an algebra. Then the part of degree two of the center of A contains a canonical two-dimensional subspace $F\Omega_1 + F\Omega_2$, which, in the case that σ has infinite order, generates the whole center [9]. Then we will show that for certain pairs $\Omega, \Omega' \in F\Omega_1 + F\Omega_2$ there is an equivalence between $\operatorname{proj}^\circ A/A\Omega$ and $\operatorname{proj}^\circ A/A\Omega'$ (see Theorem 7.4.1 for a precise statement).

As a result we obtain a new, intrinsic, description (if σ has infinite order) of the fat points of A of higher multiplicity (see Proposition 7.5.2). These fat points were essentially discovered by Sklyanin [15], but as was pointed out in [16, §3] the reason for their existence was somewhat of a mystery.

Our approach to proving the translation principle for A is based upon a generalization of the Bernstein-Beilinson theorem. However, whereas $\mathcal{D}^\lambda$ is an ordinary sheaf of algebras on $\mathbb{P}^1$, we are forced to work with the more complicated notion of an $\mathcal{O}_{\mathbb{P}^1}$-bimodule algebra.

To define this notion let X be an F-scheme of finite type. In [5] an $\mathcal{O}_X$-bimodule was defined as an $\mathcal{O}_{X\times_F X}$-module satifying certain finiteness conditions. An $\mathcal{O}_X$-bimodule algebra is, roughly speaking, an algebra object in the tensor category of $\mathcal{O}_X$-bimodules. Our substitute for the Bernstein-Beilinson theorem is as follows. Assume that

$$\mathcal{B} = \mathcal{O}_X \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots$$

is a graded left noetherian bimodule algebra and put $B = \Gamma(X, \mathcal{B})$. Assume in addition that $(\mathcal{B}_n)_n$ is an ample sequence (see definition 5.1). Then there are inverse equivalences

$$B\text{-qgr} \underset{\Gamma(X,-)}{\overset{\mathcal{B}\otimes_B -}{\rightleftarrows}} \mathcal{B}\text{-qgr}$$

The proof of this result (Theorem 5.2) is a direct generalization of the proof of [5, Thm 3.12].

The bimodule algebras we need are those coming from linear A-modules. Recall that a linear A-module of dimension l is a graded A-module having the Hilbert series of a polynomial algebra in l variables. It has been shown in [22][21] that if A is a Sklyanin algebra associated to the triple $(E, \sigma, \mathcal{L})$ then linear A-modules may be associated to effective divisors on E. Now effective diviors on E of degree l correspond to points on $S^l E$, the l-fold symmetric product of E.

We will construct a bimodule algebra $\mathcal{B}$ on $X = S^l E$ in such a way that for $y \in X$ one has that $\Gamma(F(y) \otimes_{\mathcal{O}_X} \mathcal{B})$ is precisely the right linear module associated to A. (See Theorem 6.4.1). If $l = 1$ this is a standard construction which leads to the twisted homogeneous coordinate ring of the triple $(E, \sigma, \mathcal{L})$ [3][4][5]. In fact this doesn't depend on A being a Sklyanin algebra. On the other hand if $l > 1$ then the hypothesis that A is a Sklyanin algebra seems to be essential.

In general $\mathcal{B}_1$ is, on the left and on the right, a vector bundle of degree l on $\mathcal{O}_X$, and $\mathcal{B}$ should be regarded as a substitute for the symmetric algebra $S(\mathcal{B}_1)$, which is not defined. This may be used to explain to some extent the similarity of the algebras $A/A\Omega$, $\Omega \in F\Omega_1 + F\Omega_2$ with the homogeneous coordinate rings of quadrics of rank 3 or 4 in $\mathbb{P}^3$. Indeed these may be these may be described as $\Gamma(\mathbb{P}^1, S_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{M}))$ where $\mathcal{M}$ is either $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ in the regular case, or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ in the singular case.

2. Bimodules

First we have to discuss a particular type of quasi-coherent module.

Definition 2.1. Let $f : Y \to X$ be a morphism of finite type between noetherian schemes and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_Y$-module. We say that $\mathcal{M}$ is relatively locally finite (rlf) for f, if for all coherent $\mathcal{M}' \subset \mathcal{M}$ one has that $f \mid \operatorname{Supp} \mathcal{M}'$ is finite.

Proposition 2.2. *Let f, X, Y be as in the above definition.*

(1) *If $\mathcal{M}$ is rlf for f then $R^i f_* \mathcal{M} = 0$ for $i > 0$.*
(2) *If there is an exact sequence*
$$0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$$
of quasicoherent $\mathcal{O}_Y$ modules then $\mathcal{M}_{1,2}$ are rlf for f if and only if $\mathcal{M}$ is rlf for f.
(3) *If $(\mathcal{M}_i)_{i\in I}$ are rlf for f, for a directed set I, then $\varinjlim \mathcal{M}_i$ is rlf for f.*
(4) *If $\mathcal{Q}, \mathcal{M}$ are quasi-coherent $\mathcal{O}_Y$-modules and if $\mathcal{M}$ is rlf for f then $\mathcal{Q} \otimes_{\mathcal{O}_Y} \mathcal{M}$ is rlf for f.*
(5) *(Projection formula) If $\mathcal{M}$ is an $\mathcal{O}_Y$-module, rlf for f and $\mathcal{N}$ is a quasi-coherent $\mathcal{O}_X$-module then the natural map*
$$f_* \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \to f_*(\mathcal{M} \otimes_{\mathcal{O}_Y} f^* \mathcal{N})$$

is an isomorphism.

(6) *(Base change) If there is a pullback diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{p'} & Y \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ X' & \xrightarrow{p} & X \end{array}$$

and if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Y$-module, rlf for f then $p'^*\mathcal{M}$ is rlf for f' and the natural map

$$p^* f_* \mathcal{M} \to f'_* p'^* \mathcal{M}$$

is an isomorphism.

(7) *The functor f_* from the category of $\mathcal{O}_Y$-modules rlf for f to the category of quasicoherent $\mathcal{O}_X$-modules is faithful.*

Proof. The proof is based upon the fact that everything is compatible with direct limits, so we may use coherent modules. These have finite support. Since everything is also local on X, we are basically in an affine situation. □

Now we are ready define bimodules over schemes. For simplicity we fix a noetherian base scheme S. All our schemes will be S-schemes of finite type, and all our maps will be S-maps. The following definition is a generalization of [5, §2]

Definition 2.3. Let X, Y be Noetherian S-schemes of finite type. A $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Y$ bimodule is a quasi-coherent $\mathcal{O}_{X\times_S Y}$-module, rlf for the projections $\mathrm{pr}_{1,2} : X \times_S Y \to X, Y$.

Remark 2.0.1. This is a slight generalization of the definition of a coherent bimodule in [5]. The extra generality is needed because the bimodule algebras we will construct are in general not coherent.

Let $\mathcal{M}$ be an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Y$-bimodule. If $U \subset X$, $V \subset Y$ are open then $\mathcal{M}(U) \overset{\text{def}}{=} \mathcal{M}(\mathrm{pr}_1^{-1} U)$, $\mathcal{M}(V) \overset{\text{def}}{=} \mathcal{M}(\mathrm{pr}_2^{-1} V)$ are respectively the left and right sections of $\mathcal{M}$.

If $X = Y$ then this notation would be ambiguous, so we adopt the asymmetrical convention that $\mathcal{M}(U)$ stands for the left sections.

Note that in general $\mathcal{M}(X) = \mathcal{M}(Y)$. We refer to $\mathcal{M}(X)$ as the global sections of $\mathcal{M}$ and sometimes use the alternative notations $\Gamma(\mathcal{M})$, $\Gamma(X, \mathcal{M})$, $H^0(X, \mathcal{M})$, etc.... The derived functors of $\Gamma(\mathcal{M})$ will be denote by $H^i(X, \mathcal{M})$.

The sheaves given by $(U \subset X) \mapsto \mathcal{M}(U)$, $(V \subset Y) \mapsto \mathcal{M}(V)$ are respectively $\mathrm{pr}_{1*}\mathcal{M}$ and $\mathrm{pr}_{2*}\mathcal{M}$. We call these the left and right structures of $\mathcal{M}$. The functors pr_{1*} and pr_{2*} are exact and faithful by Proposition 2.2. So we may think of these functors as forgetful functors, just as in the affine case.

If $\mathcal{M}$ is a quasicoherent $\mathcal{O}_X$-module then we sometimes identify it with the $\mathcal{O}_X$-bimodule $\delta_*\mathcal{M}$ where $\delta : X \to X \times_S X$ is the diagonal.

There is a construction of bimodules that occurs often enough to merit a special notation. Let V be an S-scheme of finite type and assume that there are finite S-maps $\alpha : V \to X$, $\beta : V \to Y$. Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_V$-module. Then by ${}_\alpha\mathcal{H}_\beta$ we denote the bimodules $(\alpha, \beta)_*\mathcal{H}$. Often we use the following pictorial notation for this bimodule.

$$
\begin{tikzcd}[column sep=small]
& \mathcal{H} \\[-20pt]
& V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
X & & Y
\end{tikzcd}
\tag{2.1}
$$

As a rule, if one or both of the maps α or β is the identity then we use the notations $\mathcal{H}_\alpha$, ${}_\beta\mathcal{H}$ and $\mathcal{H}$.

The notation (2.1) allows one to present the same bimodule in different ways. For example if we have a commutative diagram

$$
\begin{tikzcd}
& V \arrow[ddl, "\alpha"'] \arrow[d, "\delta"] \arrow[ddr, "\beta"] & \\
& V' \arrow[dl, "\alpha'"] \arrow[dr, "\beta'"'] & \\
X & & Y
\end{tikzcd}
\tag{2.2}
$$

then ${}_{\alpha'}(\delta_*\mathcal{H})_{\beta'}$ and ${}_\alpha\mathcal{H}_\beta$ are the same bimodule.

Similarly, in case there is a commutative diagram

$$
\begin{tikzcd}
& V'' \arrow[ddl, "\alpha''"'] \arrow[d, "\epsilon"] \arrow[ddr, "\beta''"] & \\
& V \arrow[dl, "\alpha"] \arrow[dr, "\beta"'] & \\
X & & Y
\end{tikzcd}
$$

where ϵ is an isomorphism then ${}_\alpha(\mathcal{H})_\beta$ and ${}_{\alpha''}(\epsilon^*\mathcal{H})_{\beta''}$ are the same bimodule. This a special case of (2.2) since $\epsilon^* = (\epsilon^{-1})_*$.

If $\mathcal{M}$ is an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Y$-bimodule and $\mathcal{N}$ is an $\mathcal{O}_S$-central $\mathcal{O}_Y - \mathcal{O}_Z$-bimodule then the tensor product of $\mathcal{M}$ and $\mathcal{N}$ is an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Z$-bimodule, defined by

$$
\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N} = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^* \mathcal{M} \otimes_{\mathcal{O}_{X\times_S Y \times_S Z}} \mathrm{pr}_{23}^* \mathcal{N})
\tag{2.3}
$$

It is also possible to define tensor products of bimodules with ordinary quasi-coherent modules. For example if $\mathcal{M}$ is an $\mathcal{O}_X$-module and $\mathcal{N}$ is and $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Y$-bimodule then

$$
\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} = \mathrm{pr}_{2*}(\mathrm{pr}_1^* \mathcal{M} \otimes_{\mathcal{O}_{X\times_S Y}} \mathcal{N})
$$

This definition can be derived from (2.3) if we consider $\mathcal{N}$ as a bimodule in the canonical way.

Proposition 2.4. *The tensor product of bimodules is associative and satisfies MacLane's pentagon axiom* [10].

Proof. We will consider the most difficult case, namely the tensor product of three bimodules. Let X_1, X_2, X_3, X_4 be S-schemes of finite type. For $I \subset \{1, \dots 4\}$ denote by X_I the product $\times_{i\in I} X_i$ over S and for $I \subset J \subset \{1, \dots, 4\}$ let pr_I^J be the natural projection $X_J \to X_I$. Let $\mathcal{M}_{ii+1}$ be an $\mathcal{O}_S$-central $\mathcal{O}_{X_i} - \mathcal{O}_{X_j}$-bimodule. Then $(\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_2}} \mathcal{M}_{23}) \otimes_{\mathcal{O}_{X_3}} \mathcal{M}_{34}$ is by definition

(2.4)
$$\mathrm{pr}^{134}_{14*}(\mathrm{pr}^{134*}_{13}\,\mathrm{pr}^{123}_{13*}(\mathrm{pr}^{123*}_{12}\,\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_{123}}} \mathrm{pr}^{123*}_{23}\,\mathcal{M}_{23}) \otimes_{\mathcal{O}_{X_{134}}} \mathrm{pr}^{134}_{34*}\,\mathcal{M}_{34})$$

From base change for the pullback

$$\begin{tikzcd} X_{1234} \arrow[r] \arrow[d] & X_{123} \arrow[d] \\ X_{134} \arrow[r] & X_{13} \end{tikzcd}$$

it follows that

$$\mathrm{pr}^{134*}_{13}\,\mathrm{pr}^{123}_{13*} = \mathrm{pr}^{1234}_{134*}\,\mathrm{pr}^{1234*}_{123}$$

hence (2.4) becomes

$$\begin{aligned} (2.5)\quad & \mathrm{pr}^{134}_{14*}(\mathrm{pr}^{1234}_{134*}\,\mathrm{pr}^{1234*}_{123}(\mathrm{pr}^{123*}_{12}\,\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_{123}}} \mathrm{pr}^{123*}_{23}\,\mathcal{M}_{23}) \otimes_{\mathcal{O}_{X_{134}}} \mathcal{M}_{34}) \\ &= \mathrm{pr}^{134}_{14*}(\mathrm{pr}^{1234}_{134*}(\mathrm{pr}^{1234*}_{12}\,\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_{1234}}} \mathrm{pr}^{1234*}_{23}\,\mathcal{M}_{23}) \otimes_{\mathcal{O}_{X_{134}}} \mathrm{pr}^{134*}_{34}\,\mathcal{M}_{34}) \end{aligned}$$

Applying the projection formula for pr^{1234}_{134} yields that (2.5) is equal to

$$\begin{aligned} & \mathrm{pr}^{134}_{14*}(\mathrm{pr}^{1234}_{134*}(\mathrm{pr}^{1234*}_{12}\,\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_{1234}}} \mathrm{pr}^{1234*}_{23}\,\mathcal{M}_{23} \otimes_{\mathcal{O}_{X_{1234}}} \mathrm{pr}^{1234*}_{134}\,\mathrm{pr}^{134*}_{34}\,\mathcal{M}_{34})) \\ &= \mathrm{pr}^{1234}_{14*}(\mathrm{pr}^{1234*}_{12}\,\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_{1234}}} \mathrm{pr}^{1234*}_{23}\,\mathcal{M}_{23} \otimes_{\mathcal{O}_{X_{1234}}} \mathrm{pr}^{1234*}_{34}\,\mathcal{M}_{34}) \end{aligned}$$

A similar comutation shows that we get the same result if we start from $\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_2}} (\mathcal{M}_{23} \otimes_{\mathcal{O}_{X_3}} \mathcal{M}_{34})$. This yields a natural isomorphism between $(\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_2}} \mathcal{M}_{23}) \otimes_{\mathcal{O}_{X_3}} \mathcal{M}_{34})$ and $\mathcal{M}_{12} \otimes_{\mathcal{O}_{X_2}} (\mathcal{M}_{23} \otimes_{\mathcal{O}_{X_3}} \mathcal{M}_{34})$. An easy, but very tedious verification shows that this isomorphism satifies the pentagon axiom. □

Proposition 2.5. *The tensor product of bimodules is right exact.*

Proof. Left to the reader. □

Remark 2.6. Bimodules on schemes have many properties which are similar to the affine case. Examples of this are Propositions 2.4,2.5. Proofs of such statements are routine, and in the sequel we will often omit them. Also all kinds of compatibilies will be used implictly. An example of such a compatibility is the easily verified fact that the left module structure of a tensor product of two bimodules depends only on the left module structure of the second one.

Lemma 2.7. (1) *Assume that there is a diagram*

$$\begin{tikzcd} & \overset{\mathcal{J}}{V} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathcal{H}\ X & & Y \end{tikzcd}$$

where $\mathcal{H}$, $\mathcal{J}$ are respectively quasi-coherent $\mathcal{O}_X$ and $\mathcal{O}_V$-modules. Then

$$\mathcal{H} \otimes_{\mathcal{O}_X} {}_\alpha\mathcal{J}_\beta = {}_\alpha(\alpha^*\mathcal{H} \otimes_{\mathcal{O}_V} \mathcal{J})_\beta$$

(Recall that $\mathcal{H}$ is really ${}_{\mathrm{id}}\mathcal{H}_{\mathrm{id}}$)

(2) *Assume that there is a diagram*

$$\begin{tikzcd}[column sep=small]
& \mathcal{H} & & \mathcal{J} & \\[-20pt]
& V \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & W \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
X & & Y & & Z
\end{tikzcd}$$

where $\mathcal{H}$, $\mathcal{J}$ are respectively quasi-coherent $\mathcal{O}_V$ and $\mathcal{O}_W$-modules. Then

$${}_{\alpha}\mathcal{H}_{\beta} \otimes_{\mathcal{O}_Y} {}_{\gamma}\mathcal{J}_{\delta} = {}_{\alpha\circ\mathrm{pr}_1}(\mathrm{pr}_1^* \mathcal{H} \otimes_{\mathcal{O}_{V\times_Y W}} \mathrm{pr}_2^* \mathcal{J})_{\delta\circ\mathrm{pr}_2}$$

Proof. We prove the second statement. In order to be able to write down the various maps we rename X, Y, Z, V, W as X_1, X_2, X_3, X_4, X_5 and we write the projections $X_4 \times_{X_2} X_5 \to X_4, X_5$ as p, q.

