Modules of covariants

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1 The Hochster-Roberts theorem

Let G be a reductive group over $\mathbb{C}$ and let W be a finite dimensional representation of W. Put $R = SW$, the symmetric algebra of W. Then it follows from the famous Hochster-Roberts theorem [7] that R^G is Cohen-Macaulay. In this setting this means that there exist homogeneous $r_1, \dots, r_h \in R^G$ such that

- The $\mathbb{C}$-algebra R_0 generated by $r_1, \dots, r_h$ is a polynomial algebra.
- R^G is a finitely generated free R_0-module.

Hence one may view this result as giving, at least in principle, a unique representation for each invariant. Indeed if $m_1, \dots, m_p$ is a basis of R^G over R_0 then each $f \in R^G$ may be written uniquely as $\sum P_i(r_1, \dots, r_h)m_i$ where the P_i are polynomials in h variables.

The original Hochster-Roberts theorem had a long and complicated proof, based upon reduction mod p. Subsequent simplifications by Kempf yielded the following result.

Theorem 1.1 [9] *Assume that there is a pure homomorphism $S \to T$ between commutative rings of finite type over a field. Then if T is regular, S is Cohen-Macaulay.*

Pure means that for any $M \in S$-mod the induced map $M \to T \otimes_S M$ is injective. This happens for example if S is a direct summand of T as S-module. This is the case relevant for invariant theory. Indeed, the Reynolds operator defines an R^G-linear splitting for the inclusion $R^G \hookrightarrow R$.

Nowadays there are short proofs of stronger versions of the Hochster-Roberts theorem. One of these is due to Boutot [2]. He proves that if T/S is pure and T has rational singularities then S has rational singularities. This implies the Hochster-Roberts theorem since there are implications

$$\text{regular} \Rightarrow \text{rational singularities} \Rightarrow \text{Cohen-Macaulay}.$$

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Boutot's proof, which is very short, uses the deep Grauert-Riemenschneider vanishing theorem, whose proof uses analysis.

Another new proof for the Hochster-Roberts theorem is due to Hochster and Huneke [6] who prove that "F-regularity" is preserved under pure maps. Again this implies the original Hochster-Roberts theorem since

$$\text{regular} \Rightarrow \text{F-regular} \Rightarrow \text{Cohen-Macaulay}$$

This new approach is still based upon reduction mod p. For a self contained summary of the proof of Hochster and Huneke, see [11].

One has the impression that the Hochster-Roberts theorem is a very subtle result. This feeling is strengthened when one considers the obvious generalization to covariants which turns out to fail rather drastically (Theorem 4.3). Therefore I would propose the following question.

Question 1.2 *Does there exist a proof for the Hochster-Roberts theorem which does not use reduction mod p and which is completely algebraic (and hence does not use results such as the Grauert-Riemenschneider vanishing theorem) ?*

In the setting of invariant theory the answer is known to be yes in the case that G is a torus [5][16] and in the case that W is sufficiently big [18].

2 Covariants

We let G, W be as before and we let U be another finite dimensional G-representation. Then the module of covariants associated to U is defined as $(R \otimes_{\mathbb{C}} U)^G$. It is easy to see that this is a finitely generated R^G-module. The definition is clearly compatible with direct sums, so henceforth we assume that U is irreducible. An element of some $(R \otimes U)^G$ is called a covariant. It corresponds to an equivariant polynomial map $W^* \to U$.

The notion of covariants stems from the classical invariant theory of forms. In modern terms this would correspond to $G = SL_n(\mathbb{C})$, $W = (S^u\mathbb{C}^n)^*$, $U = S^v\mathbb{C}^n$. One of the reasons that classical invariant theorists studied covariants rather than just invariants is probably that these allowed for greater flexibility. Indeed the forms themselves represent canonical covariants, and from these other covariants can be derived using equivariant differential operators such as transvectants.

More recently the interest in modules of covariants has revived because of their connection with the theory of inhomogeneous linear diophantine equations [16] and with the theory of PI-algebras and trace rings [13].

Given the Hochster-Roberts theorem, and the fact that $R \otimes U$ is a free R-module, it is rather natural to pose the following question.

Question 2.1 *Is $(R \otimes U)^G$ a Cohen-Macaulay R^G-module ?*

So we are asking whether $(R \otimes U)^G$ is a finitely generated free module over a polynomial ring in R^G. It is rather easy to see that the answer to this question is "no" in general.

Example 2.2 *Let $G = T = \mathbb{C}^*$, a one-dimensional torus and let $\chi : T \to \mathbb{C}^* : \alpha \mapsto \alpha$ be the identity character. Define $U = L_{\chi^{-1}}$, $W = L_\chi \oplus L_\chi \oplus L_{\chi^{-1}}$ where L_χ is the one-dimensional T-representation associated to χ. Then $R = \mathbb{C}[x,y,z]$, $M \overset{\text{def}}{=} U \otimes R = \mathbb{C}[x,y,z]$ where T acts on R and M as follows : take $\alpha \in T$, $f \in R$, $g \in M$. Then $\alpha \cdot f = f(\alpha x, \alpha y, \alpha^{-1} z)$, $\alpha \cdot g = \alpha^{-1} f(\alpha x, \alpha y, \alpha^{-1} z)$ $R^T = \mathbb{C}[xz, yz] \cong \mathbb{C}[s,t]$, $M^T = xR^T + yR^T = z^{-1}(xzR^T + yzR^T) \cong (s,t) \subset \mathbb{C}[s,t]$. So M^T is not free over R^T, which is a polynomial ring, and hence M^T is not Cohen-Macaulay.*

3 Isotypical components

There is a direct sum decomposition $R = \oplus R^G_\chi$ where the sum is over all irreducible characters of G and where R^G_χ denotes the sum of all irreducible subrepresentations of R with character χ. Let χ be the character of an irreducible representation U. Then it is easily seen that $R^G_\chi = (R \otimes U^*)^G \otimes U$ so that the Cohen-Macaulayness of $(R \otimes U^*)^G$ is equivalent to that of R^G_χ.

4 Some basic results

In this section we indicate the extremal situations with regard to question 2.1. Indeed, we will show that sometimes all modules of covariants for a given (G, W) are Cohen-Macaulay, but that usually this is only true for a finite number of them. The latter result was proved by Brion in [3]. We give a somewhat weaker version with a simple proof.

Theorem 4.1 [12, §5.1] *Every R^G_χ is Cohen-Macaulay if and only if the quotient map from $\operatorname{Spec} R$ to $\operatorname{Spec} R^G$ is equidimensional (that is : the generic fiber has the same dimension as the zero-fiber).*

Proof. Assume that $\operatorname{Spec} R \to \operatorname{Spec} R^G$ is equidimensional. Let $R_0 \subset R^G$ be a graded polynomial ring such that R^G/R_0 is finite. Then SpecR$\to$ $\operatorname{Spec} R_0$ is an equidimensional map between smooth varieties, and hence is flat by [1]. Since R^G_χ is a direct summand of R this implies that R^G_χ is flat over R_0, and since R^G_χ is in addition finitely generated this yields that R^G_χ/R_0 is free. The converse is proved by reversing the above argument. $\square$

Remark 4.2

- *The above result applies when G is finite.*
- *If G is semi-simple then, up to trivial summands, there are only a finite number of W such that $\operatorname{Spec} R \to \operatorname{Spec} R^G$ is equidimensional* [12, §5.9].
- *If $\operatorname{Spec} R \to \operatorname{Spec} R^G$ is equidimensional and G is connected then it is conjectured that R^G is itself a polynomial ring. This would imply that all modules of covariants are actually free, and hence R/R^G would also be free.*

Now we state a weak version of Brion's finiteness result. For the full version we refer to [3, §4.2]. Define $X = \operatorname{Spec} R(\cong W^*)$ and $X_s = \{x \in X \mid G_x = \mathrm{triv}\}$.