So we have the following diagram

$$\begin{tikzcd}[column sep=small]
& & X_4 \times_{X_2} X_5 \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
& X_4 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & X_5 \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
X_1 & & X_2 & & X_3
\end{tikzcd}$$

Then ${}_{\alpha}\mathcal{H}_{\beta} \otimes_{\mathcal{O}_{X_2}} {}_{\gamma}\mathcal{J}_{\delta}$ is by definition

$$\mathrm{pr}^{123}_{13*}(\mathrm{pr}^{123*}_{12}(\alpha,\beta)_*\mathcal{H} \otimes_{\mathcal{O}_{X_{123}}} \mathrm{pr}^{123*}_{23}(\gamma,\delta)_*\mathcal{J}) \tag{2.6}$$

Using base change for the pullback

$$\begin{tikzcd}
X_4 \arrow[d, "{(\alpha,\beta)}"'] & X_4 \times X_3 \arrow[l, "\mathrm{pr}_4^{43}"'] \arrow[d, "{(\alpha,\beta)\times 1}"] \\
X_1 \times X_2 & X_1 \times X_2 \times X_3 \arrow[l, "\mathrm{pr}_{12}^{123}"']
\end{tikzcd}$$

we see that (2.6) is equal to

$$\mathrm{pr}^{123}_{13*}(((\alpha,\beta)\times 1)_* \mathrm{pr}_4^{43*}\mathcal{H} \otimes_{\mathcal{O}_{X_{123}}} \mathrm{pr}^{123*}_{23}(\gamma,\delta)_*\mathcal{J}) \tag{2.7}$$

Applying the projection formula to (2.7) yields

$$\begin{aligned}
&\mathrm{pr}^{123}_{13*}((\alpha,\beta)\times 1)_*(\mathrm{pr}_4^{43*}\mathcal{H} \otimes_{\mathcal{O}_{X_{43}}} ((\alpha,\beta)\times 1)^* \mathrm{pr}^{123*}_{23}(\gamma,\delta)_*\mathcal{J}) \\
&\qquad\qquad = (\alpha\times 1)_*(\mathrm{pr}_4^{43*}\mathcal{H} \otimes_{\mathcal{O}_{X_{43}}} (\beta\times 1)^*(\gamma,\delta)_*\mathcal{J})
\end{aligned}$$

Now using base change for

$$\begin{tikzcd}
X_4 \times X_3 \arrow[d, "\beta\times 1"'] & X_4 \times_{X_2} X_5 \arrow[l, "{(p,\delta\circ q)}"'] \arrow[d, "q"] \\
X_2 \times X_3 & X_5 \arrow[l, "{(\gamma,\delta)}"']
\end{tikzcd}$$

we may further simplify to

$$(\alpha\times 1)_*(\mathrm{pr}_4^{43*}\mathcal{H} \otimes_{\mathcal{O}_{X_{43}}} (p,\delta\circ q)_* q^*\mathcal{J})$$

and using the projection formula for $(p, \delta \circ q)$ this is equal to

$$(\alpha \times 1)_*(p,\delta\circ q)_*((p,\delta\circ q)^* \operatorname{pr}_4^{43*} \mathcal{H} \otimes_{\mathcal{O}_{X_4 \times_{X_2} X_5}} q^*\mathcal{J}) \\ = (\alpha\circ p, \delta\circ q)_*(p^*\mathcal{H} \circ_{\mathcal{O}_{X_4\times_{X_2} X_5}} q^*\mathcal{J}) \quad \square$$

Now we discuss stalks of bimodules at generic points of irreducible schemes. If $\mathcal{M}$ is an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_Y$-bimodule with X, Y irreducible with generic points η, ζ then the stalks of $\mathcal{M}$ at η and ζ are the corresponding stalks of the left and right module structure of $\mathcal{M}$. That is

$$\mathcal{M}_\eta = \varinjlim_{\substack{U\subset X\\ \text{open}}} \mathcal{M}(U\times_S X)$$
$$\mathcal{M}_\zeta = \varinjlim_{\substack{V\subset X\\ \text{open}}} \mathcal{M}(X\times_S V)$$

Lemma 2.8. *Assume that $X = Y$ and let $\mathcal{M}$ be an $\mathcal{O}_S$-central $\mathcal{O}_X - \mathcal{O}_X$-bimodule. Then $\mathcal{M}_\eta$ and $\mathcal{M}_\zeta$ are canonically isomorphic.*

Proof. Of course, here $\eta = \zeta$. So the notations $\mathcal{M}_\eta$, $\mathcal{M}_\zeta$ only serve to distinguish left and right.

We will contruct the isomorphism in the case that $\mathcal{M}$ is coherent. The general case follows by taking direct limits. Let $Z = \operatorname{Supp}\mathcal{M}$. The projection maps $\operatorname{pr}_{1,2} : Z \to X$ are finite and in particular closed. We consider $\mathcal{M}$ as a sheaf on Z. Thus

$$\mathcal{M}_\eta = \varinjlim_{\substack{U\subset X\\ U \text{ open}}} \mathcal{M}(U\times_S X) = \varinjlim_{\substack{U\subset X\\ U \text{ open}}} \mathcal{M}((U\times_S X)\cap Z)$$

and by symmetry

$$\mathcal{M}_\zeta = \varinjlim_{\substack{V\subset X\\ V \text{ open}}} ((X\times_S V)\cap Z)$$

To show that these two directed systems are cofinal we need

$$\forall U\subset X \text{ open} : \exists V\subset X \text{ open} : (X\times_S V)\cap Z \subset U\times_S X \tag{2.8}$$

and a symmetric statement with U and V interchanged.

Given U we may find V satisfying (2.8) by

$$V = X - \operatorname{pr}_2(Z - \operatorname{pr}_1^{-1} U) \quad \square$$

Lemma 2.9. *Let X, η, S be as in the previous lemma. Assume that $\mathcal{M}, \mathcal{N}$ are two $\mathcal{O}_S$-central $\mathcal{O}_X$-bimodules. Then*

$$(\mathcal{M}\otimes_X \mathcal{N})_\eta = \mathcal{M}_\eta \otimes_{F(X)} \mathcal{N}_\eta$$

Proof. Left to the reader. $\square$

Now we discuss pullbacks of bimodules under closed immersions. Let $\mathcal{M}$ be an $\mathcal{O}_S$-centralizing $\mathcal{O}_X - \mathcal{O}_Y$ bimodule and let $\alpha : X' \to X$, $\beta : Y' \to Y$ be closed immersions. Then $(\alpha,\beta)^*\mathcal{M}$ is the pullback of $\mathcal{M}$ under the product map $(\alpha,\beta) : X'\times_S Y' \to X\times_S Y$. One verifies that $(\alpha,\beta)^*\mathcal{M}$ is an $\mathcal{O}_{X'} - \mathcal{O}_{Y'}$-bimodule.

However in general the left module structure on $(\alpha,\beta)^*\mathcal{M}$ will not be the same as the pullback of the left module structure on $\mathcal{M}$ under α. That is, in general, the natural map

$$\alpha^* \operatorname{pr}_{1*}\mathcal{M} \to \operatorname{pr}_{1*}(\alpha,\beta)^*\mathcal{M}$$

is not an isomorphism. A similar statement holds for right module structures :

$$\beta^* \operatorname{pr}_{2*}\mathcal{M} \to \operatorname{pr}_{2*}(\alpha,\beta)^*\mathcal{M}$$

is in general not an isomorphism.

The case where both these arrows are isomorphisms represents a particularly desirable situation which we now investigate further.

Definition 2.10. Let $\alpha,\beta,\mathcal{M}$ be as above. We say that $\mathcal{M}$ is adapted to (α,β) if

$$\mathcal{M}\otimes_{\mathcal{O}_{X\times_S Y}}\mathcal{O}_{\alpha(X')\times_S Y}\in\mathcal{O}_{\alpha(X')\times_S\beta(Y')}\text{-mod} \tag{2.9}$$

$$\mathcal{M}\otimes_{\mathcal{O}_{X\times_S Y}}\mathcal{O}_{X\times_S\beta(Y')}\in\mathcal{O}_{\alpha(X')\times_S\beta(Y')}\text{-mod} \tag{2.10}$$

Proposition 2.11. *If $\mathcal{M}$ is adapted to (α,β) then the canonical maps*

$$\alpha^* \operatorname{pr}_{1*}\mathcal{M} \to \operatorname{pr}_{1*}(\alpha,\beta)^*\mathcal{M}$$
$$\beta^* \operatorname{pr}_{2*}\mathcal{M} \to \operatorname{pr}_{2*}(\alpha,\beta)^*\mathcal{M}$$

are isomorphisms.

Proof. Assume that $\mathcal{M}$ is adapted to (α,β). We look at the following diagram.

$$\begin{tikzcd}
 & & X'\times_S Y' \arrow[dl, "{(1,\beta)}"'] \arrow[dr, "{(\alpha,1)}"] & & \\
 & X'\times_S Y \arrow[dl, "\operatorname{pr}_1"'] \arrow[dr, "{(\alpha,1)}"] & & X\times_S Y' \arrow[dl, "{(1,\beta)}"'] \arrow[dr, "\operatorname{pr}_2"] & \\
X' \arrow[dr, "\alpha"'] & & X\times_S Y \arrow[dl, "\operatorname{pr}_1"'] \arrow[dr, "\operatorname{pr}_2"] & & Y' \arrow[dl, "\beta"] \\
 & X & & Y &
\end{tikzcd}$$

The two lower squares are pullbacks. Thus

$$\alpha^* \operatorname{pr}_{1*}\mathcal{M} = \operatorname{pr}_{1*}(\alpha,1)^*\mathcal{M}$$

By hypothesis, $(\alpha,1)^*\mathcal{M}$ is an $\mathcal{O}_{\alpha(X')\times_S\beta(Y')}$-module. Hence

$$\begin{aligned}\alpha^* \operatorname{pr}_{1*}\mathcal{M} &= \operatorname{pr}_{1*}(1,\beta)_*(1,\beta)^*(\alpha,1)^*\mathcal{M}\\ &= \operatorname{pr}_{1*}(\alpha,\beta)^*\mathcal{M}\end{aligned}$$

The fact that the other map is an isomorphism is proved in a similar way. □

Proposition 2.12. *Let $\alpha,\beta,X,Y,X',Y',\ldots$ be as above*

(1) *The category of $\mathcal{O}_S$-central $\mathcal{O}_X-\mathcal{O}_Y$-bimodules, adapted to (α,β) is closed under quotients and direct sums.*

(2) *Assume that $\mathcal{M}$ is adapted to α,β and let $\mathcal{N}\subset\mathcal{M}$ be a submodule such that the maps*

$$\alpha^*\operatorname{pr}_{1*}\mathcal{N}\to\alpha^*\operatorname{pr}_{1*}\mathcal{M}$$
$$\beta^*\operatorname{pr}_{2*}\mathcal{N}\to\beta^*\operatorname{pr}_{2*}\mathcal{M}$$

are injections, then $\mathcal{N}$ is also adapted to (α,β) and the map $(\alpha,\beta)^*\mathcal{N}\to(\alpha,\beta)^*\mathcal{M}$ is injective. (The point of this result is that it gives a criterion solely in terms of the left and right module structures on $\mathcal{N}$).

(3) *Assume that $\gamma: Z'\to Z$ is a third closed immersion and $\mathcal{N}$ is an $\mathcal{O}_S$-central $\mathcal{O}_Y-\mathcal{O}_Z$-bimodule such that $\mathcal{M}$ is adapted to (α,β) and $\mathcal{N}$ is adapted to (β,γ). Then $\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{N}$ is adapted to (α,γ) and*

$$(\alpha,\gamma)^*(\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{N})=(\alpha,\beta)^*\mathcal{M}\otimes_{\mathcal{O}_{Y'}}(\beta,\gamma)^*\mathcal{N} \tag{2.11}$$

Proof. (1) Clear

(2) We use base change for the square

$$\begin{tikzcd} X' & X'\times_S Y \arrow[l, "\operatorname{pr}_1"'] \arrow[d, "{(\alpha,1)}"'] \\ X \arrow[from=u, "\alpha"'] & X\times_S Y \arrow[l, "\operatorname{pr}_1"'] \end{tikzcd}$$

Thus $\operatorname{pr}_{1*}(\alpha,1)^*\mathcal{N}\to\operatorname{pr}_{1*}(\alpha,1)^*\mathcal{M}$ is injective.
Now $(\alpha,1)^*\mathcal{N}$ and $(\alpha,1)^*\mathcal{M}$ are $\mathcal{O}_{X'\times_S Y}$-modules, and since pr_{1*} is fully faithful, we obtain that $(\alpha,1)^*\mathcal{N}\to(\alpha,1)^*\mathcal{M}$ is injective.
Now by hypothesis $(\alpha,1)^*\mathcal{M}$ is in $\mathcal{O}_{X'\times\beta(Y')}$-mod and hence the same holds for $(\alpha,1)^*\mathcal{N}$. But then (2.9) is true for $\mathcal{N}$. The proof that (2.10) is true is similar.
To show that $(\alpha,\beta)^*\mathcal{N}\to(\alpha,\beta)^*\mathcal{M}$ is injective we may apply the faithful functor pr_{1*}. But then it follows from the fact that by hypothesis $\alpha^*\operatorname{pr}_{1*}\mathcal{N}\to\alpha^*\operatorname{pr}_{1*}\mathcal{M}$ is injective.

(3) We have to show first that

$$(\alpha,1)^*(\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{N})$$

is an $\mathcal{O}_{X'}-\mathcal{O}_{\gamma(Z')}$-bimodule, and that

$$(1,\gamma)^*(\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{N})$$

is an $\mathcal{O}_{\alpha(X')}-\mathcal{O}_{Z'}$-bimodule.
We prove the first claim. We have to compute

$$(\alpha,1)^*\operatorname{pr}_{13*}(\operatorname{pr}_{12}^*\mathcal{M}\otimes_{\mathcal{O}_{X\times_S Y\times_S Z}}\operatorname{pr}_{23}^*\mathcal{N}) \tag{2.12}$$

Looking at the pullback diagram

$$\begin{tikzcd} X'\times_S Y\times_S Z \arrow[r, "{(\alpha,1,1)}"] \arrow[d, "\operatorname{pr}_{13}"'] & X\times_S Y\times_S Z \arrow[d, "\operatorname{pr}_{13}"'] \\ X'\times_S Z \arrow[r, "{(\alpha,1)}"] & X\times_S Z \end{tikzcd}$$

yields that (2.12) is equal to

(2.13)
$$\begin{aligned}\mathrm{pr}_{13*}(\alpha,1,1)^*(\mathrm{pr}_{12}^*\mathcal{M}\otimes_{\mathcal{O}_{X\times_S Y\times_S Z}}\mathrm{pr}_{23}^*\mathcal{N})\\
&=\mathrm{pr}_{13*}((\alpha,1,1)^*\,\mathrm{pr}_{12}^*\mathcal{M}\otimes_{\mathcal{O}_{X'\times_S Y\times_S Z}}(\alpha,1,1)^*\,\mathrm{pr}_{23}^*\mathcal{N})\\
&=\mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha,1)^*\mathcal{M}\otimes_{\mathcal{O}_{X'\times_S Y\times_S Z}}\mathrm{pr}_{23}^*\mathcal{N})\end{aligned}$$

Now by hypothesis $\mathrm{pr}_{12}^*(\alpha,1)^*\mathcal{M}$ is an $\mathcal{O}_{X'\times_S\beta(Y')\times_S Z}$-module. So (2.13) is equal to

$$\begin{aligned}(2.14)\quad &\mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha,1)^*\mathcal{M}\otimes_{\mathcal{O}_{X'\times_S\beta(Y')\times_S Z}}\mathcal{O}_{X'\times_S\beta(Y')\times_S Z}\otimes_{\mathcal{O}_{X'\times_s Y\times Z}}\mathrm{pr}_{23}^*\mathcal{N})\\
&=\mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\alpha,1)^*\mathcal{M}\otimes_{\mathcal{O}_{X'\times_S\beta(Y')\times_S Z}}\mathrm{pr}_{23}^*(\mathcal{O}_{\beta(Y')\times_S Z}\otimes_{\mathcal{O}_{Y\times_S Z}}\mathcal{N}))\end{aligned}$$

Now by hypothesis $\mathcal{O}_{\beta(Y')\times_S Z\otimes_{\mathcal{O}_{Y\times_S Z}}}\mathcal{N}$ is an $\mathcal{O}_{\beta(Y')\times_S\gamma(Z')}$-module. This yields easily that (2.14) is an $\mathcal{O}_{X'\times_S\gamma(Z')}$-module.
The formula (2.11) is proved by a similar computation. □

We conclude with the following observation.

Proposition 2.13. *Assume that we have finite S-maps*

$$\begin{tikzcd} & Z \arrow[dl, "u"'] \arrow[dr, "v"] & \\ X & & Y \end{tikzcd}$$

and let $\alpha : X'\to X$, $\beta : Y'\to Y$ be as above. If $u^{-1}\alpha(X') = v^{-1}\beta(Y')$ (scheme theoretically) then any bimodule of the form ${}_u\mathcal{H}_v$ with $\mathcal{H}$ a quasi-coherent $\mathcal{O}_V$-module is adapted to α,β.

Furthermore put $Z' = u^{-1}\alpha(X')$ and let $\gamma' : Z'\to Z$ be the inclusion map. Let $u' : Z'\to X'$, $v' : Z'\to Y'$ be induced by u and v. Then $(\alpha,\beta)^*{}_u(\mathcal{H})_v = {}_{u'}(\gamma^*\mathcal{H})_{v'}$.

Proof. Clear. □

3. Bimodule algebras

Throughout X will be of finite type over a noetherian base scheme S. Unless specified otherwise all $\mathcal{O}_X$-bimodules are $\mathcal{O}_S$-symmetric.

Definition 3.1. (1) An $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$ is an algebra in the category of $\mathcal{O}_X$-bimodules. That is there is a unit map $1 : \mathcal{O}_X \to \mathcal{B}$ and a product map $\mu : \mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{B}\to\mathcal{B}$, satisfying the usual compatibilities.
(2) A (left) $\mathcal{B}$-module $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, together with a left $\mathcal{O}_X$-linear multiplication map $\mu_{\mathcal{M}} : \mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{M}\to\mathcal{M}$ again satisfying the usual compatibilities. The corresponding category will be denoted by $\mathcal{B}$-qch.
Right and twosided $\mathcal{B}$-modules are defined similarly.

Remark 3.2. (1) It is clear that $\mathcal{B}$-qch is an abelian category.
(2) The reader should note that a bimodule algebra is not a local notion. That is if $U\subset X$ is open then $\mathcal{B}(U)$ does not in general carry an algebra structure. A similar remark holds for $\mathcal{B}$-modules.
(3) It is, just as in the case of $\mathcal{O}_X$-bimodules, easy to see that the forgetful functor of $\mathcal{B}$-bimodules to left $\mathcal{B}$-modules, is exact and faithful.

A useful construction is twisting a bimodule algebra. Let $\mathcal{Z}$ be an invertible $\mathcal{O}_X$-bimodule, as defined in [5]. That is there exists a bimodule $\mathcal{Z}'$ such that $\mathcal{Z}\otimes_X \mathcal{Z}^{-1}\cong \mathcal{O}_X$, $\mathcal{Z}^{-1}\otimes_X \mathcal{Z}\cong \mathcal{O}_X$. Then

$$\mathcal{B}'=\mathcal{Z}\otimes_{\mathcal{O}_X}\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{Z}^{-1}$$

is also a bimodule algebra and the fuctor

$$\mathcal{M}\mapsto \mathcal{Z}\otimes_{\mathcal{O}_X}\mathcal{M}$$

defines an equivalence between $\mathcal{B}$-qch and $\mathcal{B}'$-qch.

Definition 3.3. A $\mathcal{B}$-module is coherent if there exists a coherent $\mathcal{O}_X$-module $\mathcal{M}_0$ together with a surjective map $\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{M}_0\to\mathcal{M}$ in $\mathcal{B}$-qch

The justification of this definition is given by Proposition 3.5 below, which depends upon the following lemma.

Lemma 3.4. *("Change of rings.") There is an isomorphism between*

$$\operatorname{Hom}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{M}_0,\mathcal{N})$$

and

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}_0,\mathcal{N})$$

given by sending $f\in\operatorname{Hom}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{M}_0,\mathcal{N})$ to the composition

$$\mathcal{M}_0\xrightarrow{\cong}\mathcal{O}_X\otimes_{\mathcal{O}_X}\mathcal{M}_0\xrightarrow{1\otimes\mathrm{id}}\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{M}_0\xrightarrow{f}\mathcal{N}$$

The inverse is given by sending $g\in\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}_0,\mathcal{N})$ to the composition

$$\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{M}_0\xrightarrow{1\otimes g}\mathcal{B}\otimes_{\mathcal{O}_X}\mathcal{N}\xrightarrow{\mu_{\mathcal{M}}}\mathcal{N}$$

Proposition 3.5. *$\mathcal{B}$ is left noetherian (that is, noetherian in the category of left $\mathcal{B}$-modules) if and only if, every left ideal in $\mathcal{B}$ is coherent.*

Proof. Assume that $\mathcal{B}$ is left noetherian and let $I\subset\mathcal{B}$ be a left ideal that is not coherent. We construct a strictly ascending chain $0=I_0\subsetneq I_1\subsetneq\cdots$ of coherent ideals in I.