Theorem 4.3 *Assume that the generic orbit in X is closed. and that* $\operatorname{codim}(X - X_s, X) \geq 2$. *Then there are only a finite number of irreducible U such that $(R \otimes U)^G$ is Cohen-Macaulay.*

Proof. Let $R_0 \subset R^G$ be a graded polynomial ring such that R^G/R_0 is finite. Define $\Omega = \operatorname{Hom}(R^G, R_0)$. This is in a natural way a graded R^G-module which is isomorphic to the dualizing module of R^G, suitably shifted. It follows from [10] that $\Omega = (R \otimes L_\chi)^G(q)$ for some $\chi \in \operatorname{Hom}(G, \mathbb{C}^*)$ and some $q \in \mathbb{N}$.

Let $M = (R \otimes U)^G$. Then $\tilde{M} = \operatorname{Hom}_{R_0}(M, R_0)$ is the (shifted) Cohen-Macaulay dual of M. We have

$$\begin{aligned}
\tilde{M} &= \operatorname{Hom}_{R_0}(M, R_0) \\
&= \operatorname{Hom}_{R^G}(M, \Omega) \\
&= \operatorname{Hom}_{R^G}((R \otimes U)^G, (R \otimes L_\chi)^G(q)) \\
&= (R \otimes U^* \otimes L_\chi)^G(q)
\end{aligned} \tag{1}$$

The last equality is true on $X_s/\!\!/G$ by descent and because of the fact that the quotient map $X_s \to X_s/\!\!/G$ is a principal G-bundle (this follows from the Luna slice theorem). Then this equality is true on $X/\!\!/G$ because $X \to X/\!\!/G$ contracts no divisor and hence all modules of covariants are reflexive [3, §1.3].

The only thing we actually need from (1) is that the dual of an arbitrary module of covariants lives in degree $\geq -q$.

Assume now that M is Cohen-Macaulay. Then M is graded free over R_0. So as a graded R_0-module

$$M = \bigoplus_{i=1}^{n} R_0(-a_i)$$

for suitable $a_i \in \mathbb{N}$. Since $\tilde{M} = \operatorname{Hom}_{R_0}(M, R_0) = R_0(a_i)$ lives in degrees $\geq -q$ we deduce that $a_i \leq q$ from which it follows that M contains an element of degree $\leq q$. Hence U^* is contained in $\oplus_{i=0}^{q} S^i W$. So there are only a finite number of possibilities. □

Remark 4.4

- *In* [19]*, in the case that G is a torus, necessary and sufficient conditions are given for the existence of only a finite number of Cohen-Macaulay modules of covariants.*

- *The proof of 4.3 yields a necessary condition for $(R \otimes U)^G$ to be Cohen-Macaulay. This condition is in general far from sufficient. However consider the following case* [3] $G = \mathrm{Sl}_n(\mathbb{C})$, $W = (\mathbb{C}^n)^{n+1}$. *Then the hypothesis of Theorem 4.3 are satisfied, and furthermore R^G is a polynomial ring. Thus we may take $q = 0$. Then it follows that the only U for which $(R \otimes U)^G$ is Cohen-Macaulay is the trivial representation.*

- *If G is semisimple and connected then, up to trivial summands, there are only a finite number of W for which the hypothesis of Theorem 4.3 are not satisfied.*

5 The torus case

In this section we assume that $G = T = (\mathbb{C}^*)^s$ is an s-dimensional torus. Let $\gamma_1, \gamma_2, \dots, \gamma_d \in X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$ be the weights of W. So $R = \mathbb{C}[x_1, \dots, x_d]$ with, for $z \in T$, $z \cdot x_i = \gamma_i(z) x_i$. Let $\chi \in X(T)$. Then there exist $\Phi = (\phi_{ij})_{ij}$, $\alpha = (\alpha_i)_i$ such that $\gamma_i(z_1, \dots, z_s) = z_1^{\phi_{1i}} \cdots z_s^{\phi_{si}}$, $\chi(z_1, \dots, z_s) = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ Then it follows that

$$\begin{aligned} R^T &= \oplus_{\Phi\beta=0, \beta\geq 0} \mathbb{C} x^{\beta_1} \cdots x_d^{\beta_d} \subset R \\ R^T_\chi &= \oplus_{\Phi\beta=\alpha, \beta\geq 0} \mathbb{C} x^{\beta_1} \cdots x_d^{\beta_d} \subset R \end{aligned}$$