Assume that I_k has been constructed. Since I_k is coherent $I\neq I_k$.

Now I and I/I_k are by definition quasi-coherent $\mathcal{O}_X$-modules, so we may find a coherent $\mathcal{O}_X$-module $J\subset I$ such that the composition $J\to I\to I/I_k$ is non-zero. Thus there exists an element

$$f\in\operatorname{Hom}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{O}_X}J,I)=\operatorname{Hom}_{\mathcal{O}_X}(J,I)$$

mapping to a non-zero element of

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{B}\otimes_{\mathcal{O}_X}J,I/I_k)=\operatorname{Hom}_{\mathcal{O}_X}(J,I/I_0)$$

We put $I_{k+1}=I_k+\operatorname{im} f$. Then by construction I_{k+1} is coherent and $I_{k+1}\supsetneq I_k$.

Assume now conversely that every ideal in $\mathcal{B}$ is coherent and that $I_0\subset I_1\subset I_2\cdots$ is an ascending chain of $\mathcal{B}$-ideals. Put $I=\bigcup_i I_i$. By hypothesis I is coherent and hence there exists a surjective map $f:\mathcal{B}\otimes_{\mathcal{O}_X}J\to I$, with J coherent. This corresponds to a map $J\to I$ in $\mathcal{O}_X$-qch. Since J is coherent this map must factor through some $J\to I_k$ and hence f itself factors through I_k. Since f is surjective, we obtain $I_k=I$. □

Proposition 3.6. (1) *Quotients of coherent $\mathcal{B}$-modules are coherent.*
(2) *Extensions of coherent $\mathcal{B}$-modules are coherent.*

(3) *Assume that either X is affine or that $\mathcal{B}$ is right flat over $\mathcal{O}_X$. If $\mathcal{B}$ is left noetherian then submodules of coherent $\mathcal{B}$-modules are coherent.*
(4) *If $\mathcal{B}$ is left noetherian then a $\mathcal{B}$-module is noetherian if and only if it is coherent.*

Remark 3.0.2. The restrictions on $\mathcal{B}$ in the third statement in the above proposition are rather unfortunate and can probably be removed, allthough I don't see how to do this.

Proof of Proposition 3.6. (1) This is trivial.
(2) Let
$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$
be an exact sequence of $\mathcal{B}$-modules with $\mathcal{M}'$, $\mathcal{M}''$ coherent. So there are surjective maps $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{M}'_0 \to \mathcal{M}$, $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{M}''_0 \to \mathcal{M}$ with $\mathcal{M}'_0$, $\mathcal{M}''_0$ coherent $\mathcal{O}_X$-modules. We may lift $\mathcal{M}''_0$ to a coherent $\mathcal{O}_X$-module $\mathcal{M}_0 \subset \mathcal{M}$, mapping surjectively to $\mathcal{M}''_0$. Then $\mathcal{M}$ is a quotient of $\mathcal{B} \otimes_{\mathcal{O}_X} (\mathcal{M}'_0 \oplus \mathcal{M}_0)$.
(3) It suffices to prove this for coherent modules of the form $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{M}_0$ with $\mathcal{M}_0$ a coherent $\mathcal{O}_X$-module.
Let $\mathcal{C}$ be the class of coherent $\mathcal{O}_X$-modules such that for $\mathcal{M}_0 \in \mathcal{C}$ one has that every submodule of $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{M}_0$ is coherent. Then (1) and (2) imply that $\mathcal{C}$ is closed under quotients and extensions. Furthermore by Proposition 3.5 : $\mathcal{O}_X \in \mathcal{C}$. If X is affine then this implies that $\mathcal{C} = \mathcal{O}_X$-coh. If $\mathcal{B}$ is right flat then it is easily seen that $\mathcal{C}$ is also closed under taking submodules. Then it is an easy exercise, left to the reader, to show that $\mathcal{C}$ contains again all coherent $\mathcal{O}_X$-modules.
(4) In a non-coherent $\mathcal{B}$-module $\mathcal{M}$ one may construct a strictly ascending chain of submodules, exactly as in the proof of Proposition 3.5. Hence noetherian implies coherent.
Conversely let $\mathcal{M}$ be coherent and let there be an ascending chain $\mathcal{M}_0 \subset \cdots \subset \mathcal{M}_i \subset \cdots$. Then by (3), $\mathcal{N} = \bigcup \mathcal{M}$ is coherent. But then as in the proof of Proposition 3.5, some $\mathcal{M}_i$ must be equal to $\mathcal{N}$. □

Corollary 3.7. *If $\mathcal{B}$ is left noetherian and either X is affine or $\mathcal{B}$ is right flat then the category of coherent $\mathcal{B}$-modules is abelian. We denote it by $\mathcal{B}$-coh.*

Let $\mathcal{M}$, $\mathcal{N}$ be respectively $\mathcal{O}_X - \mathcal{O}_Y$ and $\mathcal{O}_Y - \mathcal{O}_Z$-bimodules. Then there is a natural map $\Gamma(\mathcal{M}) \otimes_{\mathcal{O}(Y)} \Gamma(\mathcal{N}) \to \Gamma(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})$ defined as the composition

$$\begin{aligned}
\Gamma(\mathcal{M}) \otimes_{\mathcal{O}(Y)} \Gamma(\mathcal{N}) &\overset{\mathrm{def}}{=} \mathcal{M}(X \times_S Y) \otimes_{\mathcal{O}(Y)} \mathcal{N}(Y \times_S Z) \\
&\xrightarrow{\mathrm{pr}_{12}^* \otimes \mathrm{pr}_{23}^*} \mathrm{pr}_{12}^* \mathcal{M}(X \times_S Y \times_S Z) \otimes_{\mathcal{O}(X \times_S Y \times_S Z)} \mathrm{pr}_{23}^* \mathcal{M}(X \times_S Y \times_S Z) \\
&\to (\mathrm{pr}_{12}^* \mathcal{M} \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}} \mathrm{pr}_{23}^* \mathcal{N})(X \times_S Y \times_S Z) \\
&= \mathrm{pr}_{13*}(\mathrm{pr}_{12}^* \mathcal{M} \otimes_{\mathcal{O}_{X \times_S Y \times_S Z}} \mathrm{pr}_{23}^* \mathcal{N})(X \times_S Z) \\
&\overset{\mathrm{def}}{=} \Gamma(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})
\end{aligned}$$

Thus if $\mathcal{B}$ is an $\mathcal{O}_X$-bimodule algebra and $B = \Gamma(\mathcal{B})$ then taking global sections induces a multiplication map $B \otimes_{\mathcal{O}(S)} B \to B$. Similarly the unit map $1 : \mathcal{O}_X \to \mathcal{B}$ defines a unit element $1 \in B$. It is easily verified that in this way B becomes an $\mathcal{O}(S)$-algebra.

Similarly if $\mathcal{M}$ is a $\mathcal{B}$-module then $M = \Gamma(\mathcal{M})$ is a B-module. This defines a functor $\Gamma : \mathcal{B}\text{-qch} \longrightarrow B\text{-mod}$. There is a functor in the other direction sending M to $\mathcal{B} \otimes_B M$. Here $\mathcal{B} \otimes_B M$ is the sheaf associated to the presheaf $U \mapsto \mathcal{B}(U) \otimes_B M$.

The functors Γ and $\mathcal{B} \otimes_B -$ are adjoint. That is

$$\operatorname{Hom}_B(M, \Gamma(\mathcal{N})) = \operatorname{Hom}_{\mathcal{B}}(\mathcal{B} \otimes_B M, \mathcal{N})$$

for all $M \in B$-mod and $\mathcal{M} \in \mathcal{B}$-qch.

4. Graded bimodule algebras

All above constructions may be graded. A graded $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$ is an $\mathcal{O}_X$-bimodule algebra, equipped with a decomposition $\oplus_{n\in\mathbb{Z}}\mathcal{B}_n$ as $\mathcal{O}_X$-bimodules such that $\mu = \oplus\mu_{m,n}$ with $\mu_{m,n}$ mapping $\mathcal{B}_m \otimes_{\mathcal{O}_X} \mathcal{B}_n \longrightarrow \mathcal{B}_{m+n}$ and with $1(\mathcal{O}_X) \subset \mathcal{B}_0$.

Similarly, a graded $\mathcal{B}$-module $\mathcal{M}$ is equipped with a decomposition $\mathcal{M} = \oplus\mathcal{M}_n$ and $\mu_{\mathcal{M}} = \oplus_{m,n}(\mu_{\mathcal{M}})_{m,n}$ with $(\mu_{\mathcal{M}})_{m,n} : \mathcal{B}_m \otimes \mathcal{M}_n \longrightarrow \mathcal{M}_{m+n}$.

We may define graded noetherian, graded coherent etc ... in the same way as the corresponding ungraded notions. However the standard arguments [11] shows that most of the time the graded notions are equivalent with the ungraded ones, so in this paper we will usually talk about noetherian, coherent etc ... , dropping the adjective "graded."

The category of graded $\mathcal{B}$-modules will be denoted by $\mathcal{B}$-Gr and the full subcategory of coherent objects (if $\mathcal{B}$ is left noetherian) will be denoted by $\mathcal{B}$-gr. Similarly if B is a graded ring then B-Gr and B-gr denote respectively the categories of graded and finitely generated graded B-modules.

The category $\mathcal{B}$-Gr and B-Gr are equipped with the shift functor. E.g. if $\mathcal{M} \in \mathcal{B}$-Gr then $\mathcal{M}(n)$ is graded with $\mathcal{M}(n)_m = \mathcal{M}_{m+n}$.

5. Projective geometry

In this section we impose some natural conditions on X and $\mathcal{B}$ coming from algebraic geometry. That is, we assume

(1) The base scheme S is equal to the spectrum of an algebraically closed field F.
(2) X is proper of finite type over F.
(3) $\mathcal{B}$ is graded of the form
$$\mathcal{O}_X \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots$$
(4) $\mathcal{B}$ is left noetherian.
(5) The category of coherent left $\mathcal{B}$-module is abelian. This holds in particular if $\mathcal{B}$ is right flat over $\mathcal{O}_X$ (see cor. 3.7).

Note that these conditions imply that all $\mathcal{B}_n$ are coherent over $\mathcal{O}_X$. More generally for any coherent graded $\mathcal{B}$-module $\mathcal{M}$ one has that all $\mathcal{M}_n$ are coherent $\mathcal{O}_X$-modules.

We say that a coherent graded $\mathcal{B}$-module $\mathcal{M}$ is torsion if $\mathcal{M}_n = 0$ for $n \geq 0$. The corresponding category is denoted by "tors". The closure under direct limits in $\mathcal{B}$-Gr of tors is denoted by Tors.

If B is a graded ring of the form $F \oplus B_1 \oplus B_2 \oplus \cdots$ then then we define tors, Tors $\subset B$-gr as above, but with coherent replaced by finitely generated. Finally we introduce, as in [2], QGr and qgr as an abbreviation for Gr/Tors and gr/tors.

An important notion, suggested by [7, Thm III.3.1], for comparing $\mathcal{B}$ and $B = \Gamma(\mathcal{B})$ is defined below.

Definition 5.1. We say that a sequence $(\mathcal{B}_n)_n$ of coherent $\mathcal{O}_X$ bimodules is ample if for any coherent $\mathcal{O}_X$-module one has that for $n \gg 0$, $\mathcal{B}_n \otimes_{\mathcal{O}_X} \mathcal{M}$ is generated by global sections and $H^q(X, \mathcal{B}_n \otimes_{\mathcal{O}_X} \mathcal{M}) = 0$ for $q > 0$.

Now we state and prove the main result of this section.

Theorem 5.2. *Let $\mathcal{B}$ be as above (in particular $\mathcal{B}$ is left noetherian) and assume that $(\mathcal{B}_n)_n$ is an ample sequence. Then $B = \Gamma(\mathcal{B})$ is noetherian and there are inverse equivalences*

$$B\text{-QGr} \underset{\Gamma(-)}{\overset{\mathcal{B}\otimes_B -}{\rightleftarrows}} \mathcal{B}\text{-QGr}$$

Moreover under these equivalences B-qgr corresponds to $\mathcal{B}$-qgr.

Remark 5.0.3. Note that we do not claim that B satifies the condition χ_1 [2, def. 3.7]. In fact there is no reason why χ_1 should hold, as can already be seen in the case that X is a point and $B = \mathcal{B}$ is one of the rings occuring in [20].

Similar examples can be made by letting X be a projective curve and $\mathcal{B}_n = {}_1(\mathcal{L}_n)_{\sigma^n}$ where $\mathcal{L}_n$ is a line bundle on X and σ is a fixed automorphism of X. See [19].

This shows that allthough Theorem 5.2 is closely related to [2, Thm 5.2], it does not quite follow from it.

Proof of Theorem 5.2. The following proof is an adaptation of the proof of [5, Thm 3.12] The result follows from step 12 and corollaries 2,3,4,5.

Step 1. *If $\mathcal{M}$ is a coherent $\mathcal{B}$-module then for $n \gg 0$, $\mathcal{M}_n$ is generated by global sections and for $q \geq 1$, $H^q(X, \mathcal{M}) \in \mathcal{B}$-Gr is finite dimensional.*

Since by hypothesis $\mathcal{B}$ is left noetherian we may construct a resolution

$$\cdots \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{M}^{(1)} \to \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{M}^{(0)} \to \mathcal{M} \to 0$$

with the $\mathcal{M}^{(i)}$, coherent, graded $\mathcal{O}_X$-modules.

Since X has finite cohomological dimension it suffices to show that for $n \gg 0$, $\mathcal{B}_n \otimes_{\mathcal{O}_X} \mathcal{M}^{(i)}$ is generated by global sectons and $H^q(X, \mathcal{B}_n \otimes_{\mathcal{O}_X} \mathcal{M}^{(i)}) = 0$ for $q \geq 1$. But this follows precisely from the fact that $(\mathcal{B}_n)_n$ is an ample sequence.

Corollary 1. *$\Gamma(-)$ defines an exact functor form $\mathcal{B}$-QGr to B-QGr.*

This follows from 1 by taking direct limits.

Step 2. *Let $\mathcal{M}$ be a coherent $\mathcal{B}$-module and let $m \in \mathbb{N}$ be such that for $n \geq m$ $\mathcal{M}_n$ is generated by global sections (such an m exists by Step 1). Then the induced maps*

$$(\mathcal{B} \otimes_F \Gamma(\mathcal{M}_m))(-m) \oplus \cdots \oplus (\mathcal{B} \otimes_F \Gamma(\mathcal{M})_{m+k}))(-m-k) \to \mathcal{M}$$

has torsion cokernel for $k \gg 0$.

This follows from the fact that $\mathcal{B}$ is left noetherian.

Step 3. *Assume that $\mathcal{M}$ is a coherent $\mathcal{B}$-module. Then $\Gamma(\mathcal{M})$ is a finitely generated B-module.*

According to step 2 there exists a map $\mathcal{P} \to \mathcal{M}$ with torsion cokernel, and with $\mathcal{P}$ a finite sum of shifts of $\mathcal{B}$. Now according to step 1 $\Gamma(\mathcal{P}) \to \Gamma(\mathcal{M})$ has finite cokernel. But $\Gamma(\mathcal{P})$ is a sum of shifts of B and it follows that $\Gamma(\mathcal{M})$ is finitely generated.

Corollary 2. *$\Gamma(-)$ sends $\mathcal{B}$-qgr to B-qgr.*

Step 4. *Assume that M is in B-gr. Then $\mathcal{B} \otimes_B M$ is in $\mathcal{B}$-gr.*

Proof. This follows by the fact that M is a quotient of a finite direct sum of shifts of B, and right exactness of $\mathcal{B} \otimes_B -$. □

Corollary 3. *$\mathcal{B} \otimes_B -$ maps B-qgr to $\mathcal{B}$-qgr.*

Step 5. *Let $M \in B$-Gr. Then the canonical map $M \to \Gamma(\mathcal{B} \otimes_B M)$ has torsion kernel and cokernel.*

We write M as a quotient

$$P_1 \to P_0 \to M \to 0$$

where the $P_{0,1}$ are direct sums of shifts of B. Then we can construct a diagram

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \Gamma(\mathcal{B} \otimes_B P_1) & \longrightarrow & \Gamma(\mathcal{B} \otimes_B P_0) & \longrightarrow & \Gamma(\mathcal{B} \otimes_B M) & \longrightarrow & 0 \end{array}$$

By inspection the left and the middle vertical arrows are isomorphisms. From corollary 1 it follows that the bottom row is exact modulo torsion. Hence the right vertical map is an isomorphism modulo torsion.

Step 6. *If T is a torsion B-module then $\mathcal{B} \otimes_B T$ is torsion.*

Since $\mathcal{B} \otimes_B -$ commutes with direct limits, it suffices to prove this for T finite dimensional. But then it suffices to prove it for F. To prove it for F it suffices to prove it for one particular non-zero finite dimensional T since F is a quotient of every such T.

By step 2 we may construct a map with non-zero torsion cokernel

$$\mathcal{P} \to \mathcal{B} \tag{5.1}$$

where $\mathcal{P}$ is a $\mathcal{P}$ is a direct sum of shifts of B. Let T be the cokernel of the map $\Gamma(\mathcal{P}) \to \Gamma(\mathcal{B})$. By step 1 T is finite dimensional.

$\mathcal{B} \otimes_B -$ is right exact and so

$$\mathcal{B} \otimes_B \Gamma(\mathcal{P}) \to \mathcal{B} \otimes_B \Gamma(\mathcal{B}) \to \mathcal{B} \otimes_B T \to 0$$

is exact.

Now $\mathcal{B} \otimes_B \Gamma(\mathcal{P}) = \mathcal{P}$, $\mathcal{B} \otimes_B \Gamma(\mathcal{B}) = \mathcal{B}$. Comparing with (5.1) yields that $\mathcal{B} \otimes_B T = T$ which is torsion.

Step 7. *If $\mathcal{M} \in \mathcal{B}$-Gr and if $\Gamma(\mathcal{M})$ is torsion then $\mathcal{M}$ is torsion.*

Again it suffices to assume that $\mathcal{M}$ is coherent. Thanks to step 2 we may construct an exact sequence, up to torsion

$$\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M} \to 0 \tag{5.2}$$

where $\mathcal{P}_0$, $\mathcal{P}_1$ are finite sums of shifts of $\mathcal{B}$. So in fact we have an exact sequence of the form

$$\operatorname{coker}(\mathcal{P}_1 \to \mathcal{P}_0) \to \mathcal{M} \to \text{torsion} \to 0 \tag{5.3}$$

Corollary 1 implies that, after applying $\Gamma(-)$, (5.2) remains exact up to torsion. So by hypothesis $\operatorname{coker}(\Gamma(\mathcal{P}_1) \to \Gamma(\mathcal{P}_0))$ is torsion. But then by step 6 $\mathcal{B} \otimes_B \operatorname{coker}(\Gamma(\mathcal{P}_1) \to \Gamma(\mathcal{P}_0)) = \operatorname{coker}(\mathcal{P}_1 \to \mathcal{P}_0)$ is torsion. Then (5.3) shows what we want.

Step 8. *If $M \to N$ is a map in B-Gr with torsion kernel then the kernel of $\mathcal{B} \otimes_B M \to \mathcal{B} \otimes_B N$ is also torsion.*

Let $\mathcal{K} = \ker(\mathcal{B} \otimes_B M \to \mathcal{B} \otimes_B N)$. Applying $\Gamma(-)$ and step 5 yields that $\Gamma(\mathcal{K})$ is torsion. Hence by step 7, $\mathcal{K}$ is torsion.

Corollary 4. *$\mathcal{B} \otimes_B -$ defines an exact functor*

$$B\text{-QGr} \to \mathcal{B}\text{-QGr}$$

which is a left inverse to $\Gamma(-)$.

This follows from steps 6,7,8

Step 9. *Let $\mathcal{M} \in \mathcal{B}$-Gr. Then the canonical map $\mathcal{M} \to \mathcal{B} \otimes_B \Gamma(\mathcal{M})$ has torsion kernel and cokernel.*

We may assume that $\mathcal{M}$ is coherent. Thanks to step 2 there exists a complex

$$\mathcal{P}_1 \to \mathcal{P}_0 \to \mathcal{M} \to 0$$

with torsion homology, where $\mathcal{P}_1$, $\mathcal{P}_0$ are finite sums of shifts of $\mathcal{B}$. Now $\mathcal{P}_{0,1} \to \mathcal{B} \otimes_B \Gamma(\mathcal{P}_{0,1})$ are isomorphic by inspection.

Hence the same holds for the maps $\mathcal{M} \otimes \mathcal{B} \otimes_B \Gamma(\mathcal{M})$ because Γ and $\mathcal{B} \otimes_B -$ are exact modulo torsion (corollaries 1 and 4).

Corollary 5. *$\mathcal{B} \otimes_B -$ is also a right inverse to $\Gamma(-)$.*

Step 10. *$B_{\geq 1}$ is finitely generated.*

Since $B_{\geq 1} = \Gamma(\mathcal{B}_{\geq 1})$ this follows from step 3.

Step 11. *Assume that there is an exact sequence in B-Gr*

$$0 \to M' \to M \to M'' \to 0$$

with M finitely generated and M'' torsion. Then M' is finitely generated.

Since M is finitely generated, the same is true for M''. Hence M'' is finite-dimensional. Therefore there exist k such that $B_{\geq 1}^k M \subset M'$. It is easy to see that $M'/B_{\geq 1}^k M \subset M/B_{\geq 1}^k M$ is finite dimensional.

Now since M is finitely generated, the same holds for $B_{\geq 1}^k M$ by step 10. Hence M' is also finitely generated.

Step 12. *B is left noetherian.*

Let $I \subset B$ be a graded left ideal and let $\mathcal{J} = \operatorname{im}(\mathcal{B} \otimes_B I \to \mathcal{B})$. Taking global sections yields maps

$$I \to \Gamma(\mathcal{B} \otimes_B I) \to \Gamma(\mathcal{J}) \hookrightarrow B$$

Now by step 5, $\operatorname{coker}(I \to \Gamma(\mathcal{B} \otimes_B I))$ is torsion and by corollary 1, $\Gamma(\mathcal{B} \otimes_B I) \to \Gamma(\mathcal{J})$ is torsion. Hence $\Gamma(\mathcal{J})/I$ is torsion. Since $\mathcal{B}$ is left noetherian, $\mathcal{J}$ is coherent and hence by step 3 $\Gamma(\mathcal{J})$ is finitely generated. Then by step 11 I is finitely generated. $\square$

6. Bimodule algebras associated to Sklyanin algebras

6.1. Sklyanin algebras. We recall the definition and some properties of the r-dimensional Sklyanin algebras.

Let E be an fixed elliptic curve over our algebraically closed ground field F, let $\sigma : E \to E$ be a translation by an element $\tau \in J(E)$ and let $\mathcal{L}$ be an invertible sheaf of E of degree $r \geq 3$. Put $V = H^0(E, \mathcal{L})$, a vectorspace of dimension r.

In order to reduce possible confusion below, the projection maps $E \times E \to E$ will in this special situation be denoted by $p_{1,2}$. We use the notation $\mathcal{L} \boxtimes \mathcal{L}$ for $p_1^*\mathcal{L} \otimes_{\mathcal{O}_{E\times E}} p_2^*\mathcal{L}$. If π, ρ are automorphism on E then on $E \times E$ we denote by $[\pi, \rho]$ the map $(p, q) \mapsto (\pi(p), \rho(q))$. Let $\Delta \subset E \times E$ be the diagonal. Then $\Gamma_\pi = [1, \pi]\Delta$ is the graph of the automorphism π of E.

Define $\mathcal{L}' = (\sigma^{-1})^*\mathcal{L}$, $V' = H^0(E, \mathcal{L}')$, $\pi = \sigma^{r-2}$, $\theta = \sigma^{-2}$.

There is an isomorphism [22] (determined up to a scalar)

$$\phi : [1, \theta^{-1}]^*(\mathcal{L}' \boxtimes \mathcal{L}' \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Delta)) \to \mathcal{L} \boxtimes \mathcal{L} \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Gamma_\pi)$$

Now $H^0(E \times E, \mathcal{L}' \boxtimes \mathcal{L}' \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Delta))$ contains the $r(r-1)/2$-dimensional subspace $\bigwedge^2 V'$, spanned by the sections of the form $x \otimes y - y \otimes x$; $x, y \in V'$.

This yields a canonical $r(r-1)/2$-dimensional subspace inside

$$H^0(E \times E, \mathcal{L} \boxtimes \mathcal{L} \otimes_{\mathcal{O}_{E\times E}} \mathcal{O}(-\Gamma_\pi)) \subset H^0(E \times E, \mathcal{L} \boxtimes \mathcal{L}) = V \otimes V$$

given by

$$R = \phi([1, \theta^{-1}]^*(\bigwedge^2 V'))$$

According to [13], R is the space of quadratic relations defining the Sklyainin algebra $A(E, \sigma, \mathcal{L}) = TV/(R)$, where TV is the tensor algebra of V.

In [22] the following result was proved.

Theorem 6.1.1. *Let $A = A(E, \sigma, \mathcal{L})$. Then*

(1) *A has the Hilbert series of a polynomial ring in r variables.*
(2) *A is Koszul.*
(3) *A is Noetherian.*
(4) *A satisfies the Auslander condition.*
(5) *A is Cohen-Macaulay*

For the definition of "Cohen-Macaulay" and the "Auslander condition" the reader is referred to [9].

Now, as in [22] we assume that $\mathcal{L} = \mathcal{O}(d)$ where d is a divisor on E. If we define $d' = \sigma d$ then $\mathcal{L}' = \mathcal{O}(d')$. We see that the isomorphism ϕ, defined above, is now multiplication with a rational function h on $E \times E$ such that

$$d \times E + E \times d - \Gamma_\pi + (h) = d' \times E + E \times \theta(d') - \Gamma_\theta$$

Now let f be a rational function on $E\times E$ contained in $H^0(E\times E,\mathcal{O}(d\times E+E\times d)) = V\otimes V$. Then it is easy to see (see the proof of [22, Prop. 4.3.2]) that $f\in R$ if and only if

$$
\begin{aligned}
&\forall z\in E : f(z,\pi z)=0\\
&\forall w,z\in E : (hf)(\theta^{-1}w,z)+(hf)(\theta^{-1}z,w)=0
\end{aligned}
\tag{6.1}
$$

In characteristic different from two the second condition alone is already sufficient, but we won't use this.

Put $A = A(E,\sigma,\mathcal{L})$. A graded A-module M is linear of dimension l if it is generated in degree zero and has the same Hilbert series as a polynomial ring in l variables. Linear modules of dimensions 1,2 and 3 are respectively called point,line and plane modules.

In [21][22] it is shown that one may associate linear modules to effective divisors. Let D be an effective divisor of degree l on E with $\mathcal{O}(D)\not\cong\mathcal{L}$. Then $W = H^0(E,\mathcal{L}(-D))\subset H^0(E,\mathcal{L}) = A_1$ is a subvector space of dimension $r-l$ and we put $M(D) = A/AW$. Then $M(D)$ is a linear left module for A. Similarly $M_r(D) = A/WA$ is a linear right module for A.

6.2. A computation. In this section we introduce a few notations and we compute certain fiber products. Since the results are rather intuitive and probably well known the reader is advised, on first reading, to go directly to §6.3.

In this section E is a smooth curve over the algebraically closed base field F, but the results will only be applied in the case that E is an elliptic curve. l will be some positive integer.

Let $X = S^lE$. If X is affine then $\mathcal{O}(E)$ is defined as $(\mathcal{O}(E)^{\otimes l})^{S_l}$ where S_l is the l'th symmetric group, acting naturally. The general case follows by gluing. It is well-known that S^lE is smooth.

Let $Y = E\times S^{l-1}E$, $Y_2 = E^2\times S^{l-2}E$. By $\psi : E^{a+b}\times S^{b-c}E\to E^a\times S^bE$ we denote the natural map obtained from the inclusion of structure sheaves.

We define maps $\delta : Y\to Y\times_F Y$, $\epsilon : Y_2\to Y\times_F Y$ as follows : δ is the diagonal map $y\mapsto (y,y)$ and $\epsilon(y) = (\psi(y),\psi(\omega y))$ where $\omega : Y_2\to Y_2$ sends (y_0,y_1,y_2) to (y_1,y_0,y_2). Put $\bar{\Delta} = \Delta\times S^{l-2}E\subset Y_2$.

Proposition 6.2.1. (1) *The maps δ, ϵ are closed immersions.*
(2) *The canonical inclusion $Y\times_X Y\hookrightarrow Y\times_F Y$ identifies $Y\times_X Y$ with $\operatorname{im}\delta\cup\operatorname{im}\epsilon$.*
(3) $\operatorname{im}\delta\cap\operatorname{im}\epsilon = \epsilon(\bar{\Delta})$*, scheme-theoretically.*

Proof. **Step 1.** *It is clear that δ is a closed immersion, so we concentrate on ϵ. According to the definition $\epsilon(y_0,y_1,y_2) = (y_0,\psi(y_1,y_2),y_1,\psi(y_0,y_2))$. So ϵ is, up to permuting the factors of $E^2\times(S^{l-2}E)^2$, given by the composition of $(y_0,y_1,y_2)\mapsto(y_0,y_2,y_1,y_2)$ with (α,α), where $\alpha : E\times S^{l-2}E\to E\times S^{l-1}E : (y_0,y_2)\mapsto(y_0,\psi(y_0,y_2))$. Since the first map is obviously a closed immersion, it suffices to show that α is a closed immersion. This follows from the following lemma.*

Sublemma. *Let A be a commutative F-algebra and let α^* be the composition of*

$$A\otimes(A^{\otimes l})^{S_l}\to A\otimes A\otimes(A^{\otimes(l-1)})^{S_{l-1}}\to A\otimes(A^{\otimes(l-1)})^{S_{l-1}}$$

where the first map is obtained from the inclusion $(A^{\otimes l})^{S_l}\hookrightarrow A\otimes(A^{\otimes(l-1)})^{S_{l-1}}$ and the second from the multiplication $A\otimes A\to A$. Then α^* is a surjection.

Proof. If V is an F-vectorspace then $S^l V = (V^{\otimes l})^{S_l}$ is, as an F-vectorspace, generated by the symmetrizations of the elements

$$\underbrace{a\otimes\cdots\otimes a}_{u_1}\otimes\underbrace{b\otimes\cdots\otimes b}_{u_2}\otimes\cdots\otimes\underbrace{c\otimes\cdots\otimes c}_{u_t}$$

where $l = u_1 + \cdots + u_t$, $u_i > 0$. By symmetrization we mean a sum over $S_l/S_{u_1}\times\cdots\times S_{u_t}$.

To prove the lemma it suffices to prove that $1\otimes S^{l-1}A$ is in the image of α^*.

Let $u_1+\cdots+u_t \le l-1$, $u_1 > 0$. We work with the following induction hypothesis.

Hyp($u_1,\dots,u_t$) For all $a, b, \dots, c \in A$ the symmetrization of

$$1\otimes\underbrace{a\otimes\cdots\otimes a}_{u_1}\otimes\underbrace{b\otimes\cdots\otimes b}_{u_2}\otimes\cdots\otimes\underbrace{c\otimes\cdots\otimes c}_{u_t}\otimes\underbrace{1\otimes\cdots\otimes 1}_{l-1-u_1-\cdots-u_t}$$

is in the image of α^*.

Clearly the hypothesis is true for $t = 0$. Assume now that the hypothesis is true for

$$\begin{gathered}u_1-1, u_2,\dots,u_t\\ u_1, u_2-1,\dots,u_t\\ \vdots\\ u_1,u_2,\dots,u_t-1\end{gathered}$$

(if $u_i - 1 = 0$ then we drop it). Then we claim that it is true for $u_1,\dots,u_t$.

To this end we look at the image of the element of $A\otimes(A^{\otimes l})^{S_l}$ given by

$$1\otimes \mathrm{Symmetrization}(\underbrace{a\otimes\cdots\otimes a}_{u_1}\otimes\underbrace{b\otimes\cdots\otimes b}_{u_2}\otimes\cdots\otimes\underbrace{c\otimes\cdots\otimes c}_{u_t}\otimes 1\otimes\cdots\otimes 1)$$

which is a sum of the form

$$\begin{aligned}
&a\otimes \mathrm{Symmetrization}(\underbrace{a\otimes\cdots\otimes a}_{u_1-1}\otimes\underbrace{b\otimes\cdots\otimes b}_{u_2}\otimes\cdots\otimes\underbrace{c\otimes\cdots\otimes c}_{u_t}\otimes 1\otimes\cdots\otimes 1)\\
&+b\otimes \mathrm{Symmetrization}(\underbrace{a\otimes\cdots\otimes a}_{u_1}\otimes\underbrace{b\otimes\cdots\otimes b}_{u_2-1}\otimes\cdots\otimes\underbrace{c\otimes\cdots\otimes c}_{u_t}\otimes 1\otimes\cdots\otimes 1)\\
&\qquad\vdots\\
&+c\otimes \mathrm{Symmetrization}(\underbrace{a\otimes\cdots\otimes a}_{u_1}\otimes\underbrace{b\otimes\cdots\otimes b}_{u_2}\otimes\cdots\otimes\underbrace{c\otimes\cdots\otimes c}_{u_t-1}\otimes 1\otimes\cdots\otimes 1)\\
&+1\otimes \mathrm{Symmetrization}(\underbrace{a\otimes\cdots\otimes a}_{u_1}\otimes\underbrace{b\otimes\cdots\otimes b}_{u_2}\otimes\cdots\otimes\underbrace{c\otimes\cdots\otimes c}_{u_t}\otimes 1\otimes\cdots\otimes 1)
\end{aligned}$$

By induction, all but the last terms are already in the image of α^*. But then the last one is also in the image. □

Step 2. *Now we show that $Y\times_X Y$ is reduced. This follows from the fact that Y and X are smooth and ψ is finite. Therefore ψ is flat* [1, V cor. 3.6]. *Hence $\mathcal{O}_{Y\times_X Y}$ is a torsion free $\mathcal{O}_X$-module. Consequently $\mathcal{O}_{Y\times_X Y}\hookrightarrow \mathcal{O}_{Y\times_X Y}\otimes_{\mathcal{O}_X} F(X) = F(Y)\otimes_{F(X)} F(Y)$.*

Now $F(Y)/F(X)$ is a subextension of $F(E^l)/F(X)$ and by definition $F(X) = F(E^l)^{S_l}$. So $F(E^l)/F(X)$ is Galois and thus $F(Y)/F(X)$ is separable. This implies that $F(Y)\otimes_{F(X)} F(Y)$ has no nilpotent elements. So $Y\times_X Y$ is reduced.

Step 3. *Since $Y\times_X Y$ is reduced, it suffices to compute the closed points of $Y\times_X Y$, which lie in the image of $Y\times_F Y$. This is the set quadruples $y = (y_0, y_2, y_1, y_3)$ with the property that $\psi(y_0, y_2) = \psi(y_1, y_3)$.*

If $y_0 = y_1$ then $y_2 = y_3$ which means that y is in the image of δ. If, on the other hand, $y_0 \neq y_1$ then there exists $y_4 \in S^{l-2}E$ such that $y_2 = \psi(y_1, y_4)$, $y_3 = \psi(y_0, y_4)$. Hence $y \in \operatorname{im}\epsilon$.

Step 4. *Now we prove that $\epsilon(\bar{\Delta}) = \operatorname{im}\delta\cap\operatorname{im}\epsilon$. It suffices to show that for any commutative F-algebra R one has*

$$\epsilon(\bar{\Delta}(R)) = \delta(Y(R))\cap\epsilon(Y_2(R))$$

Assume now that y_0, y_1, y_2 are R-points. Then it is easy to see that if $\epsilon(y_0, y_1, y_2)\in \operatorname{im}\delta$ one has $y_0 = y_1$. Thus $(y_0, y_1, y_2)\in\bar{\Delta}(R)$. □

6.3. Bimodule algebras associated to linear modules. In this section we keep the notations of the previous sections. In particular we fix the meaning of $E, \sigma, \mathcal{L}, l, d$ as in §6.1. In order to avoid confusion below let us specify once and for all that if X is a variety over F, $\sigma\in\operatorname{Aut}(X)$ and $f\in F(X)$ then in accordance with the usual conventions : $\sigma^*(f) = f\circ\sigma$ and $\sigma(f) = f\circ\sigma^{-1}$.

Put $A = A(E,\sigma,\mathcal{L})$. Let $X = S^l E$ as in §6.2. If $y = \overline{(y_1,\dots,y_l)}\in S^l E$ then we may associate to y the divisor

$$D_y = (y_1)+\cdots+(y_l)$$

on E. This defines a one-one correspondence between the F-points on $S^l E$ and the effective divisors on E of degree l.

In this section we will construct a graded $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$ together with a map $A\to H^0(X,\mathcal{B})$ such that for all $y\in S^l E$ one has

$$M_r(D_y) = H^0(X, F(y)\otimes_{\mathcal{O}_X}\mathcal{B})$$
$$M(D_y) = H^0(X, \mathcal{B}\otimes_{\mathcal{O}_X} F(\pi y))$$

Define $v: E^l\to E^l$ by

$$v(y)_i = \begin{cases}\pi(y_i) & \text{if } i\text{=}1\\ \theta(y_i) & \text{otherwise}\end{cases}$$

Now that in the notation of [22] one has $v = u_1^{-1}$. The map v is also defined on Y and Y_2 and will there be denoted by the same letter.

Define $\mathcal{H} = p_1^*\mathcal{L} = \mathcal{O}(p_1^{-1}d)$ where $p_1 : Y\to E$ is the projection on the first factor. Our main object of study will be the $\mathcal{O}_X$-bimodule $\mathcal{M} = {}_{\psi}\mathcal{H}_{\psi v}$.

Clearly $\Gamma(\mathcal{M})$ is equal to V. Let $T\mathcal{M} = \mathcal{O}_X\oplus\mathcal{M}\oplus\mathcal{M}\otimes\mathcal{M}\otimes\cdots$ be the tensor algebra of $\mathcal{M}$. The isomorphism $V = \Gamma(\mathcal{M})$ extends to a map

$$TV\to\Gamma(T\mathcal{M}) \tag{6.2}$$

Now we will construct a sub-bimodule $\mathcal{R}\subset\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{M}$ such that under (6.2) R maps to $\Gamma(\mathcal{R})$. Then we put $\mathcal{B} = T\mathcal{M}/(\mathcal{R})$, with obvious notations. Then (6.2) factors through a map $\eta: A\to\Gamma(\mathcal{B})$.