So R^T, R^T_χ have a basis that is indexed by the solutions of a system of respectively homogeneous and inhomogeneous linear diophantine equations. This observation was used by Stanley to make a connection between combinatorics and commutative algebra [16].

Since R is $\mathbb{Z}^d$-graded in the natural way, and this grading is compatible with the T-action on R, we may define the $\mathbb{Z}^d$-graded Hilbert series of R^T_χ.

$$F(R^T_\chi, u) = \sum_{\beta\in\mathbb{Z}^d} \dim(R^T_\chi)_\beta u_1^{\beta_1} \cdots u_d^{\beta_d} = \sum_{\Phi\beta=\alpha} u_1^{\beta_1} \cdots u_d^{\beta_d}$$

Thus $F(T^T_\chi, u)$ is the generating function for the solutions to $\Phi\beta = \alpha$. It is easily seen to be a rational function.

The Hochster-Roberts theorem, and also the proof of Brion's result, suggest that in order for R^T_χ to be Cohen-Macaulay, χ should in some sense be small compared to W. An appropriate notion of smallness was found by Stanley.

Definition 5.1 *A character $\chi \in X(T)$ is strongly critical for (T, W) if and only if $\chi = \sum_{i=1}^d \mu_i \gamma_i$ in $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, with $\mu_i \in]-1, 0]$.*

Theorem 5.2 [15] *Assume that χ is strongly critical for (T, W). Then*

1. *R^T_χ is Cohen-Macaulay.*
2. *Assume that the generic orbit in W^* is closed. Then the following identity holds for the $\mathbb{Z}^d$-graded Hilbert series of R^T_χ (The "Monster reciprocity theorem.")*

$$F(R^T_\chi, u^{-1}) = \pm u_1 \cdots u_d F(R^T_{-\chi-\sum\gamma_i}, u) \qquad (2)$$

The reciprocity theorem implies "reciprocity" results for the solutions of $\Phi\beta = \alpha$ [14].

Stanley proved Theorem 5.2.2 first in [14] by writing $F(R^T_\chi, u)$ as an integral over a complex torus (the "Molien-Weyl" formula). Then repeatedly applying the residue theorem yields the result.

In [15][16] the results 5.2.1 and 5.2.2 are proved using commutative algebra. It turns out that 5.2.1 implies 5.2.2.

Remark 5.3 *The terminology of strongly critical weights is mine, but the notion was introduced by Stanley in* [15] *as a special case of so-called "critical" weights, a slightly more general concept.*

6 Strongly critical characters in the general case

In this section we let G be general again and we let $T \subset G$ be a maximal torus. Let $\chi : G \to \mathbb{C}$ be an irreducible character and let $\chi \mid T = \chi_1 + \cdots + \chi_l$ where the χ_i are characters of T. Let Ω be the set of roots of G.

Definition 6.1 *χ is strongly critical for (G, W) if and only if, for all $S \subset \Omega$ and for all $i \in \{1, \ldots d\}$, it is true that*

$$\chi_i - \sum_{\rho \in S} \rho \tag{3}$$

is strongly critical for (T, W).

Again the strongly critical characters are those that are sufficiently close to the trivial representation.

The motivation for definition 6.1 is the following result by Stanley.

Theorem 6.2 [15] *Assume that χ is strongly critical for (G, W) and that the generic T-orbit in W^* is closed. Then the $\mathbb{Z}$-graded Hilbert series of R^G_χ has the property*

$$F(R^G_\chi, u^{-1}) = \pm u^d F(R^G_\psi, u) \tag{4}$$

where $\psi(g) = \chi(g^{-1})\lambda(g^{-1})$, with λ the character of $\Lambda^d W$ (so in the notation of §5, $\lambda = \sum \gamma_i$).

Proof. We will only sketch the proof. If we expand the integrand in the Molien-Weyl formula for $F(R^G_\chi, u)$, then we see that it is a sum of $F(R^T_\mu, u)$ where the μ's are weights of the form (3). Then (2) applies to each of these terms, and one obtains (4) by summing. $\square$

If one compares Theorem 5.2 with Theorem 6.2 then one is naturally lead to the following conjecture.

Conjecture 6.3 *If χ is strongly critical for (G, W) then R^G_χ is Cohen-Macaulay.*

This conjecture was proved by the author under some reasonable additional hypotheses [18]. For example it is true if G is semi-simple, and the generic orbit of W^* has finite stabilizer.

7 Local cohomology

Let us specialize to our setting the standard local cohomology criterion for a module to be Cohen-Macaulay.

Lemma 7.1 *R^G_χ is Cohen-Macaulay if and only if $H^i_{(R^G)^+}(R^G_\chi) = 0$ for $i = 0, \ldots, h-1$, where h is the Krull dimension of R^G.*

There are many equivalent definitions of local cohomology. One of them is as follows. Choose $f_1, \ldots, f_k$ homogeneous such that $\mathrm{rad}(f_1, \ldots, f_k) = (R^G)^+$. Then $H^*_{(R^G)^+}(R^G_\chi)$ is the homology of the complex

$$0 \to R^G_\chi \to \bigoplus_i (R^G_\chi)_{f_i} \to \bigoplus_{j>i} (R^G_\chi)_{f_i f_j} \to \cdots \to (R^G_\chi)_{f_1 \cdots f_k} \to 0 \qquad (5)$$

with standard alternating boundary maps. Suppose now that $G = T$ is a torus. Then Stanley found a combinatorial way to describe (5). Since T is a torus we may choose the $f_1, \ldots, f_k$ to be monomials and then (5) is compatible with the $\mathbb{Z}^d$-grading on R. Furthermore if we fix $\beta \in \mathbb{Z}^d$ and we restrict (5) to degree β then it is easy to see that one obtains the chain complex of a simplicial complex. Then further results in algebraic topology are used to study these simplicial complexes. This eventually leads to a precise criterion for R^T_χ to be Cohen-Macaulay, and also to the proof of Theorem 5.2.

As beautiful as this method is, it has the drawback of being restricted to the torus case. Since our goal is to study the general case, a different approach is needed.

Let G be general again. Comparing (5) with the complex

$$L^{\cdot}: \quad 0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{j>i} R_{f_i f_j} \to \cdots \to (R^G_\chi)_{f_1 \cdots f_k} \to 0$$

we find that $H^*_{(R^G)^+}(R^G_\chi) = H^*(L^{\cdot})^G_\chi = H^*_I(R)^G_\chi$ where $I = \mathrm{rad}(f_1, \ldots, f_k)$. This formula was stated in [9]. It essentially says that $H^*_I(R)$ encodes the information about the Cohen-Macaulay property for all modules of covariants. Of course, given the difficulty of computing $H^*_I(R)$ for an arbitrary ideal $I \subset R$, this doesn't appear to be much progress. Things become clearer when we go to the associated varieties. Put $X \cong \mathrm{Spec}\, R (\cong W^*)$. Then we have $V(I) = X^u$ where X^u is the so-called null-cone or unstable locus in X. That is

$$X^u = \{x \in X \mid o \in \overline{Gx}\}$$

So we have $H^*_I(R) = H^*_{X^u}(X, \mathcal{O}_X)$. This leads us to following more general formulation of our original question.