We adopt from now on the convention that $Y \times_X Y$ refers to the fiber product of

$$\begin{tikzcd} Y \arrow[dr, "\psi"] & & Y \arrow[dl, "\psi"'] \\ & X & \end{tikzcd}$$

and $Y \times'_X Y$ to the fiber product of

$$\begin{tikzcd} Y \arrow[dr, "\psi v"] & & Y \arrow[dl, "\psi"'] \\ & X & \end{tikzcd}$$

To compute $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}$ we use the diagrammatic method of §2. The diagram

$$\begin{tikzcd} & & Y \times'_X Y \arrow[dl] \arrow[dr] & & \\ & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi v"] & & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi v"] & \\ X & & X & & X \end{tikzcd} \tag{6.3}$$

may be completed to

$$\begin{tikzcd} Y \arrow[r, "v"] & Y \arrow[rr, "y\to(y,y)", "\delta"'] & & Y \times_X Y \arrow[dl] \arrow[dr] \arrow[dd, "v^{-1}\times 1"] & & Y_2 \arrow[ll, "(\psi(y),\psi(\omega y))\leftarrow y"', "\epsilon"] & Y_2 \arrow[l, "v"'] \\ & & Y \arrow[dd, "v^{-1}"'] & & Y \arrow[dd, "1"] & & \\ & & & Y \times'_X Y \arrow[dl] \arrow[dr] & & & \\ & & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi v"] & & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi v"] & & \\ & X & & X & & X & \end{tikzcd} \tag{6.4}$$

According to Proposition 6.2.1 δ, ϵ are closed immersions and $Y \times_X Y = \operatorname{im}\delta \bigcup \operatorname{im}\epsilon$, $\operatorname{im}\delta \cap \operatorname{im}\epsilon = \epsilon(\overline{\Delta})$.

The diagram (6.4) simplifies to

$$\begin{tikzcd} & Y \arrow[r, "y\to(y,vy)", "\alpha"'] & Y \times'_X Y \arrow[dl] \arrow[dr] & Y_2 \arrow[l, "(\psi(y),\psi(\omega v y))\leftarrow y"', "\beta"] & \\ & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi v"] & & Y \arrow[dl, "\psi"'] \arrow[dr, "\psi v"] & \\ X & & X & & X \end{tikzcd} \tag{6.5}$$

Here α,β are still closed immersions, $\operatorname{im}\alpha\cup\operatorname{im}\beta = Y\times'_X Y$ and $\operatorname{im}\alpha\cap\operatorname{im}\beta=\beta v^{-1}\bar\Delta$. On $Y\times'_X Y$ there is an exact sequence of $\mathcal{O}_{Y\times'_X Y}$-modules

$$0\to\mathcal{O}_{\beta(Y_2)}(-\beta v^{-1}\bar\Delta)\xrightarrow{i}\mathcal{O}_{Y\times'_X Y}\to\mathcal{O}_{\alpha(Y)}\xrightarrow{j}0 \tag{6.6}$$

Here j is given by the restriction map and i is the natural inclusion. If we restrict (6.6) to the complement of $\beta v^{-1}\bar\Delta$ then i is split by the restriction map $\mathcal{O}_{Y'\times_X Y'}\to\mathcal{O}_{\beta(Y_2)}$.

By 2.7(2)

$$\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{M}={}_{\psi\circ\mathrm{pr}_1}\mathcal{U}_{\psi v\circ\mathrm{pr}_2}$$

where $\mathcal{U}=\mathrm{pr}_1^*\mathcal{H}\otimes_{\mathcal{O}_{Y\times'_X Y}}\mathrm{pr}_2^*\mathcal{H}$.

Tensoring (6.6) with $\mathcal{U}$ yields an exact sequence.

$$0\to\mathcal{U}\otimes_{\mathcal{O}_{Y\times'_X Y}}\mathcal{O}_{\beta(Y_2)}(-\beta v^{-1}\bar\Delta)\to\mathcal{U}\to\mathcal{U}\otimes_{\mathcal{O}_{Y\times'_X Y}}\mathcal{O}_{\alpha(Y)}\to 0$$

which gives rise to a corresponding exact sequence of bimodules with middle term ${}_{\psi\circ\mathrm{pr}_1}\mathcal{U}_{\psi v\circ\mathrm{pr}_2}$ and with end terms

$$\begin{aligned}{}_{\psi\circ\mathrm{pr}_1}(\mathcal{U}\otimes\mathcal{O}_{\alpha(Y)})_{\psi v\circ\mathrm{pr}_2}&\cong{}_{\psi\circ\mathrm{pr}_1\circ\alpha}(\alpha^*\mathcal{U})_{\psi v\circ\mathrm{pr}_2\circ\alpha}\\&\cong{}_{\psi}(\mathcal{H}\otimes v^*\mathcal{H})_{\psi v^2}\\&\cong{}_{\psi}\mathcal{O}((d+\pi^{-1}d)\times S^{l-1}E)_{\psi v^2}\end{aligned}\tag{6.7}$$

Similarly

(6.8)

$$\begin{aligned}{}_{\psi\circ\mathrm{pr}_1}(\mathcal{U}\otimes\mathcal{O}_{\beta(Y)}(-\beta v^{-1}\bar\Delta))_{\psi v\circ\mathrm{pr}_2}&\cong{}_{\psi\circ\mathrm{pr}_1\circ\beta}(\beta^*\mathcal{U})(-v^{-1}(\bar\Delta))_{\psi\circ\mathrm{pr}_2\circ\beta}\\&\cong{}_{\psi}(\psi^*\mathcal{H}\otimes(\omega v)^*\psi^*\mathcal{H}\otimes\mathcal{O}(-v^{-1}\bar\Delta))_{\psi vw}\\&\cong{}_{\psi}\mathcal{O}((d\times E+E\times\theta^{-1}(d)-\Gamma_{\theta^{-1}\pi})\times S^{l-2}E)_{\psi vw}\end{aligned}$$

where $w=\omega v\omega$. In the notation of [22] : $w=u_2^{-1}$.

We put

$$\begin{aligned}\mathcal{M}\,\overline{\otimes}\,\mathcal{M}&={}_{\psi}\mathcal{O}((d+\pi^{-1}d)\times S^{l-1}E)_{\psi v^2}\\\mathcal{M}\,\underline{\otimes}\,\mathcal{M}&={}_{\psi}\mathcal{O}((d\times E+E\times\theta^{-1}(d)-\Gamma_{\theta^{-1}\pi})\times S^{l-2}E)_{\psi vw}\end{aligned}\tag{6.9}$$

Localizing at η, the generic point of X yields a similar exact sequence of $F(X)$-bimodules. Taking into account lemma 2.9, and the canonical isomorphisms $\mathcal{M}_\eta={}_1F(Y)_{v^*}$, $(\mathcal{M}\,\overline{\otimes}\,\mathcal{M})_\eta={}_1F(Y)_{v^{*2}}$, $(\mathcal{M}\,\underline{\otimes}\,\mathcal{M})={}_1F(Y_2)_{v^*w^*}$, we obtain a commutative diagram with exact rows.

(6.10)

$$\begin{array}{ccccccccc}0&\longrightarrow&\mathcal{M}\,\underline{\otimes}\,\mathcal{M}&\longrightarrow&\mathcal{M}\otimes\mathcal{M}&\longrightarrow&\mathcal{M}\,\overline{\otimes}\,\mathcal{M}&\longrightarrow&0\\&&\downarrow&&\downarrow&&\downarrow&&\\0&\longrightarrow&F(Y_2)_{v^*w^*}&\xrightarrow{i}&F(Y)_{v^*}\otimes_{F(X)}F(Y)_{v^*}&\xrightarrow{j}&F(Y)_{v^{*2}}&\longrightarrow&0\end{array}$$

The objects in the lower exact sequence are, strictly speaking, appropriate constant sheaves, but we will identify these with ordinary $F(X)$-bimodules.

From the observations following (6.6) and the canonical isomorphisms (6.7)(6.8) it follows that j is given by $\alpha^*:a\otimes b\mapsto av^*(b)$ and i is split by $\beta^*:a\otimes b\mapsto a(\omega v)^*(b)$. Note that to increase readability we have suppressed ψ^* in the description of α^* and β^*.

Now let $p \in F(E^2)$ be the following rational function on E^2

$$p(z,w) = \frac{h(w,\theta z)}{h(z,\theta w)}$$

A trite computation shows that

$$(p) = (\theta^{-1}(d) - d) \times E + E \times (d - \theta^{-1}(d)) + \Gamma_{\theta\pi^{-1}} - \Gamma_{\theta^{-1}\pi}$$

We consider p also as a function on $Y_2 = E^2 \times S^{l-2}E$ by evaluating it on the first two factors.

Since $\omega v w = v w \omega$, the map $\tilde{\omega} : F(Y_2)_{v^*w^*} \longrightarrow F(Y_2)_{v^*w^*} : a \mapsto p\omega^* a$ is $F(X)$-linear on both sides and satifies $\tilde{\omega}^2 = 1$. Furthermore a computation with divisors, using (6.9) shows that $\tilde{\omega}$ sends $\mathcal{M} \underline{\otimes} \mathcal{M}$ to itself. We define $\mathcal{R} = \ker(\tilde{\omega} + \mathrm{id})$.

The following lemma completes the construction of the algebra morphism $A \longrightarrow \Gamma(\mathcal{B})$.

Lemma 6.3.1. *Under the natural map $V \otimes_F V \longrightarrow \Gamma(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})$, the image of R lies in $\Gamma(\mathcal{R})$.*

Proof. We consider the elements of V as a rational function on $Y = E \times S^{l-1}E$ depending only on the first factor. Likewise elements of $V \otimes_F V$ are considered as rational functions on $Y_2 = E^2 \times S^{l-2}E$.

Step 1. *The composition*

$$R \longrightarrow \Gamma(\mathcal{M} \otimes \mathcal{M}) \longrightarrow \Gamma(\mathcal{M} \overline{\otimes} \mathcal{M})$$

is zero.

This follows from the fact that the composition

$$R \subset V \otimes_F V \longrightarrow F(Y)_{v^*} \otimes_{F(X)} F(Y)_{v^*} \xrightarrow{a\otimes b \mapsto a v^* b} F(Y)_{v^{*2}}$$

is zero since R vanishes on Γ_π.

Step 2. *By the previous step we know that R maps to $\Gamma(\mathcal{M} \underline{\otimes} \mathcal{M})$. In fact we have the following. Let ϕ be the composition*

$$R \longrightarrow \Gamma(\mathcal{M} \underline{\otimes} \mathcal{M}) \longrightarrow F(Y_2)_{v^*w^*}$$

Then, for $f \in R \subset V \otimes V$, considered as a rational function on $E \times E$, we have

$$\phi(f)(y_0, y_1, y_2) = f(y_0, \theta y_1) \tag{6.11}$$

To prove this we use the explicit splitting of i in the diagram (6.9). We then have to compute the composition

$$V \otimes V \longrightarrow F(Y)_{v^*} \otimes_{F(X)} F(Y)_{v^*} \xrightarrow{a\otimes b \mapsto a(\omega v)^* b} F(Y_2)_{v^*w^*}$$

We claim that if $f \in V \otimes V$ then under the above composition, f is sent to the right handside of (6.11). It suffices to verify this for $f = a \otimes b$. Then

$$\begin{aligned}(a(\omega v)^* b)(y_0, y_1, y_2) &= a(y_0, y_1, y_2)((\omega v)^* b)(y_0, y_1, y_2) \\ &= a(y_0) b((\omega v)(y_0, y_1, y_2)) = a(y_0) b(\theta y_1) = (a \otimes b)(y_0, \theta y_1)\end{aligned}$$

Step 3. *If $f \in R$ then $\tilde{\omega}(\phi(f)) = -\phi(f)$. This is now a simple verification using* (6.1).

$$\tilde{\omega}(\phi(f))(y_0, y_1, y_2) = p(y_0, y_1)\omega(\phi(f))(y_0, y_1, y) = p(y_0, y_1)\phi(f)(y_1, y_0, y_2)$$

$$= p(y_0, y_1) f(y_1, \theta y_0) = \frac{h(y_1, \theta y_0)}{h(y_0, \theta y_1)} f(y_1, \theta y_0) = -f(y_0, \theta y_1) = -\phi(f)(y_0, y_1, y_2) \quad \square$$

6.4. The connection with linear modules. We keep the notations of the previous sections. We prove the following result.

Theorem 6.4.1. *Let $y \in X$. Then*

$$M_r(D_y) = H^0(X, F(y) \otimes_{\mathcal{O}_X} \mathcal{B})$$
$$M(D_y) = H^0(X, \mathcal{B} \otimes_{\mathcal{O}_X} F(\pi y))$$

In particular $\mathcal{B}_n$ is, as a right and as a left $\mathcal{O}_X$-module, a vector bundle of rank $\binom{l+n-1}{n}$.

Proof. **Step 1.** *Let $\eta \in X$ be the generic point of X and let $\mathcal{B}_\eta$ be the stalk at η. By lemmas 2.8 and 2.9 it follows that this is an F-algebra, and that it doesn't matter whether we take stalks on the left or on the right.*

We will construct an F-algebra morphism

$$\bar{\psi} : \mathcal{B}_\eta \to H_l$$

where H_l is defined as follows :

$$H_l = F(E^l)[t_1, \dots, t_l]$$

with relations

$$t_j f = u_j(f) t_j$$
$$t_j t_i = \mu_{ij} t_i t_j$$

where $f \in F(E^l)$, $i, j \in \{1, \dots, l\}$, $i \neq j$,

$$u_j : E^l \to E^l : (y_1, \dots, y_l) \mapsto (\theta^{-1} y_1, \dots, \pi^{-1} y_j, \dots, \theta^{-1} y_l)$$

and $\mu_{ij} \in F(E^l)$, $\mu_{ij}(y) = p(y_i, y_j)$.

In [22] *a similar algebra was defined :*

$$Z_l = F(E^l)[t_1, \dots, t_l]$$

with relations

$$t_j f = u_j(f) t_j$$
$$t_j t_i = t_i t_j \lambda_{ij}$$

where

$$\lambda_{ij} = \theta(\mu_{ij})$$

Here and below $\theta, \pi : E^l \to E^l$ stand for the diagonal actions of $\theta, \pi : E \to E$.

The algebras H_l and Z_l are in fact isomorphic as F-algebras. The isomorphism $Z_l \to H_l$ is given by $t_i \mapsto t_i$, $f \mapsto \pi(f)$, for $f \in F(E^l)$.

In [22] *there was also defined a F-algebra homomorphism $\phi_l : A \to Z_l$ which sends $a \in V = A_1$ to $\sum t_i y_i$ where for $y \in E^l$, $a_i(y) = a(y_i)$. In this section we work with H_l instead of with Z_l since it simplifies some formulas.*

To construct $\bar{\psi}$, we need a description of $\mathcal{B}_\eta$. Clearly

$$\mathcal{B}_\eta = T_{F(X)}\mathcal{M}_\eta/(\mathcal{R}_\eta)$$

where $\mathcal{M}_\eta = F(Y)_{v^*}$.

Hence $\mathcal{M}_\eta \otimes_{F(X)} \mathcal{M}_\eta = F(Y)_{v^*} \otimes_{F(X)} F(Y)_{v^*}$ and according to the discussion after (6.6) *there is an isomorphism of $F(X)$-bimodules*

$$F(Y)_{v^*} \otimes_{F(X)} F(Y)_{v^*} \xrightarrow{(\alpha^*,\beta^*)} F(Y)_{v^{*2}} \oplus F(Y_2)_{v^*w^*}$$

and from the description of $\mathcal{R}$ in §6.3 it follows that $f \in F(Y)_{v^*} \otimes_{F(X)} F(Y)_{v^*}$ will be in $\mathcal{R}_\eta$ iff

$$\alpha^*(f) = 0 \tag{6.12}$$

$$\tilde{\omega}(\beta^*(f)) + \beta^*(f) = 0 \tag{6.13}$$

For the computations below we record the following explicit descriptions of α^* and β^*. Let $f \in F(Y) \otimes_F F(Y)$ be a lifting of $f \in F(Y)_{v^*} \otimes_{F(X)} F(Y)_{v^*}$ and consider this f as a function on $Y^2 = E \times S^{l-1}E \times E \times S^{l-1}E$. Then

$$\alpha^*(f)(y_0, y_1) = f(y_0, y_1, \pi y_0, \theta y_1) \tag{6.14}$$

$$\beta^*(f)(y_0, y_1, y_2) = f(y_0, \psi(y_1, y_2), \theta(y_1), \psi(\pi y_0, \theta y_2)) \tag{6.15}$$

(It suffices to check this for $f = a \otimes b$.) Now we define

$$\psi_i : E^l \to Y : (y_1, \dots, y_l) \mapsto (y_i, \psi(y_1, \dots, \hat{y}_i, \dots, y_l))$$

and we construct a map

$$\bar{\psi} : T_{F(X)}\mathcal{M}_\eta \to H_l$$

as follows : $\bar{\psi} \mid F(X)$ is the usual inclusion $F(X) \hookrightarrow F(E^l)$, and if $a \in \mathcal{M}_\eta = {}_1F(Y)_{v^*}$ then

$$\bar{\psi}(a) = \sum_{i=1}^{l} \psi_i^*(a) t_i$$

The commutativity of the diagram

$$\begin{array}{ccc} E^l & \xrightarrow{u_i^{-1}} & E^l \\ {\scriptstyle \psi_i}\downarrow & & {\scriptstyle \psi_i}\downarrow \\ Y & \xrightarrow{v} & Y \end{array}$$

implies that $\bar{\psi}$ is compatible with the left and right $F(X)$-module structure on $\mathcal{M}_\eta$ and hence that $\bar{\psi}$ defines a map $T_{F(X)}\mathcal{M}_\eta \to H_l$.

We know show that $\bar{\psi}$ factors through a map $\mathcal{B}_\eta \to H_l$. That is $\bar{\psi}(\mathcal{R}_\eta) = 0$.

Let $f \in \mathcal{M}_\eta \otimes_{F(X)} \mathcal{M}_\eta$. We compute

$$\bar{\psi}(f) = \sum_{i,j=1}^{l} f_{ij} t_i t_j$$

where

$$f_{ii}(y) = \alpha^*(f)(y_i, \psi(y_1, \dots, \hat{y}_i, \dots, y_l))$$

$$(i \neq j) \qquad f_{ij}(y) = \beta^*(f)(y_i, y_j, \psi(y_1, \dots, \hat{y}_i, \dots, \hat{y}_j, \dots, y_l))$$

(again one verifies this for $f = a \otimes b$ and then one uses the fact that α^*, β^* are given by (6.14)(6.15)).

Assume now that $f \in \mathcal{R}_\eta$. Then $f_{ii} = 0$ by (6.12) and $f_{ij}t_it_j + f_{ji}t_jt_i = 0$ by (6.13).

Step 2. *Let "dim" stand for right or left dimension. Then*

$$\dim_{F(X)}(\mathcal{B}_\eta)_n \geq \binom{n+l-1}{n} \tag{6.16}$$

Let us look at right dimension. The inequality obviously holds if $\mathcal{B}_\eta$ is replaced by $H_l(\cong Z_l)$ and $F(X)$ by $F(E^l)$. See [22]. Hence it suffices to show that H_l, as left $\mathcal{B}_\eta$-module, is generated by $(H_l)_0 = F(E^l)$.