Question 7.2 *What is the structure of $H^*_{X^u}(X, \mathcal{O}_X)$ as G-module ?*

At this stage, it is useful to make another generalization, which turns out to clarify matters. Let $\mathcal{D}_X$ be the sheaf of differential operators on X. Then it is standard that the (sheaffifications of) $H^*_{X^u}(X, \mathcal{O}_X)$ are holonomic $\mathcal{D}_X$-modules with regular singularities. Thus we ask

Question 7.3 *What is the structure of $H^*_{X^u}(X, \mathcal{O}_X)$ as $(G, \mathcal{D}_X)$-module ?*

A very useful fact is that if G is connected then the G-structure on $H^*_{X^u}(X, \mathcal{O}_X)$ is encoded in the $\mathcal{D}_X$-structure [20, §3.1].

8 Geometric Invariant Theory

Insight in the structure of X^u is provided by geometric invariant theory. Let $T \subset B \subset G$ be a maximal torus and a Borel subgroup of G. Let $\lambda \in Y(T)$ be a one-parameter subgroup of T. Define

$$X_\lambda = \{x \in X \mid \lim_{t \to 0} \lambda(t)x = 0\}$$

$$P_\lambda = \{g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$$

It is easy to see that X_λ is a linear subspace of X and $P_\lambda X_\lambda = X_\lambda$. In addition it was shown by Kempf [8] that P_λ is a parabolic subgroup of G.

Now the Hilbert-Mumford theorem yields that

$$X^u = \bigcup_{P_\lambda \supset B} G X_\lambda$$

Furthermore the projection map $G \times^{P_\lambda} X_\lambda \to G X_\lambda$ is usually a resolution of singularities.

9 The torus case revisited

Here we indicate how one may answer question 7.3 in the torus case. Notation is as in §5 and §7. By §8 : $X^u = \bigcup_{\lambda \in Y(T)} X_\lambda$. Let $\gamma_1, \ldots, \gamma_d$ be the weights of W, with corresponding weight vectors $w_1, \ldots, w_d$. Then $X_\lambda \subset X (\cong W^*)$ is the linear span of those w_i^* for which $\langle \lambda, \gamma_i \rangle < 0$, where $\langle\ ,\ \rangle$ denotes the canonical pairing between $Y(T)$ and $X(T)$. This description of X_λ still makes sense if $\lambda \in Y(T)_{\mathbb{R}}$, so we extend the definition of X_λ accordingly. If $U \subset Y(T)_{\mathbb{R}}$ then we put $X_U = \bigcup_{\lambda \in U} X_\lambda$. If $B \subset Y(T)_{\mathbb{R}}$ is the open unit ball for some norm on $Y(T)_{\mathbb{R}}$ then with this notation $X^u = X_B$. The key lemma is the following

Lemma 9.1 *Assume that $U_1, U_2 \subset Y(T)_{\mathbb{R}}$ are convex and that $U_1 \cup U_2$ is also convex. Then*

$$X_{U_1} \cap X_{U_2} = X_{U_1 \cap U_2}$$

Lemma 9.1 leads to a Mayer-Vietoris sequence

$$\to H^i_{X_{U_1\cap U_2}}(X,\mathcal{O}_X) \to H^i_{X_{U_1}}(X,\mathcal{O}_X)\oplus H^i_{X_{U_2}}(X,\mathcal{O}_X) \to H^i_{X_{U_1\cup U_2}}(X,\mathcal{O}_X)\to$$

This observation makes it plausible that by "cutting up" B in small pieces, the computation of $H^*_{X^u}(X,\mathcal{O}_X)$ may somehow be reduced to that of $H^*_{X_\sigma}(X,\mathcal{O}_X)$ for $\sigma \subset B$ so small that there exist $\lambda \in \sigma$ such that $X_\lambda = X_\sigma$. The appropriate technical tool to do this is to use a certain spectral sequence which is shown to degenerate [19].

Now we will state the main result from [19] but first we have to introduce some more notation. We put $s = \dim T$ and if $\lambda, \mu \in Y(T)_{\mathbb{R}}$ then we put $\lambda \sim \mu$ if $X_\lambda = X_\mu$. We let Λ be a set of representants for $B/\sim$ and if $\lambda \in \Lambda$ then we put $B_\lambda = \{\mu \in B \mid \mu \sim \lambda\}$ and $\Phi_\lambda = \overline{B_\lambda} - B_\lambda$. It is easy to see that Φ_λ may be given the structure of a finite CW-complex.