To this end we look at the composition

$$A \xrightarrow{\eta} \mathcal{B}_\eta \xrightarrow{\bar{\psi}} H_l \tag{6.17}$$

where the first map is obtained from the map $\eta : A \to \Gamma(\mathcal{B})$ defined in §6.3. Then the image of $a \in V = A_1$ under (6.17) is given by

$$\sum_{i=1}^{l} a_i t_i$$

where, for $y \in E^l$, $a_i(y) = a(y_i)$. Here we have considered a as a rational function on E.

We now easily verify that the following diagram is commutative

$$\begin{tikzcd} & A \arrow[dl, "\phi_l"'] \arrow[dr, "\bar{\psi}\circ\eta"] & \\ Z_l \arrow[rr, "\cong"] & & H_l \end{tikzcd}$$

Then by [22, Thm A.3], Z_l, and hence H_l is generated by $F(E^l)$ as a left A-module. So this is certainly true as a left $\mathcal{B}_\eta$-module.

Step 3. *Let $y \in X$. We show that the compositions*

$$A \to \Gamma(\mathcal{B}) \to \Gamma(\mathcal{B} \otimes_{\mathcal{O}_X} F(y)) \tag{6.18}$$

$$A \to \Gamma(\mathcal{B}) \to \Gamma(F(y) \otimes_{\mathcal{O}_X} \mathcal{B}) \tag{6.19}$$

are surjective.

We will concentrate on (6.18), (6.19) being similar. To prove (6.18) we have to show that for all $n \in \mathbb{N}$, $\mathcal{B}_n$, as a right module is generated by $V^{\otimes n}$.

Since $\mathcal{B}_n$ is a quotient of $\mathcal{M}^{\otimes n}$, it suffices to prove this for $\mathcal{M}^{\otimes n}$. Now it is an easy exercise to show that $\mathcal{M}$ is generated on the right, as well as on the left, by V. Then induction and the following lemma finishes the proof.

Sublemma. *Assume that Y, Z are F-schemes, $\mathcal{M}$ and $\mathcal{O}_Y$-module and $\mathcal{N}$ an $\mathcal{O}_Y-\mathcal{O}_Z$-bimodule. Assume furthermore that $\mathcal{M}$ is generated by a vector space $V \subset \Gamma(\mathcal{M})$ and that $\mathcal{N}$ is generated on the right by a vector space $W \subset \Gamma(\mathcal{N})$. Then $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ is generated by $V \otimes_F W$.*

Proof. We have a surjection of $\mathcal{O}_Y$-modules

$$V \otimes_F \mathcal{O}_Y \to \mathcal{M} \to 0$$

This yields a surjection

$$V \otimes_F \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N} \to 0$$

Since $V \otimes_F \mathcal{N}$ is generated by $V \otimes_F W$, this shows what we want. $\square$

Step 4. *Under the maps* (6.18)(6.19) *introduced in step 3,* $\Gamma(\mathcal{B} \otimes_{\mathcal{O}_X} F(y))$ *becomes a quotient of* $M(D_{\pi^{-1}y})$ *and* $\Gamma(F(y) \otimes_{\mathcal{O}_X} \mathcal{B})$ *a quotient of* $M_r(D_y)$.

Let us concentrate on $\Gamma(\mathcal{B} \otimes_{\mathcal{O}_X} F(y))$, $\Gamma(F(y) \otimes_{\mathcal{O}_X} \mathcal{B})$ is similar but easier.

Let $W = \ker(V \to \Gamma(\mathcal{M} \otimes_{\mathcal{O}_X} F(y))$. First we have to compute $\mathcal{M} \otimes_{\mathcal{O}_X} F(y)$. Recall that $\mathcal{M} = {}_\psi\mathcal{H}_{\psi v}$ where $\mathcal{H} = p_1^*\mathcal{L}$ and $p_1 : Y = E \times S^{l-1}E \to E$ is the projection on the first factor.

According to the appropriate variant of lemma 2.7(1) we find $\mathcal{M} \otimes_{\mathcal{O}_X} F(y) = \psi_*(\mathcal{H} \otimes_{\mathcal{O}_Y} \mathcal{O}_{v^{-1}\psi^{-1}y})$. Then from the exact sequence

$$0 \to \mathcal{H} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-v^{-1}\psi^{-1}y) \to \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_Y} \mathcal{O}_{v^{-1}\psi^{-1}y} \to 0$$

we deduce

$$\begin{aligned} W &= \Gamma(Y, \mathcal{H} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-v^{-1}\psi^{-1}y)) \\ &= \Gamma(Y, p_1^*\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-v^{-1}\psi^{-1}y)) \\ &= \Gamma(E, \mathcal{L} \otimes_{\mathcal{O}_E} p_{1*}\mathcal{O}_Y(-v^{-1}\psi^{-1}y)) \end{aligned}$$

and from the commutative diagrams

$$\begin{tikzcd} & Y \arrow[r, "p_1"] \arrow[d, "v"'] & E \arrow[d, "\pi"'] \\ X & Y \arrow[l, "\psi"'] \arrow[r, "p_1"] & E \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[d, "\pi"'] & Y \arrow[l, "\psi"'] \arrow[r, "p_1"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"'] \\ X & Y \arrow[l, "\psi"'] \arrow[r, "p_1"] & E \end{tikzcd}$$

we deduce

$$\begin{aligned} p_{1*}\mathcal{O}_Y(-v^{-1}\psi^{-1}y) &= p_{1*}v^*\mathcal{O}_Y(-\psi^{-1}y) \\ &= \pi^* p_{1*}\mathcal{O}_Y(-\psi^{-1}y) \\ &= p_{1*}\pi^*\mathcal{O}_Y(-\psi^{-1}y) \\ &= p_{1*}\mathcal{O}_Y(-\pi^{-1}\psi^{-1}y) \\ &= p_{1*}\mathcal{O}_Y(-\psi^{-1}\pi^{-1}y) \\ &= \mathcal{O}_E(-D_{\pi^{-1}y}) \end{aligned}$$

Thus $W = \Gamma(E, \mathcal{L}(-D_{\pi^{-1}y}))$ and hence $\Gamma(\mathcal{B} \otimes_{\mathcal{O}_X} F(y))$ is a quotient of $A/AW = M(D_{\pi^{-1}}y)$.

Step 5. *Now we conclude the proof of the theorem. According to step 2 and semi-continuity*

$$\dim \Gamma(\mathcal{B}_n \otimes_{\mathcal{O}_X} F(\pi y)) \geq \dim M(D_y)_n$$
$$\dim \Gamma(F(y) \otimes_{\mathcal{O}_X} \mathcal{B}_n) \geq \dim M_r(D_y)_n$$

Combining this with step 4 finishes the proof. □

6.5. The algebras $\mathcal{B}$ are noetherian. In this section we prove that the bimodule algebras $\mathcal{B}$, introduced in the previous sections, are noetherian. We use the reduction mod p argument used in [4][22]. However we need here a version that refers only to noetherian schemes since bimodules where only defined over noetherian schemes.

To be able to reduce mod p we have to work over a commutative base ring, say S. We will always assume that S is a noetherian domain. A family of triples $\tau_S = (E_S, \sigma_S, \mathcal{L}_S)$ is a smooth elliptic curve over S together with an S-automorphism which is a translation in each geometric fiber, and an invertible sheaf $\mathcal{L}_S$ on E_S which is of degree r in each geometric fiber.

In [22] a corresponding Sklyanin algebra was defined which is compatible with base change, and in the case that S is a field, coincides with our earlier definition.

By carefully checking the constructions in the previous sections the reader may verify that these can be globalized over S using standard techniques from algebraic geometry. See [22] for a similar but easier case.

In this way one constructs an S-flat $\mathcal{O}_S$-commuting $\mathcal{O}_{X_S}$-bimodule algebra $\mathcal{B}_S$ which is compatible with base change, and for S an algebraically closed field is the same as before.

We will use the following lemma.

Lemma 6.5.1. *Let F be a field, $(R_i, m_i)_{i\in\mathbb{N}}$ a sequence of discrete valutation rings in F such that F is the union of the quotient fields of the R_i. Assume that for all i, (R_{i+1}, m_{i+1}) dominates (R_i, m_i). Define $k = \varinjlim R_i/m_i$. Let X_{R_0} be a flat R_0-scheme of finite type. Put $X_K = X_{R_0} \times_{R_0} F$, $X_k = X_{R_0} \times_{R_0} k$. Furthermore let $\mathcal{B}_{R_0}$ be a graded R_0-commuting, R_0-flat, $\mathcal{O}_{X_{R_0}}$-bimodule algebra. We put $\mathcal{B}_F = \mathcal{B}_{R_0} \otimes_{R_0} F$, $\mathcal{B}_k = \mathcal{B}_{R_0} \otimes_{R_0} k$ which we consider respectively as $\mathcal{O}_{X_F}$ and $\mathcal{O}_{X_k}$-bimodule algebras.*

Assume that the grading on $\mathcal{B}_{R_0}$ is of the form

$$\mathcal{O}_X \oplus (\mathcal{B}_{R_0})_1 \oplus (\mathcal{B}_{R_0})_2 \oplus \cdots$$

with each $(\mathcal{B}_{R_0})_n$ coherent. Then if $\mathcal{B}_k$ is left noetherian, the same holds for $\mathcal{B}_F$.

Proof. Assume that $\mathcal{B}_k$ is left noetherian and let I be a graded left ideal in $\mathcal{B}_F$. Put $I_i = I \cap \mathcal{B}_{R_i}$ with obvious notations. Then $(I_i)_k = I_i \otimes_{R_i} k$ injects in $\mathcal{B}_k = \mathcal{B}_{R_i} \otimes_{R_i} k$. This comes down to a statement about $\mathcal{O}_{X_{R_i}}$-modules which may be checked locallly. So we obtain an ascending chain $(I_i)_k$ of left $\mathcal{B}_k$ ideals which must be stationary. So by Nakayama $I_{i+1} = I_i \otimes_{R_i} R_{i+1}$ for $i \gg 0$.

Hence $I = I_i \otimes_{R_i} F$ and it suffices to show that I_i is coherent. That is

$$\mathcal{B}_{R_i} \otimes_{\mathcal{O}_{X_{R_i}}} (I_i)_0 \oplus \cdots \oplus \mathcal{B}_{R_i} \otimes_{\mathcal{O}_{R_i}} (I_i)_n(-n) \to I_i$$

should become surjective for large n. This follows again from Nakayama and the fact that $\mathcal{B}_k$ is noetherian. □

Theorem 6.5.2. *$\mathcal{B}$ is a noetherian $\mathcal{O}_X$-bimodule algebra.*

Proof. It suffices to show than $\mathcal{B}$ is left noetherian, right noetherian is similar.

We consider first the case that σ has finite order. Define $E' = E/\langle\sigma\rangle$, $X' = S^l E'$. Then $\mathcal{M} = {}_\psi\mathcal{H}_{\psi v}$ in an $\mathcal{O}_{X'}$-commuting bimodule. Put $\mathcal{A} = \mathcal{O}_{X'} \otimes_F A$ where $\mathcal{O}_{X'}$ and A commute inside $\mathcal{A}$.

By a slight abuse of notation we may consider $\mathcal{B}$ as an $\mathcal{O}_{X'}$-bimodule. The map $A \to \Gamma(\mathcal{B})$ induces an $\mathcal{O}_{X'}$-bimodule map $\mathcal{A} \to \mathcal{B}$, and by Theorem 6.4.1 (more specifically step 3 in the proof) a surjection of left $\mathcal{A}$-modules $\mathcal{A} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \to \mathcal{B}$.

So it suffices in this case to show that $\mathcal{A}$ is noetherian. Since σ has finite order, A is finite over a finitely generated center $Z(A)$. Then $\mathcal{A}$ is finite over $\mathcal{O}_{X'} \otimes_F Z(A)$, which is obviously noetherian.

Now we assume that σ is general.

Any left ideal in $\mathcal{B}$ is defined over a countable extension of the prime field, so we may in fact assume that F itself is countably generated.

Now there exists a subring S of F, finitely generated over $\mathbb{Z}$, and a family of triples $\mathcal{T}_S = (E_S, \sigma_S, \mathcal{L}_S)$ such that $F \otimes_S \mathcal{T}_S = \mathcal{T}_F = (E, \sigma, \mathcal{L})$. Choose a maximal ideal m in S. Then S/m is finite and in particular $\sigma_{S/m} = S/m \otimes_S \sigma_S$ has finite order.

Using the fact that F is countably generated, it is easy to see that there exists a sequence of discrete valuation rings (R_i, m_i) in F as in lemma 6.5.1 such that (R_0, m_0) dominates (S, m).

Since σ_k still has finite order, $\mathcal{B}_k$ is noetherian and therefore $\mathcal{B} = \mathcal{B}_F$ is noetherian. ☐

7. Central quotients of four-dimensional Sklyanin algebras

7.1. Generalities. In this section we keep the notations of the previous sections, but we assume that $r = 4$, $l = 2$. As before σ is the translation by a certain element of $J(E)$, which we denote by τ. We make the standard assumption that $4\tau \neq 0$. If $4\tau = 0$ then $A = A(E, \sigma, \mathcal{L})$ is a twist of a polynomial algebra.

Since we will use the results of [9] we are forced to assume that F is of characteristic zero, but it is very likely that this condition may be removed once a version of [9] becomes available for more general base fields.

We normalize things in the standard way. We fix $o \in E$ such that $\mathcal{L} = \mathcal{O}(4(o))$ and we make E into an abelian variety with o as origin. Of course we also take $d = 4(o)$ so that as before $\mathcal{L} = \mathcal{O}(d)$.

It is shown in [9] than $Z(A)_2$ contains a two-dimensional subspace $F\Omega_1 + F\Omega_2$ such that the annihilator of any line module intersects $F\Omega_1 + F\Omega_2$ in one-dimensional subspace, and furthermore that any such one dimensional subspace annihilates a line module. The correspondence line modules $\leftrightarrow$ central annihilators may be made explicit as follows.

Proposition 7.1.1. [9] *Let $p, q \in E$ and write $M(p,q)$ for the linear module $M((p) + (q))$. Then*

(1) *The intersection*

$$\operatorname{Ann}_A M(p,q) \cap (F\Omega_1 + F\Omega_2) \tag{7.1}$$

depends only on $p + q$.

Put $r = p + q$ and denote by $\Omega(r)$ an arbitrary non-zero element of (7.1). Then

(2) $\Omega(r) = \Omega(r')$ *if and only if* $r = r'$ *or* $r + r' = -4\tau$.

Remark 7.1.2. The reader should note that 2τ in this paper corresponds to τ in [9].

It is shown in [9] that $A/A\Omega_1 + A\Omega_2$ is the twisted homogenous coordinate ring $B = B(E, \pi, \mathcal{L})$ associated to the triple $(E, \pi, \mathcal{L})$. That is B is the global sections of

$$\mathcal{O}_E \oplus \mathcal{L} \oplus \mathcal{L} \otimes \pi^* \mathcal{L} \oplus \cdots$$

with multiplication $a \otimes b \mapsto a\pi^{*m}(b)$ for $a \in B_m$, $b \in B_n$.

It shown in [5] that there is a category equivalence between $\mathcal{O}_E$-qch and B-QGr. Under this equivalence a (left) point module $M(y)$ for A, $y \in E$ corresponds to $F(\pi y)$.

If $M \in A$-gr is of GK-dimension l then one defines $M^\vee = \operatorname{Ext}^{4-l}(M, A)$. We will use the left A-module $M^{\vee\vee}$ below. It properties have been determined in [9, §2]. We select some of these properties in the following proposition

Proposition 7.1.3. *Let* $M \in A$-gr *be of GK-dimension* l*. Then*

(1) $M^{\vee\vee}$ *is pure.*
(2) *There is a natural map* $M \longrightarrow M^{\vee\vee}$ *whose kernel is the maximal submodule of* M *of GK dimension* $< l$ *and whose cokernel has GK dimension* $\leq l - 2$.
(3) *If GK-dimension* $M \leq 2$ *then* $M^{\vee\vee}$ *is Cohen-Macaulay.*
(4) *If* M *is Cohen-Macaulay then* $M^{\vee\vee} = M$.

This proposition may be used in combination with the following

Proposition 7.1.4. *The shifted point, line and plane modules are precisely the Cohen-Macaulay modules of multiplicity 1 and GK-dimension 1,2,3.*

For the definition of multiplictity we refer to [3]. The following proposition is implicit in [9]

Proposition 7.1.5. *The intersection of the kernels of* $(A \longrightarrow M(p,q))_{p+q=r}$ *(where* 1 *is mapped to a generator of* $M(p,q)$*) is given by* $\Omega(r)A$.

Proof. Denote this common kernel by I. Clearly $\Omega(r) \in I$. Since $A/A\Omega(r)$ is a domain [9], either $I = A\Omega(r)$ or $\operatorname{GKdim} A/I \leq 2$. Assume the latter.

Let M_0 be the maximal submodule of A/I of GKdim ≤ 1, and let $M_0 \subset M_1 \subset \cdots \subset M_n = A/I$ be a filtration such that each M_{i+1}/M_i is critical of GK-dimension 2. Fix some $M(p,q)$, $p + q = r$. Since $M(p,q)$ is a quotient of A/I there must be a non-zero, necessarily injective map, $M_i/M_{i-1} \longrightarrow M(p,q)$ for some i.

Since M has GK-dimension 2 and multiplicity 1, it follows from Propositions 7.1.3,7.1.4 that $M^{\vee\vee}$ is a shifted line module, say $M(r,s)(-k)$. Thus $M(r,s)(-k)$ is a submodule of $M(p,q)$. But then it follows from the description of maps between line modules in [9] that, given r, s, there are only a finite number of possibities for p, q.

So we find that A/I has at most a finite number of line module quotients. But this is absurd since there are clearly an infinite number of couples such that $p+q = r$. $\square$

Since $l = 2$ we have $Y = Y_2 = E^2$, $X = S^2E$. Let $z \in E$. We construct the following pullback diagram

$$\begin{array}{ccc} E & \xrightarrow{\zeta_z} & Y \\ {\scriptstyle\psi}\downarrow & & {\scriptstyle\psi}\downarrow \\ \mathbb{P} & \xrightarrow{\zeta_z} & X \end{array}$$

where ζ_z is the map $y \mapsto (y+z, -y+z)$ and $\psi : E \to \mathbb{P}^1$ is the quotient under $y \mapsto -y$.

From Proposition 2.13 it easily follows that $\mathcal{M}$ adapted to (ζ_z, ζ_z) and hence we put $\mathcal{M}_z = (\zeta_z, \zeta_z)^*\mathcal{M}$. Applying again Proposition 2.13 yields that

$$\mathcal{M}_z = {}_{\psi}(\mathcal{H}_z)_{\psi v}$$

where $v : E \to E$ sends y to $y + 2\tau$ and where $\mathcal{H}_z = t_z^*\mathcal{L}$, for t_z the translation by z. Then $\mathcal{H}_z = \mathcal{O}(d_z)$, $d_z = 4(-z)$.

It follows from Proposition 2.12(1)(3) that $T\mathcal{M}$ and $\mathcal{B}$ are adapted to (ζ_z, ζ_z), so we put $\mathcal{B}_z = (\zeta_z, \zeta_z)^*\mathcal{B}$. Furthermore, by Theorem 6.4.1, $\mathcal{R}$ is a subvectorbundle of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}$ on the left and on the right, and thus by Proposition 2.12(2), $\mathcal{R}$ is also adapted to (ζ_z, ζ_z). Thus if we put $\mathcal{R}_z = (\zeta_z, \zeta_z)^*\mathcal{R}$ then

$$\mathcal{B}_z = T\mathcal{M}_z/(\mathcal{R}_z)$$

In this way, all contstructions in §6 may be pulled over to E via ζ_z. For the benefit of the reader we present a translation table (see Table 1).

notation in §6	notation of pull-back under ζ_z	definition
Y	E	
Y_2	E	
X	$\mathbb{P}^1$	
d	d_z	$4(-z)$
$\mathcal{H}$	$\mathcal{H}_z$	$\mathcal{O}(d_z)$
ω	ω	$y \mapsto -y$
ψ	ψ	quotient under $y \mapsto -y$
v	v	$y \mapsto y + 2\tau$
$\mathcal{M}$	$\mathcal{M}_z$	${}_{\psi}(\mathcal{H}_z)_{\psi v}$
$\mathcal{R}$	$\mathcal{R}_z$	$(\zeta_z, \zeta_z)^*\mathcal{R}$
w	w	$y \mapsto y - 2\tau$
p	p_z	$\zeta_z^*(p)$
$\tilde{\omega}$	$\tilde{\omega}_z$	$y \mapsto p_z\omega^*$

TABLE 1. A translation table

By analogy to (6.10) localizing at the generic point of $\mathbb{P}^1$ yields a commutative diagram.

(7.2)

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_z \underline{\otimes} \mathcal{M}_z \arrow[r] \arrow[d] & \mathcal{M}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z \arrow[r] \arrow[d] & \mathcal{M}_z \overline{\otimes} \mathcal{M}_z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(E) \arrow[r, "i"] & F(E)_{v^*} \otimes_{F(\mathbb{P}^1)} F(E)_{v^*} \arrow[r, "j"] & F(E)_{v^*} \arrow[r] & 0
\end{tikzcd}$$

Recall that as in (6.10) j is given by $\alpha^* : a \otimes b \mapsto av^*(b)$ and i is split by β^* : $a \otimes b \mapsto a(\omega v)^*(b)$. The sections in $\mathcal{R}_z$ are those that are anti-symmetric under the map $\tilde{\omega}_z$.

Note that with the definitions in table 1, $\mathcal{B}_z$ could have been defined directly, without constructing $\mathcal{B}$ first.

There is a natural map

$$A \to \Gamma(\mathcal{B}) \to \Gamma(\mathcal{B}_z) \tag{7.3}$$

sending a section of $\mathcal{B}$ to $(\zeta_z, \zeta_z)^*(a)$. In this way $\mathcal{B}_z$-modules become A-modules.

The following proposition shows that $\mathcal{B}_z$ defines certain line modules.

Proposition 7.1.6. *If $y \in \mathbb{P}^1$ and $\tilde{y} \in E$ is a lifting of y under ψ then $\Gamma(\mathcal{B}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y))$ is isomorphic to $M(-\tilde{y} + z - 2\tau, \tilde{y} + z - 2\tau)$.*

Proof. Applying ζ_z^* shows that $\Gamma(\mathcal{B} \otimes_{\mathcal{O}_X} F(\zeta_z y))$ is isomorphic to $\Gamma(\mathcal{B}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y))$. Furthermore $\zeta_z y = \psi(\tilde{y}+z, -\tilde{y}+z)$ and hence by Theorem 6.4.1 $\Gamma(\mathcal{B} \otimes_{\mathcal{O}_X} F(\zeta_z y)) = M(\tilde{y} + z - 2\tau, -\tilde{y} + z - 2\tau)$. □

Corollary 7.1.7. *The kernel of the map (7.3) $A \to \Gamma(\mathcal{B}_z)$ is generated by $\Omega(2z - 4\tau)$.*

Proof. If $\mathcal{F}$ is a $\mathcal{O}_{\mathbb{P}^1}$-vector bundle then the natural map $\Gamma(\mathbb{P}^1, \mathcal{F}) \to \prod_y \Gamma(\mathbb{P}^1, \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y))$ is injective. So the map

$$\Gamma(\mathcal{B}_z) \to \prod_y \Gamma(\mathcal{B}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y))$$

is injective. Hence the kernel of the map $A \to \Gamma(\mathcal{B}_z)$ is by Proposition 7.1.6 the common kernel of

$$A \to M(\tilde{y} + z - 2\tau, -\tilde{y} + z - 2\tau)$$

which by Proposition 7.1.5 is generated by $\Omega(2z - 4\tau)$. □

7.2. Ampleness. In this section we verify the conditions of Theorem 5.2 for $\mathcal{B}_z$. Note that the implicit condition that $\mathcal{B}_z$ should be noetherian is true since it is true for $\mathcal{B}$ by Theorem 6.5.2.

Our main result is the following

Theorem 7.2.1. *We always have*

$$\Gamma(\mathcal{B}_z) = A/A\Omega(2z - 4\tau)$$

Furthermore if

$$-4z - 4q\tau \neq 0 \qquad \text{for } q \leq -1 \tag{7.4}$$

then $(\mathcal{B}_z)_m$ *is an ample sequence on* $\mathbb{P}^1$ *and hence we have inverse equivalences*

$$A/A\Omega(2z-4\tau)\text{-QGr} \underset{\Gamma(-)}{\overset{\mathcal{B}_z\otimes -}{\rightleftarrows}} \mathcal{B}_z\text{-QGr}$$

The proof is based upon the following lemma

Lemma 7.2.2. *Let* $n \in \mathbb{Z}$*. As a left* $\mathcal{O}_{\mathbb{P}^1}$*-module* $\mathcal{M}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)$ *is isomorphic to*

$$\begin{cases} \mathcal{O}_{\mathbb{P}^1}(n+1) \oplus \mathcal{O}_{\mathbb{P}^1}(n+1) & \text{if } -4z-4n\tau \neq 0 \\ \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(n+2) & \text{if } -4z-4n\tau = 0 \end{cases}$$

Proof. We compute

$$\begin{aligned} a_m &= \dim_k \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)) \\ &= \dim_k \Gamma(E, \mathcal{O}_E(m) \otimes_{\mathcal{O}_E} \mathcal{H}_z \otimes_{\mathcal{O}_E} v^*\mathcal{O}_E(n)) \\ &= \dim_k \Gamma(E, \mathcal{O}(2m(o) + 4(-z) + 2n(-2\tau))) \end{aligned}$$

which is equal to $2n+2m+4$, unless

$$n+m+2=0 \qquad \text{and}$$
$$2m \times (o) + 4 \times (-z) + 2n \times (-2\tau) = -4z - 4n\tau = 0$$

in E in which case we find $a_m = 1$ instead of 0.

Thus if $-4z-4n\tau \neq 0$ then the values of a_m are

$$0, \dots, 0, 2, 4, \dots$$

where the first non-zero entry occurs for $m = -n-1$. This is enough information to decide that

$$\mathcal{M}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n) = \mathcal{O}_{\mathbb{P}^1}(n+1) \oplus \mathcal{O}_{\mathbb{P}^1}(n+1)$$

If on the other hand $-4z-4n\tau = 0$ then the values of a_m are

$$0, \dots, 0, 1, 2, 4, \dots$$

where now the first non-zero values occurs for $m = -n-2$. This corresponds to

$$\mathcal{M}_z \otimes \mathcal{O}_{\mathbb{P}^1}(n) = \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}(n+2) \qquad \square$$

Proof of Theorem 7.2.1. To prove that $(\mathcal{B}_z)_m$ is an ample sequence it suffices to prove that for all $n \in \mathbb{Z}$, $(\mathcal{B}_z)_m \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)$ eventually has vanishing cohomology and becomes generated by global sections on the left. Now we claim that, as a left $\mathcal{O}_{\mathbb{P}^1}$-module $(\mathcal{B}_z)_m \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)$ may always be written as

$$\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_p) \tag{7.5}$$

with all $a_i \geq n$. Furthermore if there exists $l \in \{0, \dots, m-1\}$ such that for all $l_0 \in \{0, \dots, l\}$ one has $-4z - 4(n+l_0)\tau \neq 0$ then every $a_i \geq n+l+1$.

Since $(\mathcal{B}_z)_m \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(m)$ is a quotient of $(\mathcal{M}_z)^{\otimes m} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(m)$, and since we are on $\mathbb{P}^1$ it suffices to prove the claim for $(\mathcal{M}_z)^{\otimes m} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(m)$, but there it follows by induction. To see this, write the left module structure of $(\mathcal{M}_z)^{\otimes(m-1)} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)$ in the form (7.5) and apply lemma 7.2.2.

Hence we see that if $-4z - 4q\tau \neq 0$ for $q \leq -1$ then all $a_i \geq 0$ for $m \geq -n$. Hence we have already shown that under (7.4) $(\mathcal{B}_z)_m$ is an ample sequence.

To prove the first claim it suffices by corollary 7.1.7 to compute $\dim \Gamma(\mathbb{P}^1, (\mathcal{B}_z)_m)$. Now since we have shown above that $H^1(\mathbb{P}^1, (\mathcal{B}_z)_m) = 0$, it suffices to compute

$\chi((\mathcal{B}_z)_m)$. But Euler characteristics are constant in flat families, so we may assume that σ is the identity. But then $\mathcal{B}_z$ is just the symmetric algebra of the ordinary $\mathcal{O}_{\mathbb{P}^1}$ module $\mathcal{M}_z$, which in this case, is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ or $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$. Then a simple computation shows that $\chi((\mathcal{B}_z)_m)$ has the right value. □

7.3. Twisting. In this section we prove that the various $\mathcal{B}_z$ are connected with each other through twisting (see §3). That is we prove

Proposition 7.3.1. *Assume that $z' = z - n\tau$, $n \in \mathbb{Z}$. Then we have*

$$\mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{B}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-n) \cong \mathcal{B}_{z'} \tag{7.6}$$

Proof. The strategy of the proof will be as follows. Let $Z = n\psi(o)$, $\mathcal{Z} = \mathcal{O}(Z) \cong \mathcal{O}_{\mathbb{P}^1}(n)$, $\tilde{\mathcal{Z}} = \psi^* Z = \mathcal{O}(2n(o))$. We will show that

$$\mathcal{Z} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z \cong \mathcal{M}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{Z}$$

where $z' = z - n\tau$, and that furthermore this isomorphism is compatible with $\mathcal{R}_z$, in the sense that the composition

(7.7)
$$\mathcal{Z} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z \to \mathcal{M}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{Z} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z \to \mathcal{M}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{Z}$$

sends $\mathcal{Z} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{R}_z$ to $\mathcal{R}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{Z}$.

We compute

$$\mathcal{Z} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z = {}_{\psi}(\tilde{\mathcal{Z}} \otimes_{\mathcal{O}_E} \mathcal{H}_z)_{\psi v} = {}_{\psi}\mathcal{O}(\psi^{-1}Z + d_z)_{\psi v}$$

and

$$\mathcal{M}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{Z} = {}_{\psi}(\mathcal{H}_{z'} \otimes_{\mathcal{O}_E} v^* \tilde{\mathcal{Z}})_{\psi v} = {}_{\psi}\mathcal{O}(d_{z'} + v^{-1}\psi^{-1}Z)_{\psi v}$$

It is now easily verified that

$$\psi^{-1}Z + d_z \sim d_{z'} + v^{-1}\psi^{-1}Z$$

and hence $\mathcal{Z} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z$ and $\mathcal{M}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{Z}$ are isomorphic.

To be more precise we may construct the following commutative diagram

$$\begin{tikzcd}
\mathcal{Z} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{M}_z \arrow[r] \arrow[d] & \mathcal{M}_{z'} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{Z} \arrow[d] \\
F(\mathbb{P}^1) \otimes_{F(\mathbb{P}^1)} F(E)_{v^*} \arrow[r] \arrow[d, "\cong"'] & F(E)_{v^*} \otimes_{F(\mathbb{P}^1)} F(\mathbb{P}^1) \arrow[d, "\cong"'] \\
F(E)_{v^*} \arrow[r, "t"] & F(E)_{v^*}
\end{tikzcd}$$

where the bottom arrow is left multiplication by an element $t \in F(E)$, which is determined up to a scalar factor by

$$(t) = d_{z'} + v^{-1}\psi^{-1}\mathcal{Z} - d_z - \psi^{-1}\mathcal{Z} \tag{7.8}$$

A similar diagram may be constructed for (7.7).

$$\begin{array}{ccccc}
\mathcal{Z}\otimes\mathcal{M}_z\otimes\mathcal{M}_z & \longrightarrow & \mathcal{M}_{z'}\otimes\mathcal{Z}\otimes\mathcal{M}_z & \longrightarrow & \mathcal{M}_{z'}\otimes\mathcal{M}_{z'}\otimes\mathcal{Z}\\
\downarrow & & \downarrow & & \downarrow\\
F(\mathbb{P}^1)\otimes F(E)_{v^*}\otimes F(E)_{v^*} & \longrightarrow & F(E)_{v^*}\otimes F(\mathbb{P}^1)\otimes F(E)_{v^*} & \longrightarrow & F(E)_{v^*}\otimes F(E)_{v^*}\otimes F(\mathbb{P}^1)\\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow\\
F(E)_{v^*}\otimes F(E)_{v^*} & \xrightarrow{t\otimes 1} & F(E)_{v^*}\otimes F(E)_{v^*} & \xrightarrow{1\otimes t} & F(E)_{v^*}\otimes F(E)_{v^*}
\end{array}$$

So the composition of the lowest horizontal maps is given by multiplication with $t\otimes t$. This preserves the kernel of j in (7.2) hence we obtain a diagram

$$\begin{array}{ccc}
\mathcal{Z}\otimes\mathcal{M}_z\underline{\otimes}\mathcal{M}_z & \longrightarrow & \mathcal{M}_{z'}\underline{\otimes}\mathcal{M}_{z'}\otimes\mathcal{Z}\\
\downarrow & & \downarrow\\
F(\mathbb{P}^1)\otimes F(E) & \longrightarrow & F(E)\otimes F(\mathbb{P}^1)\\
\cong\downarrow & & \cong\downarrow\\
F(E) & \xrightarrow{tv^{-1}\omega(t)} & F(E)
\end{array}$$

To construct the lowest map we have used the explicit splitting of i, given just below diagram (7.2).

To show that multiplication by $tv^{-1}\omega(t)$ sends $\mathcal{Z}\otimes\mathcal{R}_z$ to $\mathcal{R}_{z'}\otimes\mathcal{Z}$, it suffices to verify that the following diagram is commutative, up to a scalar.

$$\begin{array}{ccc}
F(E) & \xrightarrow{\tilde{\omega}_z} & F(E)\\
{\scriptstyle tv^{-1}\omega(t)}\downarrow & & {\scriptstyle tv^{-1}\omega(t)}\downarrow\\
F(E) & \xrightarrow{\tilde{\omega}_{z'}} & F(E)
\end{array}$$

This amounts to a computation with divisors using (7.8) and the fact that

$$(p_z)=\theta^{-1}(d_z)-d_z+\omega d_z-\omega\theta^{-1}(d_z)+t_{2\tau}\Omega-t_{-2\tau}\Omega$$

where

$$\Omega=\sum_{\omega\in E_2}\omega \qquad \square$$

7.4. A translation principle. We let the notations be as before. For $z\in E$ we put $A_z=A/A\Omega(2z)$.

Assume that $z'=z-n\tau$, $n\in\mathbb{Z}$. We let the fuctor $F_{z,z'}:A_z\text{-QGr}\to A_{z'}\text{-QGr}$ be defined by the composition

(7.9)
$$A_z\text{-QGr}\xrightarrow{\mathcal{B}_{z+2\tau}\otimes-}\mathcal{B}_{z+2\tau}\text{-QGr}\xrightarrow{\mathcal{O}_{\mathbb{P}^1}(n)\otimes-}\mathcal{B}_{z'+2\tau}\text{-QGr}\xrightarrow{\Gamma(-)}A_{z'}\text{-QGr}$$

Note that the middle arrow depends upon the isomorphism (7.6) which is only determined up to scalar. It is possible to deduce from this that $F_{z,z'}$ itself is determined up to twisting with a scalar automorphism of $A_{z'}$ (such an automorphism sends $a\in A_m$ to $\lambda^m a$ if the scalar is λ). However we will not use this fact.

Theorem 7.4.1. *(Translation principle for Sklyanin algebras.) Let $z\in E$, $z'\in z+\mathbb{Z}\tau$ and assume that one of the following conditions is true.*

(1) *$z \notin \eta + \mathbb{Z}\tau$ for $\eta \in E_4$.*
(2) *τ has infinite order and $z = \eta + q\tau$, $z' = \eta + q'\tau$ with $\eta \in E^4$, $q, q' \in \mathbb{Z}$, $q, q' \leq -2$.*

*Then the fuctor $F_{z,z'}$ defined above is an equivalence. Furthermore it sends A_z-*qgr *to $A_{z'}$-*qgr *and commutes with shifts.*

Proof. The middle arrow in (7.9) is always an equivalence, according to Proposition 7.3.1 and §3. By Theorem 7.2.1 the extremal arrows are equivalences if

$$-4(z + 2\tau) - 4a\tau \neq 0 \qquad \text{for } a \leq -1 \tag{7.10}$$

$$-4(z' + 2\tau) - 4a'\tau \neq 0 \qquad \text{for } a' \leq -1 \tag{7.11}$$

If condition (1) is true then clearly (7.10)(7.11) always hold. So assume condition (2). Then, using the fact that τ has infinite order, (7.10)(7.11) are equivalent to

$$-4(q + 2) - 4a \neq 0 \qquad \text{for } a \leq -1$$
$$-4(q' + 2) - 4a' \neq 0 \qquad \text{for } a' \leq -1$$

which are trivially true if $q, q' \leq -2$.

That $F_{z,z'}$ sends A_z-qgr to $A_{z'}$-qgr and commutes with shifts follows from the fact that the functors which make up $F_{z,z'}$ have the corresponding properties. □

Remark 7.4.2. (1) In our definition A_z depends only on $2z$. However if $2z = 2z'$, $z \neq z'$ then $F_{z,z'}$ will not be defined since $\mathcal{B}_{z+2\tau} \ncong \mathcal{B}_{z'+2\tau}$. That is the reason why we defined A_z the way we did.

(2) There are other equivalences among the A_z-qgr than those given by $F_{z,z'}$. For example if $z' = z + \eta$, $\eta \in E_4$ then it follows from [16] that $A_z \cong A_{z'}$. Furthermore by [9] (see Proposition 7.1.1(2)) it follows that $A_z \cong A_{-z-2\tau}$. From this last property it folows that if $z \in \eta + \mathbb{Z}\tau$, $\eta \in E_4$ and τ has infinite order then the only A_z that is not equivalent to the the others is given by $z = \eta - \tau$.
We don't know exactly for which pairs (z, z') there is an equivalence between A_z-QGr and $A_{z'}$-QGr. Presumably the answer is similar to that for the primitive quotients of $U(\mathfrak{sl}_2)$.

Let as before $z, z' \in E$, $z' = z - n\tau$, $n \in \mathbb{Z}$. We will assume that hypothesis (1) or (2) of Theorem 7.4.1 holds. Thus $F_{z,z'}$ is an equivalence.