Theorem 9.2 *There exists a filtration on $H^i_{X^u}(X,\mathcal{O}_X)$ as $(T,\mathcal{D}_X)$-module such that*

$$\mathrm{gr}\, H^i_{X^u}(X,\mathcal{O}_X) = \oplus_{\lambda\in\Lambda} H^{d_\lambda}_{X_\lambda}(X,\mathcal{O}_X)^{\oplus u_{i,\lambda}}$$

where $u_{i,\lambda} = \dim \tilde{H}^{i+s-d_\lambda-1}(\Phi_\lambda)$, $d_\lambda = \mathrm{codim}(X_\lambda, X)$.

To apply the above theorem, one needs to know the T-action on $H^{d_\lambda}_{X_\lambda}(X,\mathcal{O}_X)$ but this can be very easily computed [19]. This yields a description of the T-structure of $H^*_{X^u}(X,\mathcal{O}_X)$ from which we can obtain the local cohomology modules $H^*_{(R^T)^+}(R^T_\chi)$. Inspection reveals that there should be a connection between the Φ_λ and the simplicial and polyhedral complexes introduced by Stanley, but the nature of this connection is not at all clear.

Example 9.3 *Let $T = (\mathbb{C}^*)^2$. Then $X(T), Y(T)$ may be identified with $\mathbb{Z}^2$. Take*

$$W = L^{\oplus 2}_{(-1,0)} \oplus L^{\oplus 2}_{(0,1)} \oplus L^{\oplus 2}_{(1,1)} \oplus L^{\oplus 2}_{(1,-1)}$$

In figure 1 we have given a graphical representation of this example. The weights of W have been indicated by fat dots. Hence there are four fat dots, each representing a weight of multiplicity two. The interior of the white area represents those χ such that R^T_χ is Cohen-Macaulay. In particular there are only a finite number of such weights, in accordance with Theorem 4.3. The interior of the region bounded by the dashed lines are the strongly critical weights. Since this is not all of the white area, we see that the converse to 5.2.1 is false.

10 The general case

Theorem may to some extent be generalized to arbitrary G. Let the notation be as in §7 and §8. We need the following conditions.

Condition (*) • *If $\lambda, \mu \in Y(T)$ such that $X_\lambda \neq X_\mu$ then $GX_\lambda \neq GX_\mu$.*

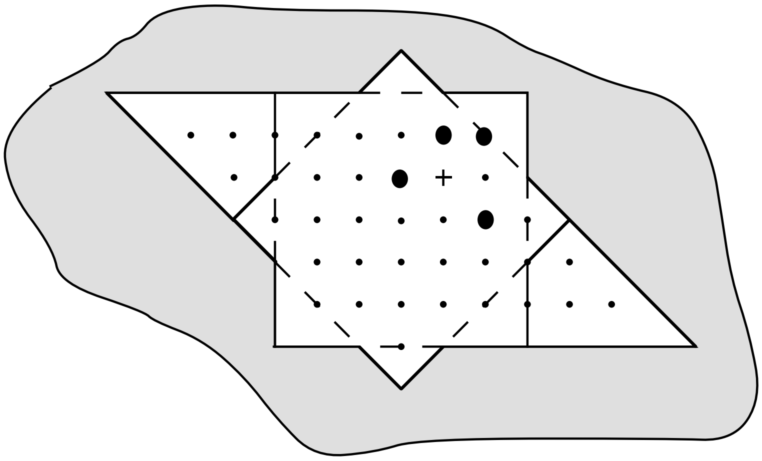

Figure 1:

- *If $\lambda \in Y(T)$ then there exist $\mu \in Y(T)$ with $X_\mu = X_\lambda$ such that the projection map $G \times^{P_\mu} X_\mu \longrightarrow GX_\mu$ is small. (A map $\pi : Y \longrightarrow X$ is said to be small if for any $n > 0$, $\operatorname{codim}\{y \in Y \mid \dim \pi^{-1} y \geq n\} > 2n$.)*

Then we have [20] :

Theorem 10.1 *Assume that condition (*) holds. Then there is a filtration on $H^i_{X^u}(X, \mathcal{O}_X)$ as $(G, \mathcal{D}_X)$-module such that*

$$\operatorname{gr} H^i_{X^u}(X, \mathcal{O}_X) = \oplus_{\lambda \in Y(T)} \mathcal{L}(GX_\lambda, X)^{\oplus u_{i,\lambda}}$$

where $u_{i,\lambda} \in \mathbb{N}$ and $\mathcal{L}(GX_\lambda, X)$ is the unique simple subquotient of $H^*_{GX_\lambda}(X, \mathcal{O}_X)$ with support equal to GX_λ.

Remark 10.2 *1. As in Theorem 9.2 one may give explicit formulas for the $u_{i,\lambda}$ in terms of the reduced cohomology groups of certain finite CW-complexes.*

2. The G-structure of $\mathcal{L}(GX_\lambda, X)$ may be explicitly computed.

11 Some examples

We conclude with a few examples that can be computed using Theorem 10.1 and remark 10.2.

First let $G = \mathrm{Sl}_2(\mathbb{C})$ and $W = \oplus_{i=1}^e S^{d_i}\mathbb{C}^2$ with all $d_i > 0$. Define for $n \geq 0$

$$s^{(n)} = \begin{cases} \dfrac{(n+1)^2}{4} & \text{if } n \text{ is odd} \\ \dfrac{n(n+2)}{4} & \text{if } n \text{ is even} \end{cases}$$

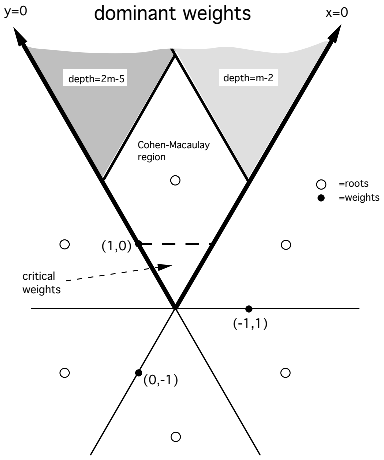

Figure 2:

and put $s = \sum s^{(d_i)}$. It follows from Stanley's criterion that $(R \otimes S^u V)^G$ in Cohen-Macaulay if $u < s - 2$. In this case it turns out that the converse is almost true. We separate the following cases

(A) $W = \mathbb{C}^2, S^2\mathbb{C}^2, \mathbb{C}^2 \oplus \mathbb{C}^2, \mathbb{C}^2 \oplus S^2\mathbb{C}^2, S^2\mathbb{C}^2 \oplus S^2\mathbb{C}^2, S^3\mathbb{C}^2, S^4\mathbb{C}^2$.

(B) All d_i are even and u is odd.

(C) All other cases.

Theorem 11.1 [4][17][20] *In case (A) the module $(R \otimes S^u\mathbb{C}^2)^G$ will always be Cohen-Macaulay. In case (B) $(R \otimes S^u\mathbb{C}^2)^G = 0$. In case (C) $(R \otimes S^u\mathbb{C}^2)^G$ is Cohen-Macaulay if and only if Stanley's criterion holds.*

Now let us look at the case $G = \mathrm{Sl}_3(\mathbb{C})$, $W = (\mathbb{C}^3)^n$. Let ω_1, ω_2 be the fundamental weights and let χ be a character of G with corresponding highest weight $x\omega_1 + y\omega_2$, $x \geq 0$, $y \geq 0$. Then R_χ^G is Cohen-Macaulay if $x < m - 3$, $y < m - 3$. On the other hand χ is strongly critical for (G, W) if $x + y + 4 < m$. This is represented graphically in figure 2.

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