Since $F_{z,z'}$ commutes with shifts, it follows from standard Morita theory that there exists a $A_{z'} - A_z$-bimodule $A_{z',z}$ such that $F_{z,z'} = A_{z',z} \otimes_{A_z} -$. In fact one has

$$A_{z',z} = F_{z,z'}(A_z) \tag{7.12}$$

Of course $A_{z',z}$ is only determined up to torsion modules.

If M is a graded vector space then we denote by $H(M, t)$ the Hilbert series of M. That is

$$H(M, t) = \sum_{n \in \mathcal{Z}} \dim(M_n) t^n$$

If M is only determined up to finite dimensional modules then $H(M, t)$ is only determined up to laurent polynomials. We will need the following lemma below.

Lemma 7.4.3. *One has*

$$\begin{aligned} H(A_{z',z},t) &= H(A_z,t) + \frac{n}{(1-t)^2} \\ &= \frac{1+t}{(1-t)^3} + \frac{n}{(1-t)^2} \qquad \text{mod. laurent pol.} \end{aligned}$$

Proof. By (7.12) and Propositions 7.3.1,

$$\begin{aligned} (A_{z',z})_m &= \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} (\mathcal{B}_z)_m) \\ &= \Gamma(\mathbb{P}^1, (\mathcal{B}_{z'})_m \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)) \end{aligned}$$

Now the $(\mathcal{B}_{z'})_m$ form an ample sequence, and hence for large m

$$\dim \Gamma(\mathbb{P}^1, (\mathcal{B}_{z'})_m \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)) = \chi((\mathcal{B}_{z'})_m \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n)) \tag{7.13}$$

However Euler characteristics are constant in flat families, so we may assume that $\tau = 0$. But in that case $(\mathcal{B}_{z'})$ is the ordinary symmetric algebra of $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Then one easily shows that (7.13) has the correct value. □

The following result describes the action of $F_{z,z'}$ on point and line modules.

Theorem 7.4.4. *Assume that $z' = z - n\tau$, $n \in \mathbb{Z}$ and that hypotheses (1) and (2) of Theorem 7.4.1 hold.*

(1) *Assume that $s, t \in E$ are such that $s+t = 2z$. Then $F_{z,z'}(M(s,t)) = M(s',t')$ where $s' = s - n\tau$, $t' = t - n\tau$.*
(2) *Assume now that $s+t = -2z-4\tau$. Then $F_{z,z'}(M(s,t)) = M(s'',t'')(n)$ with $s'' = s + n\tau$, $t'' = t + n\tau$.*
(3) *Let $M(s)$ be the point corresponding to $s \in E$. Then $F_{z,z'}(M(s)) = M(s')$, $s' = s - n\tau$.*

Proof. We first prove (1). Let $\tilde{y} \in E$ be given by $s - z$ and put $y = \psi(\tilde{y})$. Then according to Proposition 7.1.6 one has

$$M(s,t) = M(\tilde{y}+z, -\tilde{y}+z) = \Gamma(\mathcal{B}_{z+2\tau} \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y))$$

That is, under the first arrow in (7.12), $M(s,t)$ is sent to $\mathcal{B}_{z+2\tau} \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y)$.

Then under the second arrow $\mathcal{B}_{z+2\tau} \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y)$ is sent to

$$\begin{aligned} \mathcal{O}_{\mathcal{O}_{\mathbb{P}^1}}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{B}_{z+2\tau} \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y) &= \mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{B}_{z+2\tau} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(-n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y) \\ &= \mathcal{B}_{z'+2\tau} \otimes_{\mathcal{O}_{\mathbb{P}^1}} F(y) \end{aligned}$$

Finally using Prop. 7.1.6 $\mathcal{B}_{z'+2\tau}$ is sent under the last arrow to

$$M(\tilde{y}+z', -\tilde{y}+z') = M(s-n\tau, t-n\tau)$$

This finishes the proof of (1).

(3) follows quite easily from (1). We put $t = 2z - s$. Then there is an exact sequence [9]

$$0 \to M(s+2\tau, t-2\tau)(-1) \to M(s,t) \to M(s) \to 0$$

Applying $F_{z,z'}$ yields an exact sequence in $A_{z'}$-qgr

$$0 \to M(s'+2\tau, t'-2\tau)(-1) \to M(s',t') \to F_{z,z'}(M(s)) \to 0 \tag{7.14}$$

with $s' = s - n\tau$, $t' = t - n\tau$.

Looking at Hilbert series reveals that $F_{z,z'}(M(s)$ must be isomorphic in A-qgr to a point module, say $M(u')$, $u' \in E$.

To determine u' we choose $\Omega \in F\Omega_1 + F\Omega_2$, $\Omega \neq \Omega(2z)$ and look at the long exact sequence in B-qgr obtained from (7.14) by tensoring with $B = A_z/A_z\Omega = A/(\Omega_1, \Omega_2)$. We obtain

(7.15)
$$0 \to \mathrm{Tor}_1(A_z/\Omega A_z, M(u')) \to M(s'+2\tau, t'-2\tau)/\Omega M(s'+2\tau, t'-2\tau)(-1) \to \\ M(s',t')/\Omega M(s',t') \to M(u') \to 0$$

Now we use the category equivalence between B-qgr and $\mathcal{O}_E$-coh. Then (7.15) yields the following identity in the Grothendieck group of finite dimensional $\mathcal{O}_E$-modules

$$[F(u'+4\tau+2\tau)] + [F(s'+2\tau)] + [F(t'+2\tau)] \\ = [F(s'+2\tau+2\tau+2\tau)] + [F(t'-2\tau+2\tau+2\tau)] + [F(u'+2\tau)]$$

Inspection, using the hypothesis $4\tau \neq 0$, yields $u' = s'$.

Now we prove (2). Assume that $s + t = -2z - 4\tau$ and let $x \in A_1$ such that x is zero on s and t. Thus the divisor of x is given by $(s) + (t) + (a) + (b)$ We have a surjective map $A_z/A_z x \to M(s,t)$. Let $M(-1)$ be its kernel. Since by [9, Prop 6.2] A_z is a domain, we may compute the Hilbert series of M and we find that it is the Hilbert series of a line module. Furthermore M is a quotient of the kernel of $A/Ax \to M(s,t)$, which is a plane module [9, Prop 4.6]. Hence M is generated in degree zero and therefore M itself is a line module, So there exist $p, q \in E$ and an exact sequence

$$0 \to M(p,q)(-1) \to A_z/A_z x \to M(s,t) \to 0 \tag{7.16}$$

To determine p, q we tensor (7.16) with $A_z/A_z\Omega$ where $\Omega \in F\Omega_1 + F\Omega_2$, $\Omega \neq \Omega(2z)$. As above we obtain an identity in the Grothendieck group of finite dimensional $\mathcal{O}_E$-modules.

$$[F(p+2\tau+2\tau)] \oplus [F(q+2\tau+2\tau)] \oplus [F(s+2\tau)] \oplus [F(t+2\tau)] \\ = [F(a+2\tau)] \oplus [F(b+2\tau)] \oplus [F(s+2\tau)] \oplus [F(t+2\tau)]$$

Hence we can take $p = a - 2\tau$, $q = b - 2\tau$.

If we apply $F_{z,z'}$ to the exact sequence

$$0 \to A_z \xrightarrow{x} A_z \to A_z/A_z x \to 0$$

then we obtain an exact sequence, up to torsion

$$0 \to A_{z',z} \xrightarrow{x} A_{z',z} \to F_{z,z'}(A_z/A_z x) \to 0$$

So we deduce

$$F_{z,z'}(A_z/A_z x) = A_{z',z}/A_{z',z}x$$

and furthermore

$$H(A_{z',z}/A_{z',z}x, t) = \frac{1+t}{(1-t)^2} + \frac{n}{1-t} \qquad \text{mod. laurent pol.}$$

Now note that $a+b+s+t = 0$ and hence $p+q = a-2\tau+b-2\tau = -s-t-4\tau = +2z$. So by (1) the image of $M(p,q)$ under $F_{z,z'}$ is an unshifted line module. Let $K = F_{z,z'}(M(s,t))$. Applying $F_{z,z'}$ to (7.16) yields

$$H(K,t) = \frac{1+t}{(1-t)^2} + \frac{n}{1-t} - \frac{t}{(1-t)^2} = \frac{n+1-nt}{(1-t)^2} \qquad \text{mod. laurent pol.}$$

hence it follows that K has GK dimension 2 and multiplicity 1. Then by Propositions 7.1.3, 7.1.4 $K^{\vee\vee}$ a shifted line module. Looking at the Hilbert series of K learns that it must be a positive shift by n. Thus, up to torsion $K = M(c,d)(n)$ for some $c, d \in E$. Assume that $s \neq t$. Then $M(s) \oplus M(t)$ is up to torsion a quotient of $M(s,t)$. Hence by (3) $M(s - n\tau) \oplus M(t - n\tau)$ is a quotient of $M(c,d)(n)$ in A-qgr. Applying $(-)^{\vee\vee}$ gives an actual map $M(c,d)(n) \to M(s - n\tau) \oplus M(t - n\tau)$ in A-gr with finite cokernel. We deduce that $c = s + n\tau, d = t + n\tau$. If $s = t$ the we may repeat the argument with the unique non-trivial self extension of $M(s)$, annihilated by $A/(\Omega_1, \Omega_2)$, replacing the direct sum $M(s) \oplus M(t)$. This finishes the proof of (3). □

7.5. Fat points. Notations are as before. A fat point of A is an irreducible object in A-qgr of GK-dimension one. In our special case it is easy to see that the tail $(M)_{\geq n}$ of a fat point is, for $n \gg 0$, critical, without socle and has constant Hilbert series. The value of this constant is equal to the multiplicity of M.

Assume from now on that σ has infinite order. It is shown in [16] that for each $n \in \mathbb{N}$, $n \geq 2$, A has precisely four non-isomorphic fat points of multiplicity n. The proof of this result consists of two parts. First one has to construct these particular fat points and then one has to show that one has found all of them. Let us refer to these two parts as "existence" and "completeness."

It turns out that "completeness" may be handled using a generalization of the methods in [3][4]. However "existence" is based upon the ingenious, but adhoc, construction by Sklyanin in [15]. As was noted in [16, §3] the reason why this construction works is somewhat unclear.

In this section we show that one may quite naturally obtain (the existence of) the fat points of A from the translation principle. Our starting point is the fat points of multiplicity one. That is, the ordinary point modules. The point modules of an algebra, given with generators and relations may be obtained using the method exhibited in [3] which consists of "multi linearizing" the equations.

In [18] it was shown that the point modules of A are parametrized by $E \cup S$ where S is a set of four points. In [9] it was shown that a natural indexation for the elements of S is given by E_2, the points of order two on E. So we put

$$S = \{e_\omega \mid \omega \in E_2\}$$

and we denote the corresponding point modules by $M(e_\omega)$. The following theorem is [9, Thm 5.7]

Theorem 7.5.1. *Let $p, q \in E$ be such that $p + q = \omega \in E_2$. Then there is a short exact sequence*

$$0 \to M(p - 2\tau, q - 2\tau)(-1) \to M(p,q) \to M(e_\omega) \to 0 \tag{7.17}$$

This result implies in particular that $M(e_\omega)$ is annihilated by $\Omega(\omega) = \Omega(\omega - 4\tau)$ and thus $M(e_\omega) \in A_{\eta-2\tau}$-gr where $2\eta = \omega$.

Proposition 7.5.2. *Put $z = \eta - 2\tau$, $z' = \eta - (n+1)\tau$, $n \in \mathbb{N}$, $\eta \in E_4$. Then for $\omega = 2\eta$, $F_{z,z'}(M(e_\omega))$ gives a fat point of multiplicity n. Furthermore these fat points are not isomorphic in A-qgr for different $\omega \in E_2$.*

Proof. According to Theorem 7.4.1, $F_{z,z'}$ is an equivalence. So the only fact that is not immedately obvious is the multiplicity of $F_{z,z'}(M(e_\omega))$.

We apply $F_{z,z'}$ to the exact sequence (7.17). Then Theorem 7.4.4 yields an exact sequence in $A_{z'}$-qgr

$$0 \to M(p' - 2\tau, q' - 2\tau)(-1) \to M(p'', q'')(n-1) \to F_{z,z'}(M(e_\omega)) \to 0$$

Computing Hilbert series yields that $F_{z,z'}(M(e_\omega))$ has the required multiplicity. □

Remark 7.5.3. With more work it is also possible to say something in the case that σ has finite order. The functor $F_{z,z'}$ is still defined, and it may be used to construct the fat points of intermediate multiplicity [17]. However, since $F_{z,z'}$ is no longer an equivalence, it is harder to prove that the constructed objects have the required properties.

Appendix A. The translation principle for H^λ

We let the notation be as in the introduction. In this appendix we will sketch the proof of the translation principle for $\operatorname{proj}^\circ H^\lambda$. Since H^λ is related to the Sklyanin algebra one would expect to be able to give a proof similar to that of Theorem 7.4.1. This is indeed possible but some new technical problems appear since one has to work with bimodules whose support is a non-reduced scheme. In order not to introduce any more complications we have chosen to derive the translation principle for H^λ from that of U^λ.

One may define a sheaf of twisted differential operators $\mathcal{D}^\lambda$ on $\mathbb{P}^1$, which for $\lambda \in \mathbb{Z}$ corresponds to the sheaf of differential operators on $\mathcal{O}(\lambda)$ (see eg [23, §6]). One still has that if $\lambda' = \lambda - n$ then

$$\mathcal{D}^{\lambda'} \cong \mathcal{O}(-n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{D}^\lambda \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(n)$$

and this isomorphism is compatible with the natural filtrations on $\mathcal{D}^\lambda$ by order of differential operators.

Furthermore one has that $U^\lambda = \Gamma(\mathbb{P}^1, \mathcal{D}^\lambda)$ and this isomorphism is also compatible with the natural filtration on U^λ and $\mathcal{D}^\lambda$.

The Bernstein-Beilinson theorem [6], specialized to our situation, asserts the following.

Theorem A.1. *Assume that either $\lambda \in F \setminus \mathbb{Z}$ or $\lambda \in \mathbb{N}$. Then there are inverse equivalences*

$$U^\lambda\text{-mod} \underset{\Gamma(\mathbb{P}^1,-)}{\overset{\mathcal{D}^\lambda \otimes_{U^\lambda} -}{\rightleftarrows}} \mathcal{D}^\lambda\text{-qch}$$

Furthermore, if $\mathcal{M} \in \mathcal{D}^\lambda$-qch then $\mathcal{M}$ is generated by global sections and $H^q(\mathbb{P}^1, \mathcal{M}) = 0$ for $q \geq 1$.

Define $\tilde{\mathcal{D}} = \oplus_{n\in\mathbb{N}} \mathcal{D}_n^\lambda t^n$ where $\mathcal{D}_n^\lambda$ is the sheaf of (twisted) differential operators of degree $\leq n$. By what we have said above we have $H^\lambda = \Gamma(\mathbb{P}^1, \tilde{\mathcal{D}}^\lambda)$ and furthermore $\tilde{\mathcal{D}}^\lambda$-gr and $\tilde{\mathcal{D}}^{\lambda'}$-gr are equivalent. So taking into account the isomorphism $H^\lambda = H^{-\lambda-2}$, the translation principle for H^λ follows from Theorem 5.2 and the following.

Proposition A.2. *Assume that either $\lambda \in F \setminus \mathbb{Z}$ or $\lambda \in \mathbb{N}$. Then $\mathcal{D}_n^\lambda$ is an ample sequence on $\mathbb{P}^1$.*

Proof. Let $\mathcal{M}_0 \in \mathcal{O}_X$-coh and put $\tilde{\mathcal{M}} = \tilde{\mathcal{D}}^\lambda \otimes_{\mathcal{O}_X} \mathcal{M}_0$, $\mathcal{M} = \mathcal{D}^\lambda \otimes_{\mathcal{O}_X} \mathcal{M}_0$.

It is easy to see that $\widetilde{\mathcal{M}_t} \cong \mathcal{M}[t, t^{-1}]$. Since $\widetilde{\mathcal{M}_t}$ is the direct limit of

$$\tilde{\mathcal{M}} \xrightarrow{t} \tilde{\mathcal{M}} \xrightarrow{t} \tilde{\mathcal{M}} \xrightarrow{t} \cdots$$

we obtain $H^q(\mathbb{P}^1, \tilde{\mathcal{M}})_t = H^q(\mathbb{P}^1, \tilde{\mathcal{M}}_t)$. Hence the Bernstein-Beilinson theorem implies that

$$\mathcal{C} = \operatorname{coker}(\mathcal{O}_{\mathbb{P}^1} \otimes \Gamma(\mathbb{P}^1, \tilde{\mathcal{M}}) \to \tilde{\mathcal{M}}) \tag{A.1}$$

and

$$H^1(\mathbb{P}^1, \tilde{\mathcal{M}}) \tag{A.2}$$

are t-torsion. We will show that the gradings on (A.1) and (A.2) are actually right limited (which is what we need). Note that in (A.1) right limited is the same as coherent over $\mathcal{O}_{\mathbb{P}_1}$ and in (A.2) right limited means finite dimensional.

Look at $\tilde{\mathcal{M}}/t\tilde{\mathcal{M}}$. This is a graded module over the symmetric algebra $S_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{T}_{\mathbb{P}^1})$. Now $\mathcal{T} = \mathcal{O}(2)$, and so $(\mathcal{T}^{\otimes n})_n$ is certainly an ample sequence. Therefore, by Theorem 5.2, one has that (A.1)(A.2) have right limited grading if we replace $\tilde{\mathcal{M}}$ by $\tilde{\mathcal{M}}/t\tilde{\mathcal{M}}$.

Applying $R\Gamma$ to the exact sequence

$$0 \to \tilde{\mathcal{M}}(-1) \xrightarrow{t} \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}/t\tilde{\mathcal{M}} \to 0$$

yields a short exact sequence

$$H^1(\mathbb{P}^1, \tilde{\mathcal{M}})(-1) \xrightarrow{t} H^1(\mathbb{P}^1, \tilde{\mathcal{M}}) \to \text{fin. dim.} \to 0$$

Then the fact that $H^1(\mathbb{P}^1, \tilde{\mathcal{M}})$ is t-torsion, easily implies that it is finite dimensional.

For (A.1) we construct the diagram

$$\begin{tikzcd}
 & & & 0 & \\
 & & & \text{right lim.} \arrow[u] & \\
0 \arrow[r] & \tilde{\mathcal{M}}(-1) \arrow[r, "t"] & \tilde{\mathcal{M}} \arrow[r] & \tilde{\mathcal{M}}/t\tilde{\mathcal{M}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \otimes \Gamma(\mathbb{P}^1, \tilde{\mathcal{M}})(-1) \arrow[r, "t"] \arrow[u] & \mathcal{O}_{\mathbb{P}^1} \otimes \Gamma(\mathbb{P}^1, \tilde{\mathcal{M}}) \arrow[r] \arrow[u] & \mathcal{O}_{\mathbb{P}^1} \otimes \Gamma(\mathbb{P}^1, \tilde{\mathcal{M}}/t\tilde{\mathcal{M}}) \arrow[r] \arrow[u] & \text{right lim.}
\end{tikzcd}$$

This yields an exact sequence

$$\mathcal{C}(-1) \xrightarrow{t} \mathcal{C} \to \text{right lim.} \to 0$$

which again implies that the grading on $\mathcal{C}$ is right limited. □

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